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# ON $BG$-ALGEBRAS

**Abstract.** In this paper we introduce the notion of $BG$-algebras which is a generalization of $B$-algebras. We construct a $BG$-algebra from a non-empty set, which is non-group-derived. Moreover, using the notion of normal subalgebra, we obtain several isomorphism theorems of $BG$-algebra and related properties.

## 1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras ([3,4]). It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. In [1,2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: $BCH$-algebras. They have shown that the class of $BCI$-algebras is a proper subclass of the class of $BCH$-algebras. J. Neggers and H. S. Kim ([8]) introduced the notion of $d$-algebras which is another generalization of $BCK$-algebras, and then they investigated several relations between $d$-algebras and $BCK$-algebras as well as some other interesting relations between $d$-algebras and oriented digraphs. They also introduced $B$-algebras ([9, 10]), i.e., (I) $x * x = 0$; (II) $x * 0 = x$; (III) $(x * y) * z = x * (z * (0 * y))$, for any $x, y \in X$. Recently, Y. B. Jun, E. H. Roh and H. S. Kim ([5]) introduced a new notion, called a $BH$-algebra which is a generalization of $BCH/BCI/BCK$-algebras, i.e., (I); (II) and (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$, for any $x, y \in X$. In this paper we introduce the notion of $BG$-algebras which is a generalization of $B$-algebras. We construct a $BG$-algebra from a non-empty set, which is non-group-derived. Moreover, using the notion of normal subalgebra, we obtain several isomorphism theorems of $BG$-algebra and related properties.

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## 2. $BG$-algebras

A *BG-algebra* is a non-empty set $X$ with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(1) $$x * x = 0,$$

(2) $$x * 0 = x,$$

(3) $$(x * y) * (0 * y) = x,$$

for all $x, y \in X$.

EXAMPLE 2.1. Let $X := \{0, 1, 2\}$ be a set with the following table:

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 0 | 1 |
| 2 | 2 | 2 | 0 |

Then $(X; *, 0)$ is a $BG$-algebra.

THEOREM 2.2. *If $(X; *, 0)$ is a $B$-algebra, then $(X; *, 0)$ is a $BG$-algebra.*

Proof. Since $(X; *, 0)$ is $B$-algebra, the axioms (1) and (2) for $BG$-algebra are satisfied and $(x * y) * (0 * y) = x * ((0 * y) * (0 * y)) = x * 0 = x$ for any $x, y \in X$. Hence $(X; *, 0)$ is a $BG$-algebra. ■

REMARK. The converse of Theorem 2.2 does not hold. The $BG$-algebra $(X; *, 0)$ given in Example 2.1 is not a $B$-algebra, since $(0*2)*1 = 2*1 = 2$ and $0*(1*(0*2)) = 0*(1*2) = 0*1 = 1$ imply $(0*2)*1 \neq 0*(1*(0*2))$.

Thus the class of $B$-algebras is a proper subclass of $BG$-algebras.

PROPOSITION 2.3. *Let $(X; \circ, 0)$ be a group. If we define $x * y = x \circ y^{-1}$, then $(X; *, 0)$ is a $BG$-algebra.*

Proof. We see that $x * x = x \circ x^{-1} = 0$ and $x * 0 = x \circ 0^{-1} = x \circ 0 = x$. For any $x, y \in X$, we have $(x*y)*(0*y) = (x \circ y^{-1}) \circ (0 \circ y^{-1})^{-1} = x \circ (y^{-1} \circ y) = x \circ 0 = x$. Hence $(X; *, 0)$ is a $BG$-algebra. ■

From Proposition 2.3 we can see that every group $(X; \circ, 0)$ can determine a $BG$-algebra $(X; *, 0)$, called a *group-derived* $BG$-algebra. It is then a question of interest to determine whether or not all $BG$-algebras are group-derived. Example 2.1 is an example of a non-group-derived $BG$-algebra, since the only group of order 3 is $(Z_3; +, 0)$. Indeed, if we assume that Example 2.1 is group-derived, then $2 = 2*1 = 2+1^{-1} = 2+2 = 1$, a contradiction.

