DEMONSTRATIO MATHEMATICA
Vol. XLI No 3 2008

V. B. L. Chaurasia, S.C. Pandey

UNIFIED ELLIPTIC-TYPE INTEGRALS AND ASYMPTOTIC FORMULAS

Abstract. The object of the present paper is to consider a unified and extended form of certain families of elliptic-type integrals, which have been discussed in number of earlier works on the subject due to their importance and applications in problems arising in radiation physics and nuclear technology. The results obtained are of general character and include the investigations carried out by several authors. We obtain asymptotic formulas for the unified elliptic-type integrals.

1. Introduction

Elliptic integrals occur in a number of physical problems [1–2], and frequently in the form of multiple integrals. For example, the problems dealing with the computation of the radiations field off axis from certain uniform circular disc radiating according to an arbitrary angular distribution law [3], when treated with Legendre polynomials expansion method, give rise to Epstein and Hubbell [4, 21] family of elliptic-type integrals:

$$\Omega_j(k) = \int_0^\pi (1 - k^2 \cos\theta)^{-j-\frac{1}{2}}\, d\theta; \quad j = 0, 1, 2, \ldots \tag{1.1}$$

and $0 \leq k < 1$.

Elliptic integrals (1.1) have been studied and generalized by many authors notably by Kalla [5, 6] and Kalla et al. [7]. Kalla and Al-Saqabi [8] and Saxena et al. [9], Kalla et al. [10], Srivastava and Bromberg [11] and others.

Some of these generalizations are as follows.

2000 *Mathematics Subject Classification*: 26A33, 33E12, 33C45.
Key words and phrases: Elliptic-type integrals, Hypergeometric functions, Asymptotic formulas.

Kalla [5, 6] introduced the generalization of the form:

$$R_\mu(k,\alpha,\gamma)=\int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\,\sin^{2\gamma-2\alpha-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\,d\theta, \tag{1.2}$$

where $0\le k<1$, $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\mu)>-1/2$.

Results for this generalization are also derived by Glasser and Kalla [12].

Al-Saqabi [14] defined and studied the generalization given by the integral

$$B_\mu(k,m,\nu)=\int_0^\pi \frac{\cos^{2m}(\theta)\,\sin^{2\nu}(\theta)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\,d\theta, \tag{1.3}$$

where $0\le k<1$; $m\in N_0$, $\mu\in C$, $\mathrm{Re}(\mu)>-1/2$.

Asymptotic expansion of (1.3) has recently been discussed by Matera et al. [15].

The integral

$$\Lambda_\nu(\alpha,k)=\int_0^\pi \frac{\exp[\alpha\sin^2(\theta/2)]}{(1-k^2\cos\theta)^{\nu+\frac{1}{2}}}\,d\theta, \tag{1.4}$$

where $0\le k<1$, $\alpha,\nu\in R$; presents another generalization of (1.1), given by Siddiqi [16].

Srivastava and Siddiqi [13] have given an interesting unification and extension of the families of elliptic- type integrals in the following form:

$$\Lambda_{(\lambda,\mu)}^{(\alpha,\beta)}(\rho;k)=\int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\,\sin^{2\beta-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\left[1-\rho\sin^2\left(\frac{\theta}{2}\right)\right]^{-\lambda}d\theta, \tag{1.5}$$

where $0\le k<1$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\lambda,\mu\in C$, $|\rho|<1$.

Kalla and Tuan [17] generalized Eq. (1.5) by means of the following integral and also obtained its asymptotic expansion

$$\begin{aligned}\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;k)=\int_0^\pi &\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)(1-k^2\cos\theta)^{-\mu-\frac{1}{2}}\\ &\times\left(1-\rho\sin^2\left(\frac{\theta}{2}\right)\right)^{-\lambda}\left(1+\delta\cos^2\left(\frac{\theta}{2}\right)\right)d\theta,\end{aligned} \tag{1.6}$$

where $0\le k<1$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\lambda,\mu,\gamma\in C$ and either $|\rho|,|\delta|<1$ or ρ (or δ) $\in C$ whenever $\lambda=m$ or $\gamma=-m$, $m\in N_0$, respectively.

Al-Zamel et al. [18] discussed a generalized family of elliptic-type integrals in the form:

$$(1.7)\quad Z_{(\gamma)}^{(\alpha,\beta)}(k) = Z_{(\gamma_1,\cdots,\gamma_n)}^{(\alpha,\beta)}(k_1,\ldots,k_n)$$

$$= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^{n}(1-k_j^2\cos\theta)^{-\gamma_j}\,d\theta$$

$$= B(\alpha,\beta)\prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j}F_D^{(n)}\left(\beta;\gamma_1,\ldots,\gamma_n;\alpha+\beta;\frac{2k_1^2}{k_1^2-1},\ldots,\frac{2k_n^2}{k_n^2-1}\right),$$

where $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $|k_j|<1$ $\gamma_j\in C$ $j=1,\ldots,n$, $F_D^{(n)}(\cdot)$ is the Lauricella hypergeometric function of n variables [22, p.163].

