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DISTORTION THEOREMS IN THE CLASS $K_n(D)$

Abstract. This article presents the analysis of properties of functions belonging to the class $\mathrm{K}_n(D)$ earlier introduced by the author. This is a class of functions analytical in the domain D, for which the n-th divided difference $[F; z_0, \dots, z_n] \neq 0$ for any pairwise different points $z_0, \dots, z_n \in D$. For $n = 1$ the class $\mathrm{K}_1(D)$ consisting of functions univalent in the domain D is obtained.

The subclass $\tilde{\mathrm{K}}_n(\mathrm{E})$, formed by functions $F(z) = z^n + a_{n+1}z^{n+1} + \cdots$ analytical in a unit circle E is separated from class $\mathrm{K}_n(D)$ and its properties are considered.

The author touches classical and at the same time urgent questions, arising at the study of analytical functions, belonging to some class.

Introduction

Define the n-th order divided difference (see [1])

$$[F; z_0, \dots, z_n] = \frac{1}{2\pi i} \int_{\Gamma} \frac{F(\xi)\,\mathrm{d}\xi}{(\xi - z_0)\dots(\xi - z_n)},$$

where Γ is a closed rectifiable Jordan curve in domain D enclosing all the points $z_0, \dots, z_n \in D$.

DEFINITION 1. Denote by $\mathrm{K}_n(D)$, $n > 1$, the class of analytic functions F in D, such that $[F; z_0, \dots, z_n] \neq 0$ for pairwise different points $z_0, \dots, z_n \in D$ (see [2]).

Note that the class of univalent functions in D can be defined alternatively as the class of functions F, such that the first divided difference $[F; z_0, z_1] \neq 0$ for any different $z_0, z_1 \in D$ (see [3]).

Each function in $\mathrm{K}_n(D)$, $n \geq 1$, (class defined by the author) is not more than n-valent function in D (see [2]). The theory of this class covers a wide range of problems frequently encountered at the boundary between different branches of mathematics, e.g. the theory of interpolation and approximation of functions, the theory of univalent and multivalent functions, and the theory of Chebychev systems, etc.

The present work continues the investigation of the properties of the functions of the class $K_n(D)$, $n \geq 1$. We will estimate the modules of the functions and their derivatives and formulate the compactness tests for some families of functions in the class $K_n(D)$, $n \geq 1$.

1. Let us present a number of lemmas, the proofs of which can be found in [2].

LEMMA 1. *If $F \in K_n(D)$, then $F^{(n)}(z) \neq 0$ in D.*

LEMMA 2. *If $F \in K_n(D)$, then $aF(z) + P(z) \in K_n(D)$ for any $a \neq 0$ and for any polynomial P of degree not higher than $n - 1$.*

LEMMA 3. *Let the linear-fractional function $\xi = (az + b)(cz + d)^{-1}$, $ad - bc \neq 0$, maps domain D onto some domain D_0. If $F \in K_n(D_0)$, then*

$$(cz + d)^{n-1} F[(az + b)/(cz + d)] \in K_n(D).$$

If D is the unit disk which we denote by E, then, according to Lemmas 1 and 2 in the class $K_n(E)$, one can separate the class $\tilde{K}_n(E)$ of n-normalized in E functions of form

$$F(z) = z^n + \sum_{k=2}^{\infty} a_{k,n} z^{n+k-1}.$$

Let Φ_{n,δ_n} be a function analytic, n-normalized in E, and satisfying the condition

$$(1 - z^2)\Phi^{(n+1)}_{n,\delta_n}(z) - (n + 1)(\delta_n + z)\Phi^{(n)}_{n,\delta_n}(z) = 0, \quad \forall z \in E,$$

where

$$\delta_n = \sup_{F \in \tilde{K}_n(E)} \frac{1}{(n + 1)!} |F^{(n+1)}(0)|.$$

Note that

$$\frac{1}{n!}\Phi^{(n)}_{n,\delta_n}(z) = (1 + z)^{\frac{n+1}{2}(\delta_n - 1)}(1 - z)^{-\frac{n+1}{2}(\delta_n + 1)}, \tag{1}$$

$$(1 - r^2)^{n+1}\Phi^{(n)}_{n,\delta_n}(r)\Phi^{(n)}_{n,-\delta_n}(r) = 1, \quad \forall r = |z| < 1. \tag{2}$$

THEOREM 1. *If $F \in \tilde{K}_n(E)$, then*

$$\left|\ln((1 - r^2)^{\frac{n+1}{2}} \frac{F^{(n)}(z)}{n!})\right| \leq \frac{n + 1}{2}\delta_n \ln\frac{1 + r}{1 - r}, \quad \forall |z| = r < 1. \tag{3}$$

Proof. Let

$$F(z) = z^n + \sum_{k-2}^{\infty} a_{k,n} z^{n+k-1} \in \tilde{K}_n(E)$$

and Λ be the set of all linear-fractional functions of form

$$\omega = \omega(z) = (z+\zeta)\left(1+\bar{\zeta}z\right)^{-1}, \qquad \zeta \in \mathrm{E}.$$