LEMMA 2.4. *Let $(X; *, 0)$ be a $BG$-algebra. Then*

(i) *the right cancellation law holds in $X$, i.e., $x*y = z*y$ implies $x = z$,*
(ii) $0 * (0 * x) = x$ *for all $x \in X$,*

(iii) *if* $x * y = 0$*, then* $x = y$ *for any* $x, y \in X$*,*
(iv) *if* $0 * x = 0 * y$*, then* $x = y$ *for any* $x, y \in X$*,*
(v) $(x * (0 * x)) * x = x$ *for all* $x \in X$*.*

Proof. (i). Assume that $x * y = z * y$. Then $x = (x * y) * (0 * y) = (z * y) * (0 * y) = z$. (ii). In axiom (3) for $BG$-algebra, replacing $y$ by $x$, we have that $(x * x) * (0 * x) = x$ implies $0 * (0 * x) = x$. (iii). If $x * y = 0$, then we have $x * y = 0 = y * y$. By applying (i) we obtain, $x = y$. (iv). If $0 * x = 0 * y$, then $x = (x * x) * (0 * x) = (y * y) * (0 * y) = y$ by the axiom (3) for $BG$-algebra. (v). $(x * (0 * x)) * x = (x * (0 * x)) * (0 * (0 * x)) = x$ by the axiom (3) and Lemma 2.4-(ii). ∎

THEOREM 2.6. *Let* $(X; *, 0)$ *be a BG-algebra with the identity* $(x * y) * z = x * (0 * ((0 * y) * z))$ *for all* $x, y, z \in X$*. Then* $(X; *, 0)$ *is group-derived.*

Proof. Define a binary operation "$\circ$" on $X$ by

$$x \circ y := x * (0 * y).$$

Then $x \circ 0 = x * (0 * 0) = x * 0 = x$ and $0 \circ x = 0 * (0 * x) = x$ by Lemma 2.4-(ii). Thus 0 acts like an identity element on $X$. Also $x \circ (0 * x) = x * (0 * (0 * x)) = x * x = 0$ and $(0 * x) \circ x = (0 * x) * (0 * x) = 0$. So $0 * x$ behaves like a multiplicative inverse for $x$ with respect to the operation "$\circ$". We claim that $(X; \circ)$ is a semigroup. Indeed,

$$\begin{aligned} x \circ (y \circ z) &= x * (0 * (y * (0 * z))) \\ &= x * (0 * ((0 * (0 * y)) * (0 * z))) && \text{[by Lemma 2.4-(ii)]} \\ &= (x * (0 * y)) * (0 * z) && \text{[by hypothesis]} \\ &= (x \circ y) \circ z. \end{aligned}$$

Note that $x \circ y^{-1} = x * (0 * y^{-1}) = x * (0 * (0 * y)) = x * y$. Hence $(X; *, 0)$ is also a group-derived $BG$-algebra. This completes the proof. ∎

The condition $(x*y)*z = x*(0*((0*y)*z))$ in Theorem 2.6 does not hold in general. In Example 2.1 we have $(2*1)*2 = 0$, while $2*(0*((0*1)*2)) = 2$.

EXAMPLE 2.7. Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

Then $(X; *, 0)$ is a $BG$-algebra satisfying the identity $(x * y) * z = x * (0 * ((0 * y) * z))$. So $(X; *, 0)$ is a group-derived $BG$-algebra.

PROPOSITION 2.8. *Every BG-algebra is a BH-algebra.*

Proof. It follows immediately from Lemma 2.4-(iii). ∎

The converse of Proposition 2.8 need not be true in general.