Saxena and Kalla [19] have studied a family of elliptic-type integrals of the form:

$$(1.8)\quad \Omega_{(\sigma_1,\ldots,\sigma_{n-2}:\delta,\mu)}^{(\alpha,\beta)}(\rho_1,\ldots,\rho_{n-2},\delta;k)$$

$$= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^{n-2}\left[1-\rho_j\sin^2\left(\frac{\theta}{2}\right)\right]^{-\sigma_j}$$

$$\times\left[1+\delta\cos^2\left(\frac{\theta}{2}\right)\right]^{-\gamma}(1-k^2\cos\theta)^{-\mu-\frac{1}{2}}\,d\theta,$$

where $0\le k<1$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$; σ_j $(j=1,\ldots,n-2)$, $\gamma,\mu\in C$;

$$\max\left\{|\rho_j|,\left|\frac{\delta}{1+\delta}\right|,\left|\frac{2k^2}{k^2-1}\right|\right\}<1.$$

In a recent paper, Saxena and Pathan [20] investigated an extension of Eq. (1.8) in the form:

$$(1.9)\quad \Omega_{(\sigma_1,\ldots,\sigma_m,\gamma;\tau_1,\ldots,\tau_n)}^{(\alpha,\beta)}(\rho_1,\ldots,\rho_m,\delta;\lambda_1,\ldots,\lambda_n)$$

$$= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{i=1}^{m}\left[1-\rho_i\sin^2\left(\frac{\theta}{2}\right)\right]^{-\sigma_i}$$

$$\times\left[1+\delta\cos^2\left(\frac{\theta}{2}\right)\right]^{-\gamma}\prod_{j=1}^{n}(1-\lambda_j^2\cos\theta)^{-\tau_j}\,d\theta,$$

where $\min(\mathrm{Re}(\alpha),\mathrm{Re}(\beta))>0$; $|\lambda_j|<1$; $\sigma_i,\gamma,\tau_j\in C$;

$$\max\left\{|\rho_j|,\left|\frac{2\lambda_j^2}{\lambda_j^2-1}\right|,\left|\frac{\delta}{1+\delta}\right|\right\}<1\qquad(i=1,\ldots,m;\ j=1,\ldots,n).$$

Here we consider a unified and generalized form of a family of elliptic-type integrals:

$$(1.10)\quad \bar{\Omega}^{(\alpha,\beta)}_{(\lambda_i,\tau_j)}((\rho_i),(\delta_i);k_j)$$
$$= \bar{\Omega}^{(\alpha,\beta)}_{\lambda_1,\dots,\lambda_N,\tau_1,\dots,\tau_M}(\rho_1,\dots,\rho_N,\delta_1,\dots,\delta_N;k_1\dots,k_M)$$
$$= \int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)$$
$$\times\prod_{i=1}^{N}\left[1+\rho_i\sin^2\left(\frac{\theta}{2}\right)+\delta_i\cos^2\left(\frac{\theta}{2}\right)\right]^{-\lambda_i}\prod_{j=1}^{M}[1-k_j^2\cos\theta]^{-\tau_j}\,d\theta$$

where $\min(\mathrm{Re}(\alpha),\mathrm{Re}(\beta))>0,\ |k_j|<1;\ \lambda_i,\tau_j\in C;$

$$\max\left\{|\rho_i|,\ |\delta_i|,\left|\frac{2k_j^2}{k_j-1}\right|,\left|\frac{\delta_i-\rho_i}{1+\delta_i}\right|\right\}<1\qquad(i=1,\dots,N\text{ and }j=1,\dots,M).$$

For $\lambda_i=\dots=\lambda_N=0$, Eq. (1.10) reduces to Eq. (1.7). Further if we set $\tau_1=\mu+1/2$, $\tau_2=\dots\tau_M=0$ and for $i=1,\dots,N-2$, $\rho_i=-\rho_i$, $\delta_i=0$ with $\rho_{N-1}=\rho_N=\delta_{N-1}=0$, $\delta_N=\delta$, Eq. (1.10) yields the family of elliptic-type integrals introduced by Saxena and Kalla [18]. Also if we set $\rho_i=-\rho_i$, $\delta_i=0$ (for $i=1,\dots,N-1$) with $\rho_N=0$, $\delta_N=\delta$, Eq. (1.10) reduces to Eq. (1.9). So elliptic type integrals given by (1.10) includes all the generalizations discussed in (1.7), (1.8) and (1.9).