According to Lemma 3, for any fixed $\zeta \in \mathrm{E}$ the function

$$\Psi(z;\zeta) = \left(1+\bar{\zeta}z\right)^{n-1} F((z+\zeta)/\left(1+\bar{\zeta}z\right)) = \sum_{m=0}^{\infty} b_m(\zeta)\, z^m$$

is in $\tilde{K}_n(E)$. According to Lemma 1, we have $b_n(\zeta) \neq 0$, $\forall \zeta \in \mathrm{E}$. By subtracting the polynomial

$$b_0(\zeta) + b_1(\zeta)\, z + \cdots + b_{n-1}(\zeta)\, z^{n-1}$$

from $\Psi(z;\zeta)$ and dividing the result by $b_n(\zeta)$, we obtain the function $F(z;\zeta)$, which, according to Lemma 2, belongs to the class $\tilde{\mathrm{K}}_n(\mathrm{E})$. Let us expand this function in to the power series of z:

$$F(z;\zeta) = z^n + \sum_{k=2}^{\infty} a_{k,n}(\zeta)\, z^{n+k-1}.$$

Observe that $F(z;0) \equiv F(z)$ and thus, $a_{k,n}(0) = a_{k,n}$, $k = 2, 3, \ldots$. Furthermore,

$$a_{2,n}(\zeta) = -\bar{\zeta} + (1-|\zeta|^2)\frac{F^{(n+1)}(\zeta)}{(n+1)F^{(n)}(\zeta)}. \tag{4}$$

Let us substitute z for ζ. Calculations show that

$$a_{2,n}(z) = \frac{r(1-r^2)}{(n+1)z}\frac{\partial}{\partial r}\ln(1-r^2)^{\frac{n+1}{2}}\frac{F^{(n)}(z)}{n!}. \tag{5}$$

Since $|a_{2,n}(z)| \le \delta_n$, $\forall z \in \mathrm{E}$, then in view of (4), (5), we get

$$\left|\ln\left((1-r^2)^{\frac{n+1}{2}}\frac{F^{(n)}(z)}{n!}\right)\right| = \left|\int_0^r \frac{(n+1)z}{r(1-r^2)} a_{2,n}(z)\,\mathrm{d}r\right| \le \int_0^r \frac{n+1}{1-r^2}\delta_n\,\mathrm{d}r = \frac{n+1}{2}\delta_n \ln\frac{1+r}{1-r}$$

for any $|z| = r < 1$. Thus the inequality (3) is proved.

THEOREM 2. *If* $F \in \tilde{\mathrm{K}}_n(\mathrm{E})$, *then*

$$\Phi^{(n)}_{n,-\delta_n}(r) \le \left|F^{(n)}(z)\right| \le \Phi^{(n)}_{n,\delta_n}(r), \qquad \forall |z| = r < 1 \tag{6}$$

and

$$|\arg \Phi^{(n)}(z)| \le \frac{n+1}{2}\delta_n \ln\frac{1+r}{1-r}, \qquad \forall |z| = r < 1. \tag{7}$$

Proof. From the formula (1) and inequality (3), we get

$$-\ln\left((1-r^2)^{\frac{n+1}{2}}\frac{\Phi_{n,\delta_n}^{(n)}(r)}{n!}\right)\le\ln\left((1-r^2)^{\frac{n+1}{2}}\frac{|F^{(n)}(z)|}{n!}\right)$$
$$\le\ln\left((1-r^2)^{\frac{n+1}{2}}\frac{\Phi_{n,\delta_n}^{(n)}(r)}{n!}\right).$$

On the other hand taking into account (2), we can write:

$$\ln\left((1-r^2)^{\frac{n+1}{2}}\frac{\Phi_{n,-\delta_n}^{(n)}(z)}{n!}\right)\le\ln\left((1-r^2)^{\frac{n+1}{2}}\frac{|F^{(n)}(z)|}{n!}\right)$$
$$\le\ln\left((1-r^2)^{\frac{n+1}{2}}\frac{\Phi_{n,\delta_n}^{(n)}(z)}{n!}\right).$$

Hence, the inequality (6) follows. In a similar way, we get (7).

THEOREM 3. *If* $F\in\tilde{\mathrm{K}}_n(\mathrm{E})$, *then*

$$|F^{(k)}(z)|\le\Phi_{n,\delta_n}^{(k)}(r),\qquad\forall\,|z|=r<1,\ k=0,1,\dots,n.$$

Proof. Theorem 2 for $k=n$ yields the inequality

$$\left|F^{(n)}(z)\right|\le\Phi_{n,\delta_n}^{(n)}(r),\qquad\forall\,|z|=r<1.$$

Integrating this inequality along the radius of the circle E from zero to the point $z_0=r_0e^{i\gamma_0}\neq0$, we obtain

$$\left|F^{(n-1)}(z_0)\right|=\left|\int_0^{r_0}F^{(n)}(z)\,\mathrm{d}z\right|\le\int_0^{r_0}\left|F^{(n)}\left(re^{i\gamma_0}\right)\right|\mathrm{d}r\le\int_0^{r_0}\Phi_{n,\delta_n}^{(n)}(r)\,\mathrm{d}r$$
$$=\Phi_{n,\delta_n}^{(n-1)}(r_0).$$