EXAMPLE 2.9. Let $X := \{0,1,2,3\}$ be a set with the following table:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 3 | 0 | 2 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 2 | 0 | 3 |
| 3 | 3 | 3 | 1 | 0 |

Then $(X;*,0)$ is a $BH$-algebra ([5]), but it is not a $BG$-algebra, since $(2*3)*(0*3) = 1 \neq 2$.

By Theorem 2.2 and Proposition 2.8, we know the following relation:

| The class of $B$-algebras | $\subset$ | The class of $BG$-algebras | $\subset$ | The class of $BH$-algebras |
|---|---|---|---|---|

We construct a $BG$-algebra for any non-empty set as follows:

THEOREM 2.10. *Let $X$ be a set with $0 \in X$. If we define a binary operation "$*$" on $X$ by*

$$x * y := \begin{cases} y & \text{if } x = 0, \\ 0 & \text{if } x = y, \\ x & \text{otherwise}, \end{cases}$$

*for any $x, y \in X$, then $(X;*,0)$ is a $BG$-algebra.*

Proof. For any $x, y \in X$, if $x = 0$, then $(x*y)*(0*y) = (0*y)*(0*y) = 0$. Assume $x \neq 0$. If $y = x$, then $(x*y)*(0*y) = (x*x)*(0*x) = 0*(0*x) = x$. If $y \neq x$, then $(x*y)*(0*y) = x*(0*y) = x*y = x$, proving that $(X;*,0)$ is a $BG$-algebra. ∎

Using Theorem 2.10 we can construct an infinitely many $BG$-algebras. The following example is constructed by Theorem 2.10.

EXAMPLE 2.11. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 1 | 1 |
| 2 | 2 | 2 | 0 | 2 |
| 3 | 3 | 3 | 1 | 0 |

Then $(X; *, 0)$ is a $BG$-algebra.

THEOREM 2.12. *The BG-algebra $(X; *, 0)$, $|X| \geq 3$, obtained by Theorem 2.10 is non-group-derived.*

Proof. Let $a, b, c \in X$ be distinct elements and $c \neq 0$. Then $c*a = c*b = c$. If we assume that $(X; *, 0)$ is a group-derived $BG$-algebra obtained from the group $(X; \circ)$ by Theorem 2.10, then $c = c*a = c \circ a^{-1}$ and $c = c*b = c \circ b^{-1}$, i.e., $a = b$, a contradiction. This proves the theorem. ∎

## 3. Homomorphisms and quotient $BG$-algebras

Let $(X; *_X, 0_X)$ and $(Y; *_Y, 0_Y)$ be $BG$-algebras. A mapping $\varphi : X \to Y$ is called a *BG-homomorphism* if $\varphi(x *_X y) = \varphi(x) *_Y \varphi(y)$ for any $x, y \in X$. A $BG$-homomorphism $\varphi : X \to Y$ is called a *BG-isomorphism* if $\varphi$ is a bijection, and denote it by $X \cong Y$. Let $\varphi : X \to Y$ be a $BG$-homomorphism. Then the subset $\{x \in X | \varphi(x) = 0_Y\}$ of $X$ is called the *kernel* of the $BG$-homomorphism $\varphi$, and denote it by $Ker\varphi$. In this section, we discuss several isomorphism theorems discussed in [9] in view of $BG$-algebras, and also obtain some consequences of structure theorems.

DEFINITION 3.1. Let $(X; *, 0)$ be a $BG$-algebra. A non-empty subset $S$ of $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for any $x, y \in S$.

In Example 2.11, $S_1 = \{0, 1\}$ and $S_2 = \{0, 1, 2\}$ are subalgebras of $X$. We know that any subalgebra of a $BG$-algebra is also a $BG$-algebra.