2. Relations with other families of elliptic-type integrals

On comparing our result (1.10) with the definitions (1.2)–(1.6) we get the following relationships

$$(2.1)\quad \bar{\Omega}^{(\alpha,\gamma-\alpha)}_{(\underbrace{0,\dots,0}_{N},\mu+1/2,\underbrace{0,\dots,0}_{M-1})}(\rho_1,\dots,\rho_N,\ \delta_1,\dots,\delta_N;k,\underbrace{0,\dots,0}_{M-1})$$
$$=\bar{\Omega}^{(\alpha,\gamma-\alpha)}_{(\lambda_1,\dots,\lambda_N,\mu+1/2,\underbrace{0,\dots,0}_{M-1})}(\underbrace{0,\dots,0}_{N},\underbrace{0,\dots,0}_{N},\underbrace{0,\dots,0}_{M-1})=R_\mu(k,\alpha,\gamma)$$

where $0\le k<1$, $\mathrm{Re}(\gamma)$, $\mathrm{Re}(\alpha)>0$; $\mu\in C$.

$$(2.2)\quad \bar{\Omega}^{(\nu+1/2,\nu+1/2)}_{(-2m,\underbrace{0,\dots,0}_{N-1},\mu+1/2,\underbrace{0,\dots,0}_{M-1})}(-2,\rho_2,\dots,\rho_N,0,\ \delta_2,\dots,\delta_N;k,\underbrace{0,\dots,0}_{M-1})$$
$$=\bar{\Omega}^{(\nu+1/2,\nu+1/2)}_{(-2m,\lambda_2,\dots,\lambda_N,\mu+1/2,\underbrace{0,\dots,0}_{M-1})}(-2,\underbrace{0,\dots,0}_{N-1},\underbrace{0,\dots,0}_{N};k,\underbrace{0,\dots,0}_{M-1})$$
$$=2^{-2\nu}B_\mu(k,m,\nu)$$

where $0 \le k < 1$; $m \in N_0$, $\mu \in C$; $\mathrm{Re}(\nu) > -1/2$.

$$(2.3)\quad \lim_{\lambda\to\infty}\left[\bar{\Omega}^{(1/2,1/2)}_{(\lambda,\underbrace{0,\dots,0}_{N-1},\mu+1/2,\underbrace{0,\dots,0}_{M-1})}(-\rho/\lambda,\underbrace{0,\dots,0}_{N-1},\underbrace{0,\dots,0}_{N};k,\underbrace{0,\dots,0}_{M-1})\right] = \Lambda_\mu(\rho;k)$$

where $0 \le k < 1$; λ and μ in C.

$$(2.4)\quad \bar{\Omega}^{(\alpha,\beta)}_{(\lambda,\underbrace{0,\dots,0}_{N-1},\mu+1/2,\underbrace{0,\dots,0}_{M-1})}(-\rho,\rho_2\dots,\rho_N,0,\ \delta_2,\dots,\delta_N;k,\underbrace{0,\dots,0}_{M-1})$$
$$= \bar{\Omega}^{(\alpha,\beta)}_{(\lambda,\lambda_2,\dots,\lambda_N,\mu+1/2,\underbrace{0,\dots,0}_{M-1})}(-\rho,\underbrace{0,\dots,0}_{N-1},\underbrace{0,\dots,0}_{N};k,\underbrace{0,\dots,0}_{M-1}) = \Lambda^{(\alpha,\beta)}_{(\lambda,\mu)}(\rho;k)$$

where $0 \le k < 1$; $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$; $\lambda,\mu \in C$; $|\rho| < 1$.

$$(2.5)\quad \bar{\Omega}^{(\alpha,\beta)}_{(\lambda,\underbrace{0,\dots,0}_{N-2},r;\mu+1/2,\underbrace{0,\dots,0}_{M-1})}(-\rho,\rho_2,\dots,\rho_N,$$
$$\delta_1,\delta_2,\dots,\delta_{N-1},\delta;k,0,\dots,0)$$
$$= \bar{\Omega}^{(\alpha,\beta)}_{(\lambda,\lambda_2,\dots,\lambda_{N-1},\gamma;\mu+1/2,\underbrace{0,\dots,0}_{M-1})}(-\rho,\underbrace{0,\dots,0}_{N-1},\underbrace{0,\dots,0}_{M-1},\delta;k,\underbrace{0,\dots,0}_{M-1}$$
$$= \Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$$

where $0 \le k < 1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$; $\lambda,\mu,\gamma \in C$; either $|\rho| < 1$, $|\delta| < 1$ or ρ (or δ) $\in C$, whenever $\lambda = -m$ (or $\gamma = -m$); $m \in N_0$.