THEOREM 3.2. *Let $(X; *, 0)$ be a BG-algebra and $\emptyset \neq S \subseteq X$. Then the following are equivalent:*

(a) *$S$ is a subalgebra of $X$;*
(b) *$x * (0 * y), 0 * y \in S$ for any $x, y \in S$.*

Proof. (a)⇒(b). Since $S \neq \emptyset$, there exists an element $x \in S$ and so $0 = x*x \in S$. Since $S$ is closed under "$*$", $0*y \in S$ and thus $x*(0*y) \in S$.

(b)⇒(a). Since $x * y = x * (0 * (0 * y))$, $x * y \in S$ for any $x, y \in S$. ∎

A non-empty subset $N$ of $X$ is said to be *normal* ([9]) of $X$ if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$.

EXAMPLE 3.3. Let $X := \{0,1,2,3\}$ be a set with the following table:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 1 | 2 | 0 |

Then $(X;*,0)$ is a $BG$-algebra and $N = \{0,3\}$ is normal of $X$.

THEOREM 3.4. *Every normal subset $N$ of a BG-algebra $X$ is a subalgebra of $X$.*

Proof. If $x, y \in N$, then $x*0, y*0 \in N$. Since $N$ is normal, $x*y = (x*y)*(0*0) \in N$. Thus $N$ is a subalgebra of $X$. ∎

REMARK. The converse of Theorem 3.4 does not hold. Indeed, in Example 2.1, $N = \{0,2\}$ is a subalgebra of $X$, but it is not normal, since $0*2, 1*1 \in N$, while $(0*1)*(2*1) = 1 \notin N$.

LEMMA 3.5. *Let $N$ be a normal subalgebra of a BG-algebra $X$ and let $x, y \in N$. If $x*y \in N$, then $y*x \in N$.*

Proof. Let $x*y \in N$. Since $y*y = 0 \in N$ and $N$ is normal, $y*x = (y*x)*(y*y) \in N$. ∎

Let $(X;*,0)$ be a $BG$-algebra and let $N$ be a normal subalgebra of $X$. Define a relation $\sim_N$ on $X$ by $x \sim_N y$ if and only if $x*y \in N$, where $x, y \in X$. Then it is easy to show that $\sim_N$ is an equivalence relation on $X$. Denote the equivalence class containing $x$ by $[x]_N$, i.e., $[x]_N = \{y \in X | x \sim_N y\}$ and let $X/N = \{[x]_N | x \in X\}$.

THEOREM 3.6. *Let $N$ be a normal subalgebra of a BG-algebra $X$. Then $X/N$ is a BG-algebra.*

Proof. If we define $[x]_N * [y]_N := [x*y]_N$, then the operation " $*$ " is well-defined, since if $x \sim_N p$ and $y \sim_N q$, then $x*p \in N$ and $y*q \in N$ implies $(x*y)*(p*q) \in N$ by normality of $N$. So $x*y \sim_N p*q$ and so $[x*y]_N = [p*q]_N$. Note that $[0]_N = \{x \in X | x \sim_N 0\} = \{x \in X | x*0 \in N\} = \{x \in X | x \in N\} = N$. Checking remaining axioms is trivial and we omit the proof. ∎

The $BG$-algebra $X/N$ discussed in Theorem 3.6 is called the *quotient BG-algebra* of $X$ by $N$. The proofs of Theorems 3.7 and 3.10 follow from the Homomorphism Theorem for algebras ([6, p. 28–29]), and we omit the proofs.

THEOREM 3.7. *Let $N$ be a normal subalgebra of a BG-algebra $X$. Then the mapping $\gamma$ $X \to X/N$ given by $\gamma(x) := [x]_N$ is a surjective BG-homomorphism, and $Ker\gamma = N$.*

The mapping $\gamma$ discussed in Theorem 3.7 is called the *natural* (or *canonical*) $BG$-homomorphism of $X$ onto $X/N$.