3. Explicit representation and asymptotic expansion

$$(3.1)\quad \bar{\Omega}^{(\alpha,\beta)}_{(\lambda_1,\dots,\lambda_N;\tau_1,\dots,\tau_M)}(\rho_1,\dots,\rho_N,\delta_1,\dots,\delta_N;k_1,\dots,k_M)$$
$$= \prod_{j=1}^{M}(1-k_j^2)^{-\tau_j}\prod_{i=1}^{N}(1+\delta_i)^{\lambda_j}\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1}\prod_{j=1}^{M}\left[1-\frac{2\omega k_j^2}{k_j^2-1}\right]^{-\tau_j}$$
$$\times\prod_{i=1}^{N}\left[1-\frac{(\delta_i-\rho_i)\omega}{1+\delta_i}\right]^{\lambda_j}d\omega.$$

On employing the formula [22, p. 163]

$$(3.2)\quad \frac{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F_D^{(n)}[\alpha,\ \beta_1,\dots,\beta_n;\ \gamma;\ x_1,\dots,x_n]$$
$$= \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux_1)^{-\beta_1}\dots(1-ux_n)^{-\beta_n}du.$$

where $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\alpha) > 0$. (3.1) becomes

$$(3.3)\quad \bar{\Omega}^{(\alpha,\beta)}_{(\lambda_1,\ldots,\lambda_N;\tau_1,\ldots,\tau_M)}(\rho_1,\ldots,\rho_N,\delta_1,\ldots,\delta_N;k_1\ldots,k_M)$$
$$= B(\alpha,\beta)\prod_{j=1}^{M}(1-k_j^2)^{-\tau_j}\prod_{i=1}^{N}[(1+\delta_i)^{-\lambda_i}]$$
$$\times F_D^{(M+N)}\Big[\beta,\,\tau_1,\ldots,\tau_M,\,\lambda_1,\ldots,\lambda_N;$$
$$\alpha+\beta;\,\frac{2k_1^2}{k_1^2-1},\ldots,\frac{2k_M^2}{k_M^2-1},\frac{\delta_1-\rho_1}{1+\delta_1},\ldots,\frac{\delta_N-\rho_N}{1+\delta_N}\Big],$$

where $F_D^{M+N}(\cdot)$ is the Lauricella function of $(M+N)$ variables, (3.1) reduces to the result given recently Saxena and Pathan [20]. If we set $\rho_i = -\rho_i$ (for $i = 1,\ldots,m$) and $\rho_i = 0$ (for $i = m+1,\ldots,N$) with $\delta_i = 0$ ($i = 1,\ldots,m,m+1,\ldots,N$) except for $i = m_t$, where the term m_t lying between $m+1$ and N. And $\min(\mathrm{Re}(\alpha),\mathrm{Re}(\beta)) > 0$, $|k_j| < 1$; $\lambda_i, \tau_j \in C$

$$\max\left[|\rho_i|,\,|\delta_i|,\left|\frac{2k_j}{k_j^2-1}\right|,\left|\frac{\delta_i-\rho_i}{1+\delta_i}\right|\right] < 1 \quad \text{for } i=1,\ldots,N \text{ and } j=1,\ldots,M.$$

We will now derive the asymptotic expansion of the generalized elliptic-type integral (1.10). Expressing the Lauricella hypergeometric type function $F_D^{M+N}(\cdot)$ in terms of Gauss hypergeometric function, we obtain

$$(3.4)\quad F_D^{(M+N)}\Big[\beta,\,\tau_1,\ldots,\tau_M,\,\lambda_1,\ldots,\lambda_N;$$
$$\alpha+\beta;\,\frac{2k_1^2}{k_1^2-1},\ldots,\frac{2k_M^2}{k_M^2-1},\frac{\delta_1-\rho_1}{1+\delta_1},\ldots,\frac{\delta_N-\rho_N}{1+\delta_N}\Big]$$
$$=\sum_{r_1,\ldots,r_{M+N-1}=0}^{\infty}\frac{(\beta)_{r_1+r_2+\ldots+r_{M+(N-1)}}\prod_{j=1}^{M-1}(\tau_j)_{j+N}\prod_{i=1}^{N}(\lambda_i)_{r_i}}{(\alpha+\beta)_{r_1+r_2+\ldots+r_{M+N-1}}}$$
$$\times\frac{\prod_{i=1}^{N}\left(\frac{\delta_i-\rho_i}{1+\delta_i}\right)^{r_i}}{r_i\,!}\frac{\prod_{j=1}^{M-1}\left[\frac{2k_j^2}{k_j^2-1}\right]^{r_{j+N}}}{r_{j+M}\,!}$$
$$\times\,{}_2F_1\left[\beta+r_1+\ldots+r_{N+M-1},\tau_M;\alpha+\beta+r_1+\ldots+r_{M+N-1};\frac{2k_M^2}{k_M^2-1}\right].$$

If $\beta-\tau_M$ is not an integer, then by an appeal to the analytic continuation formula for the Gauss hypergeometric function [23, p.559, equation 15.3.7] namely

$$(3.5)\qquad {}_2F_1(a,b;c;z)=\frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}\,(-z)^{-a}\,{}_2F_1[a,1-c+a;\;1-b+a;\;1/z] + \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}\,(-z)^{-b}\,{}_2F_1[b,1-c+b;\;1-a+b;\;1/z],$$

where $|\arg(-z)|<\pi$; we have

$$(3.6)\quad {}_2F_1\left[\beta+r_1+\ldots+r_{M+N-1},\tau_M;\alpha+\beta+r_1+\ldots+r_{M+N-1};\frac{2k_M^2}{k_M^2-1}\right]$$

$$=\frac{\Gamma(\alpha+\beta+r_1+\ldots+r_{M+N-1})\Gamma(\tau_M-\beta-r_1-\ldots-r_{M+N-1})}{\Gamma(\tau_M)\;\Gamma(\alpha)} \times\left(\frac{1-k_M^2}{2k_M^2}\right)^{\beta+r_1+\ldots+r_{M+N-1}}$$

$$\times\;{}_2F_1\left[(\beta+r_1+\ldots+r_{M+N-1}),1-\alpha;\; 1-\tau_M+\beta+r_1+\ldots+r_{M+N-1};\;\frac{k_M^2-1}{2k_M^2}\right]$$

$$+\frac{\Gamma(\alpha+\beta+r_1+\ldots+r_{M+N-1})\Gamma(\beta-\tau_M+r_1+\ldots+r_{M+N-1})}{\Gamma(\beta+r_1+\ldots+r_{N+(M+1)})\Gamma(\alpha+\beta-\tau_M+r_1+\ldots+r_{N+M-1})} \times\left(\frac{1-k_M^2}{2k_M^2}\right)^{\tau_M}$$

$$\times\;{}_2F_1\left[\tau_M,1-\alpha-\beta-r_1-\ldots-r_{M+N-1};1-\beta-r_1-\ldots-r_{M+N-1}+\tau_M;\;\frac{k_M^2-1}{2k_M^2}\right]$$

$$(3.7)\qquad =(\alpha+\beta)_{r_1+\ldots+r_{N+M-1}}\,\Gamma(\alpha+\beta)\Gamma(\tau_M-\beta) \times\left[\frac{(1-k_M^2)}{(2k_M^2)}\right]^{\beta}\left[\frac{k_M^2-1}{2k_M^2}\right]^{r_1+\ldots+r_{M+N-1}}$$

$$\times\;{}_2F_1\left[(\beta+r_1+\ldots+r_{N+M-1}),\; 1-\alpha;\;1-\tau_M+\beta+r_1+\ldots+r_{M+N-1};\frac{k_M^2-1}{2k_M^2}\right]$$

$$+\frac{(\alpha+\beta)_{r_1+\ldots+r_{M+N-1}}(\beta-\tau_M)_{r_1+\ldots+r_{M+N-1}}}{(\alpha+\beta-\tau_M)_{r_1+\ldots+r_{M+N-1}}(\beta)_{r_1+\ldots+r_{M+N-1}}}\;\frac{\Gamma(\alpha+\beta)\,\Gamma(\beta-\tau_M)}{\Gamma(\beta)\,\Gamma(\alpha+\beta-\tau_M)} \times\left(\frac{1-k_M^2}{2k_M^2}\right)^{\tau_M}$$

$$\times\;{}_2F_1\left[\tau_M,1-\alpha-\beta-r_1-\ldots-r_{M+N-1}+\tau_M;\; 1-\beta-r_1-\ldots-r_{N+M-1}+\tau_M;\;\frac{k_M^2-1}{2k_M^2}\right].$$

If we combine (3.3), (3.4) and (3.7) then yields the result

$$(3.8)\quad \bar{\Omega}^{(\alpha,\beta)}_{(\lambda_1,\ldots,\lambda_N;\tau_1,\ldots,\tau_M)}(\rho_1,\ldots,\rho_N,\delta_1,\ldots,\delta_N;k_1\ldots,k_M)$$

$$= \frac{\Gamma(\beta)\Gamma(\tau_M-\beta)(2k_M^2)^{-\beta}(1-k_M^2)^{\beta-\tau_M}}{\Gamma(\tau_M)} \prod_{i=1}^{N}(1+\delta_i)^{\lambda_i} \prod_{j=1}^{M-1}(1-k_j^2)^{-\tau_j}$$