THEOREM 3.8. *Let $\varphi : X \to Y$ be a BG-homomorphism. Then $\varphi$ is injective if and only if $Ker\varphi = \{0_X\}$.*

Proof. Let $x, y \in X$ with $\varphi(x) = \varphi(y)$. Then $\varphi(x) * \varphi(y) = 0_Y$. So $x * y \in Ker\varphi$. Since $Ker\varphi = \{0_X\}$, $x * y = 0_X$. Thus $x = y$ by Lemma 2.4-(iii). Hence $\varphi$ is injective. The converse is trivial, and we omit the proof. ∎

THEOREM 3.9. *Let $\varphi : X \to Y$ be a BG-homomorphism. Then the kernel $Ker\varphi$ is a normal subalgebra of $X$.*

Proof. Since $0_X \in Ker\varphi$, $Ker\varphi \neq \emptyset$. Let $x * y, a * b \in Ker\varphi$. Then $\varphi(x) * \varphi(y) = 0 = \varphi(a) * \varphi(b)$. Thus by Lemma 2.3-(iii), $\varphi(x) = \varphi(y)$ and $\varphi(a) = \varphi(b)$. It follows that $\varphi((x * a) * (y * b)) = \varphi(x * a) * \varphi(y * b) = (\varphi(x) * \varphi(a)) * (\varphi(y) * \varphi(b)) = (\varphi(x) * \varphi(a)) * (\varphi(x) * \varphi(a)) = 0$. So $(x * a) * (y * b) \in Ker\varphi$. Hence $Ker\varphi$ is a normal subalgebra of $X$. ∎

By Theorem 3.7 and Theorem 3.9, if $\varphi : X \to Y$ is a $BG$-homomorphism, then $X/Ker\varphi$ is a $BG$-algebra.

THEOREM 3.10. *Let $\varphi : X \to Y$ be a BG-homomorphism. Then $X/Ker\varphi \cong Im\varphi$. In particular, if $\varphi$ is surjective, then $X/Ker\varphi \cong Y$.*

THEOREM 3.11. *Let $\varphi : X \to Y$ be a BG-epimorphism, and let $K$ be a normal subalgebra of $Y$. Then $X/\varphi^{-1}(K) \cong Y/K$.*

Proof. Let $\gamma : Y \to Y/K$ be a natural $BG$-homomorphism, and let $\mu = \gamma \circ \varphi$. Then $\mu$ is a $BG$-homomorphism and $Ker\mu = Ker(\gamma \circ \varphi) = \{x \in X |\ (\gamma \circ \varphi)(x) = [\varphi(x)]_K = K\} = \{x \in X |\ \varphi(x) \in K\} = \{x \in X |\ x \in \varphi^{-1}(K)\} = \varphi^{-1}(K)$. By Theorem 3.10, we have $X/\varphi^{-1}(K) \cong Y/K$. ∎

The proofs of Theorems 3.12, 3.13 and 3.15 follow from the Second Isomorphism Theorem ([6, p. 149–150]), and we omit the proofs.

THEOREM 3.12. *Let $N$ and $K$ be normal subalgebras of a BG-algebra $X$, and let $K \subseteq N$. Then $X/N \cong \frac{X/K}{N/K}$.*

THEOREM 3.13. *Let $X, Y$ and $Z$ be BG-algebras, and $h : X \to Y$ be a BG-epimorphism and $g : X \to Z$ be a BG-homomorphism. If $Ker(h) \subseteq Ker(g)$, then there exists a unique BG-homomorphism $f : Y \to Z$ satisfying $f \circ h = g$.*

THEOREM 3.14. *Let $X, Y$ and $Z$ be BG-algebras, and $g : X \to Z$ be a BG-homomorphism and $h : Y \to Z$ be a BG-monomorphism. If $Im(g) \subseteq Im(h)$, then there is a unique BG-homomorphism $f : X \to Y$ satisfying $h \circ f = g$.*