$$\times \sum_{r_1,\ldots,r_{N+(M-1)}=0}^{\infty} \frac{(\beta)_{r_1+\ldots+r_{N+(M-1)}}}{(1+\beta-\tau_M)_{r_1+\ldots+r_{N+(M-1)}}} \prod_{i=1}^{N}(\lambda_i)_{r_i} \prod_{j=1}^{M-1}(\tau_j)_{r_{j+N}}$$

$$\times \frac{\prod_{i=1}^{N}\left[\left(\frac{\delta_i-\rho_i}{1+\delta_i}\right)\left(\frac{k_M^2-1}{2k_M^2}\right)\right]^{r_i} \prod_{j=1}^{M-1}\left[\frac{k_j^2}{(k_j^2-1)}\,\frac{(k_M^2-1)}{2k_M^2}\right]^{r_{j+N}}}{r_1!\,r_2!\,\ldots\,r_{N+(M-1)}!}$$

$$\times {}_2F_1\left[\beta+r_1+\ldots+r_{N+M-1},(1-\alpha);\;(1-\tau_M+\beta+r_1+\ldots+r_{M+N-1});\;\frac{k_M^2-1}{2k_M^2}\right]$$

$$+\frac{\Gamma(\alpha)\Gamma(\beta-\tau_M)}{\Gamma(\alpha+\beta-\tau_M)}(2k_M^2)^{-\tau_M}\prod_{i=1}^{N}(1+\delta_i)^{\lambda_i}\prod_{j=1}^{M-1}(1-k_j^2)^{-\tau_j}$$

$$\times \sum_{r_1,\ldots,r_{N+M-1}=0}^{\infty}\frac{(\beta-\tau_M)_{r_1+\ldots+r_{N+M-1}}}{(\alpha-\beta+\tau_M)_{r_1+\ldots+r_{N+M-1}}}$$

$$\times \frac{\prod_{i=1}^{N}\left[\left(\frac{\delta_i-\rho_i}{1+\delta_i}\right)^{r_i}(\lambda_i)_{r_i}\right]\prod_{j=1}^{M-1}\left[(\tau_j)_{r_{j+N}}\left(\frac{2k_j^2}{k_j^2-1}\right)^{r_{j+N}}\right]}{r_1!\,r_2!\,\ldots\,r_{N+M-1}!}$$

$$\times {}_2F_1\left[\tau_M,1-\alpha-\beta-r_1-\ldots-r_{N+M-1}+\tau_M;\;(1-\beta-r_1-\ldots-r_{N+M-1}+\tau_M);\frac{k_M^2-1}{2k_M^2}\right].$$

(3.8) may be regarded as the asymptotic series for

$$\bar{\Omega}^{(\alpha,\beta)}_{(\lambda_1,\ldots,\lambda_N;\tau_1,\ldots,\tau_M)}(\rho_1,\ldots,\rho_N,\delta_1,\ldots,\delta_N;k_1\ldots,k_M)$$

as $k_M^2 \to 1$.

Next we consider the expansion of

$${}_2F_1\left[\beta+r_1+\ldots+r_{N+M-1},\tau_M;\;\alpha+\beta+r_1+\ldots+r_{N+M-1};\;\frac{2k_M^2}{k_M^2-1}\right]$$

appearing in (3.4), when its upper parameters differ by integers. The following two cases arise:

(i) when $\tau_M = \beta - \mu$, $\mu = 0, 1, 2, \ldots$ and
(ii) when $\tau_M = \beta + \mu$, $\mu = 0, 1, 2, \ldots$.

In both the cases the results are derived by A1-Zamel et al [18,p.21-23, equations (4.4),(4.6) and (4.7)]. By making use of results of Al-Zamel et al. [18], the asymptotic expansion can be easily derived in the above two cases. Here we are discussing only first case. The second case can be treated similarly.

That is when $\tau_M = \beta - \mu$; $\mu = 0, 1, 2, \ldots$ then by virtue of the results given by Al-zamel [18, p.21-23], we find that

$$(3.9) \quad \bar{\Omega}^{(\alpha,\beta)}_{(\lambda_1,\ldots,\lambda_N;\tau_1,\ldots,\tau_M)}(\rho_1,\ldots,\rho_N,\delta_1,\ldots,\delta_N;k_1\ldots,k_M)$$

$$= (1-\beta)_\mu (2k_M^2)^{-\beta}(1-k_M^2)^\mu \prod_{i=1}^{N}(1+\delta_i)^{\lambda_i} \prod_{j=1}^{M-1}(1-k_j^2)^{-\tau_j}$$

$$\times \sum_{r_1,\ldots,r_{N+M-1}=0}^{\infty} \frac{\prod_{i=1}^{N}[(\lambda_i)_{r_i}] \prod_{j=1}^{M-1}[(\tau_j)_{r_{j+N}}]}{r_1!\ldots r_{N+M-1}!} \prod_{i=1}^{N}\left[\left(\frac{\delta_i-\rho_i}{1+\delta_i}\right)\frac{k_M^2-1}{2k_M^2}\right]^{r_i}$$

$$\times \prod_{j=1}^{M-1}\left[\frac{k_j^2(k_M^2-1)}{k_M^2(k_j^2-1)}\right]^{r_{j+N}} \sum_{r^*=0}^{\infty} \frac{(\beta)_{r^*+r_1+\ldots+r_{N+M-1}}(1-\alpha)_{r^*}}{r^*!(r^*+\mu+r_1+\ldots+r_{N+M-1})!}\left(\frac{k_M^2-1}{2k_M^2}\right)^{r^*}$$

$$\times [\ln(2k_M^2) - \ln(1-k_M^2) + \psi(1+\mu+r^*+r_1+\ldots+r_{N+M-1}) + \psi(1+r^*)$$

$$-\psi(\beta+r^*+r_1+\ldots+r_{N+M-1}) - \psi(\alpha-r^*)] + (2k_M^2)^{\mu-\beta}\prod_{i=1}^{N}[(1+\delta_i)^{\lambda_i}]$$

$$\times \prod_{j=1}^{M-1}[(1-k_j^2)^{-\tau_j}] \sum_{r_1,\ldots,r_{N+M-1}=0}^{\infty} \prod_{i=1}^{N}\left[\frac{(\lambda_i)_{r_i}\left(\frac{\delta_i-\rho_i}{1+\delta_i}\right)^{r_i}}{r_1!\ldots r_{N+M-1}!}\right] \prod_{j=1}^{M-1}\left[\frac{2k_j^2}{k_j^2-1}\right]^{r_{j+N}}$$

$$\times \sum_{r^*=0}^{\mu+r_1+\ldots+r_{N+M-1}-1} (\beta-\mu)_{r^*} \frac{(\mu+r_1+\ldots+r_{N+M-1}-r^*-1)!}{r^*!(\alpha)_{r_1+\ldots+r_{M+N-1}+\mu-r^*}}\left(\frac{k_M^2-1}{2k_M^2}\right)^{r^*}$$

$$(3.10) \quad = (1-\beta)_\mu (2k_M^2)^{-\beta}(1-k_M^2)^\mu \prod_{i=1}^{N}[(1+\delta_i)^{\lambda_i}] \prod_{j=1}^{M-1}[(1-k_j^2)^{-\tau_j}]$$

$$\times \sum_{r^*=0}^{\infty} \sum_{r_1,\ldots,r_{N+M-1}=0}^{\infty} (\beta)_{r^*+r_1+\ldots+r_{N+M-1}} (1-\alpha)_{r^*} \prod_{i=1}^{N} [(\lambda_i)_{r_i}] \prod_{j=1}^{M-1} [(\tau_j)_{N+j}]$$

$$\times \prod_{i=1}^{N} \left[\frac{(\delta_i - \rho_i)}{(1+\delta_i)} \frac{k_M^2 - 1}{2k_M^2} \right]^{r_i} \prod_{j=1}^{M-1} \left[\frac{k_j^2}{k_M^2} \frac{k_M^2 - 1}{k_j^2 - 1} \right]^{r_{j+N}} \left(\frac{k_M^2 - 1}{2k_M^2} \right)^{r^*}$$

$$\times [\ln(2k_M^2) - \ln(1-k_M^2) + \psi(1+r^*) + \psi(1+\mu+r^*+r_1+\ldots+r_{N+M-1})$$

$$- \psi(a-r^*) - \psi(\beta+r^*+r_1+\ldots+r_{N+M-1}) + (2k_M^2)^{\mu-\beta}$$

$$\times \prod_{i=1}^{N} [(1+\delta_i)^{\lambda_i}] \prod_{j=1}^{M-1} [(1-k_j^2)^{-\tau_j}]$$

$$\times \sum_{r_1,\ldots,r_{N+M-1}=0}^{\infty} \sum_{r^*=0}^{\mu+r_1+\ldots+r_{N+M-1}-1} (\beta-\mu)_{r^*}$$

$$\times \frac{(\mu+r_1+\ldots+r_{N+M-1}-r^*-1)!}{(r^*)!(\alpha)_{r_1+\ldots+r_{N+M-1}+\mu-r^*} \, r_1! \ldots r_{N+M-1}!}$$

$$\times \prod_{i=1}^{N} \left[(\lambda_i)_{r_i} \left(\frac{\delta_i - \rho_i}{1+\delta_i} \right)^{r_i} \right] \prod_{j=1}^{M-1} \left[(\tau_j)_{j+N} \left(\frac{2k_j^2}{k_j^2 - 1} \right)^{r_j+N} \right] \left(\frac{k_M^2 - 1}{2k_M^2} \right)^{r^*}.$$

Following the same procedure one can establish asymptotic formulas for other families of elliptic-type integrals.