Proof. For each $x \in X, g(x) \in Im\ (g) \subseteq Im\ (h)$. Since $h$ is a $BG$-monomorphism, there exists a unique $y \in Y$ such that $h(y) = g(x)$. Define a map $f : X \to Y$ by $f(x) = y$. Then $h \circ f = g$. Let $x_1, x_2 \in X$, then $g(x_1 * x_2) = h(f(x_1 * x_2)) = h(f(x_1) * f(x_2))$. Since $h$ is injective, $f(x_1 * x_2) = f(x_1) * f(x_2)$. So $f$ is a $BG$-homomorphism. The uniqueness of $f$ follows from the fact that $h$ is a $BG$-monomorphism. This completes the proof. ■

THEOREM 3.15. *Let $X$ and $Y$ be BG-algebras and let $f : X \to Y$ be a BG-homomorphism. If $N$ is a normal subalgebra of $X$ such that $N \subseteq Ker(f)$, then $\bar{f} : X/N \to Y$ defined by $\bar{f}([x]_N) := f(x)$ for all $x \in X$ is a unique BG-homomorphism such that $\bar{f} \circ \gamma = f$. where $\gamma : X \to X/N$ is natural BG-homomorphism.*

COROLLARY 3.16. *Let $X$ and $Y$ be BG-algebras and let $f : X \to Y$ be a BG-homomorphism. If $N$ is a normal subalgebra of $X$ such that $N \subseteq Ker(f)$, then the following are equivalent:*

(i) *there exists a unique BG-homomorphism $\bar{f} : X/N \to Y$ such that $\bar{f} \circ \gamma = f$, where $\gamma : X \to X/N$ is the natural BG-homomorphism;*
(ii) $N \subseteq Ker(f)$.

*Furthermore, $\bar{f}$ is a BG-monomorphism if and only if $N = Ker(f)$.*

Proof. (ii)⇒(i). By Theorem 3.13.
(i)⇒(ii). If $x \in N$, then

$$f(x) = (\bar{f} \circ \gamma)(x) = \bar{f}([x]_N) = \bar{f}([0]_N) = f(0) = 0.$$

Hence $x \in Ker(f)$.

Furthermore, $\bar{f}$ is a monomorphism if and only if $Ker\ \bar{f} = \{N\}$ if and only if $f(x) = 0$ implies $[x]_N = [0]_N = N$ if and only if $Ker(f) \subseteq N$. This completes the proof. ■

THEOREM 3.17. *Let $f : X \to Y$ be a BG-homomorphism, and let $M, N$ be normal subalgebras of $X$ and $Y$, respectively, such that $f(M) \subseteq N$. Then there exists a unique BG-homomorphism $h : X/M \to Y/N$ such that $h \circ p = q \circ f$, where $p : X \to X/M$ and $q : Y \to Y/N$ are natural BG-homomorphisms.*

Proof. Define a map $h : X/M \to Y/N$ by $h([x]_M) = [f(x)]_N$. Then $h$ is well-defined. Indeed, if $[x]_M = [y]_M$ $(x, y \in X)$, then $x * y \in M$. Thus $f(x) * f(y) = f(x * y) \in f(M) \subseteq N$. So $[f(x)]_N = [f(y)]_N$. If

$[x]_M, [y]_M \in X/M$, then $h([x]_M * [y]_M) = h([x * y]_M) = [f(x * y)]_N = [f(x)]_N * [f(y)]_N = h([x]_M) * h([y]_M)$. So $h$ is a $BG$-homomorphism. On the other hand, if $x$ is any element of $X$, then since $(h \circ p)(x) = h([x]_M) = [f(x)]_N = (q \circ f)(x)$, we obtain that $h \circ p = q \circ f$. To show the uniqueness of $h$, let $k : X/M \to Y/N$ be a $BG$-homomorphism such that $k \circ p = q \circ f$. Then $k([x]_M) = k(p(x)) = q(f(x)) = h(p(x)) = h([x]_M)$ for any $[x]_M \in X/M$. Thus $k = h$. This completes the proof. ∎

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