Acknowledgement. The authors are grateful to Professor H.M. Srivastava, (University of Victoria, Canada) for his kind help and valuable suggestions in the preparation of this paper.

References

[1] J. D. Evans, J. H. Hubbell and V. D. Evans, *Exact series solution to the Epstein-Hubbell generalized elliptic-type integral using complex variable residue theory*, Appl. Math. Comp. 53 (1993), 173–189.

[2] E. L. Kaplan, *Multiple elliptic integrals*, J. Math. Phys. 29 (1950), 69–75.

[3] J. H. Hubbell, R. L. Bach, R. J. Herbold, *Radiation field from a circular disk source*, J. Res. N.B.S. 65 (1961), 249–264.

[4] M. J. Berger, J. C. Lamkin, *Sample calculation of gamma ray penetration into shelters, Contribution of sky shine and roof contamination*, J. Res. N.B.S. 60 (1958), 109–116.

[5] S. L. Kalla, *Results on generalized elliptic-type integrals*, Mathematical Structure Computational Mathematics - Mathematical Modelling (Edited by Bl. Sendov), Special Vol., pp. 216–219, Bulg. Acad. Sci., (1984).

[6] S. L. Kalla, *The Hubbell rectangular source integral and its generalizations*, Radiat. Phys. Chem., 41 (1993), 775–781.

[7] S. L. Kalla, S. Conde, J. H. Hubbell, *Some results on generalized elliptic-type integrals*, Appl. Anal. 22 (1986), 273–287.
[8] S. L. Kalla, B. Al-Saqabi, *On a generalized elliptic-type integral*, Rev. Bra. Fis. 16 (1986), 145–156.
[9] R. K. Saxena, S. L. Kalla, J. H. Hubbell, *Asymptotic expansion of a unified Elliptic-type integrals*, Math. Balkanica, 15 (2001), 387–396.
[10] S. L. Kalla, C. Leubner, J. H. Hubbell, *Further results on generalized elliptic-type integrals*, Appl. Anal. 25 (1987), 269–274.
[11] H. M. Srivastava, S. Bromberg, *Some families of generalized elliptic-type integrals*, Math. Comput. Modelling 21 (3) (1995), 29–38.
[12] M. L. Glasser, S. L. Kalla, *Recursion relations for a class of generalized elliptic-type integrals*, Rev. Tec. Ing. Univ. Zulia 12 (1989), 47–50.
[13] H. M. Srivastava, R. N. Siddiqi, *A unified presentation of certain families of elliptic-type integrals related to radiation field problems*, Radiat. Phys. Chem. 46 (1995), 303–315.
[14] B. N. Al-Saqabi, *A generalization of elliptic-type integrals*, Hadronic J. 10 (1987), 331–337.
[15] J. Matera, L. Galue, S. L. Kalla, *Asymptotic expansions for some elliptic-type integrals*, Raj. Acad. Phy. Sci. 1 (2) (2002), 71–82.
[16] R. N. Siddiqi, *On a class of generalized elliptic-type integrals*, Rev. Brasileira Fis. 19 (1989), 137–147.
[17] S. L. Kalla, V. K. Tuan, *Asymptotic formulas for generalized Elliptic-type integrals*, Comput. Math. Appl., 32 (1996), 49–55.
[18] A. Al-Zamel, V. K. Tuan, S. L. Kalla, *Generalized Elliptic-type integrals and asymptotic formulas*, Appl. Math. Comput., 114 (2000), 13–25.
[19] R. K. Saxena, S. L. Kalla, *A new method for evaluating Epstein-Hubbell generalized elliptic-type integrals*, Int. J. Appl. Math. 2 (2000), 732–742.
[20] R. K. Saxena, M. A. Pathan, *Asymptotic formulas for unified Elliptic-type integrals*, Demonstratio Math. 36 (3) (2003), 579–589.
[21] L. F. Epstein, J. H. Hubbell, *Evaluation of a generalized elliptic-type integral*, J. Res. N.B.S. 67 (1963), 1–17.
[22] A. M. Mathai, R. K. Saxena, *The H-Function with Application in Statistics and other Disciplines*, Halsted Press, New York, (1978).
[23] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, (1972).

V. B. L. Chaurasia
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF RAJASTHAN
JAIPUR, RAJASTHAN, INDIA

S.C. Pandey
DEPARTMENT OF MATHEMATICS
BANASTHALI VIDYAPITH
JAIPUR CAMPUS
JAIPUR, RAJASTHAN, INDIA
e-mail: sharedpandey@yahoo.co.in

Received August 20, 2007.