By virtue of the arbitrariness of the choice of z_0, we arrive at the inequality

$$\left|F^{(n-1)}(z)\right|\le\Phi_{n,\delta_n}^{(n-1)}(r),\quad\forall\,|z|=r<1.$$

The continuation of the process convince us in of the validity of Theorem 3. Note that for any function $F\in\tilde{\mathrm{K}}_n(\mathrm{E})$ we have also proven the inequality

$$|F(z)|\le\Phi_{n,\delta_n}(r),\quad\forall\,|z|=r<1.$$

2. Let us formulate and prove the following two general distortion theorems, analogous to those established by Koebe (see [4]) for univalent functions.

THEOREM 4. *For a given domain D and a compact $H \subset D$, there exists a positive number M such that the double inequality*

$$\frac{1}{M} \le \frac{\left|F^{(n)}(z_1)\right|}{\left|F^{(n)}(z_2)\right|} \le M, \quad \forall F(z) \in \mathrm{K}_n(D), \ \forall z_1, z_2 \in H \tag{8}$$

is valid.

Proof. Without losing generality we can assume that H is a bounded, closed domain in D. In fact, any compact can be included in some bounded, closed domain H', $H' \in D$, so if inequality (8) is satisfied for H', it is also satisfied with the same M for any compact in H'.

So, let H be a bounded, closed domain in D and h be the distance from this domain to the boundary of the domain D. Let us cover the (z)-plane by a square grid with the grid size of $h/4$, $h > 0$. The closed squares of the grid, which include the points from the set H form a closed domain $\bar{D}_0$, such that $H \subset \bar{D}_0$ and $\bar{D}_0 \subset D$.

Let z_1 and z_2 be two arbitrary points in the set H. Then, there exists a sequence of points $\zeta_1 = z_1, \zeta_2, \ldots, \zeta_p = z_2$ in D, such that any two successive points in this sequence are located in neighbouring squares and the number p of such points in this sequence does not exceed the number N of all squares, comprising the closed set $\bar{D}_0$. The distance between points ζ_k and ζ_{k+1}, $1 \le k \le p-1$, does not exceed $h/\sqrt{2}$. Note that $h/\sqrt{2} < 3h/4$. The circles $|z-\zeta_k| \le h$, $k = 1, \ldots, p$ are located completely inside domain D. Therefore, the function

$$\Psi_k(\zeta) = \frac{n!(F(\zeta_k + h\zeta) - \mathrm{P}(\zeta))}{h^n F^{(n)}(\zeta_k)} = \zeta^n + \sum_{m=2}^{\infty} c_m(\zeta_k)\zeta^{n+m-1},$$

where $\mathrm{P}(\zeta)$ is a polynomial of power not exceeding $n-1$, will belong to the class $\tilde{\mathrm{K}}_n(\mathrm{E})$ for $k = 1, \ldots, p-1$. Differentiating the function $\Psi_k(\zeta) n$ times with respect to ζ and taking into account Theorem 1, we obtain the inequalities

$$\left|\frac{F^{(n)}(\zeta_k + h\zeta)}{F^{(n)}(\zeta_k)}\right| \le \Phi^{(n)}_{n,\delta_n}(r), \qquad \forall |\zeta| = r < 1, \ k = 1, \ldots, p-1. \tag{9}$$

In particular, if $h\zeta = \zeta_{k+1} - \zeta_k$, then $4|\zeta| < 3$ and the inequalities (9) become

$$\frac{\left|F^{(n)}(\zeta_{k+1})\right|}{\left|F^{(n)}(\zeta_k)\right|} \le \Phi_{n,\delta_n}\left(\frac{3}{4}\right) = g, \qquad k = 1, \ldots, p-1.$$

Multiplying these inequalities, we get the inequality

$$\frac{|F^{(n)}(z_2)|}{|F^{(n)}(z_1)|} \le g^{p-1} \le g^{N-1} = M. \tag{10}$$

The positive number M is independent of the choice of the function $F(z)$ in the class $\mathrm{K}_n(D)$, as well as of the choice of points z_1, z_2 in the set H. By interchanging the points z_1, z_2 in (10) we obtain the second inequality from (8).

THEOREM 5. *For a given domain D and a compact $H \subset D$, there exist constants $b_0 = 1, b_1, \ldots, b_n$ such that there holds the inequality*

$$(11) \qquad |F(z_2)| \leq \sum_{m=0}^{n} b_m |F^{(m)}(z_1)|, \qquad \forall F(z) \in \mathrm{K}_n(D),\ \forall z_1, z_2 \in H.$$

Proof. As in the proof of Theorem 4, we suppose that H is a bounded, closed domain inside D. Let us use the same constructions as in the proof of Theorem 4. First, note that

$$(12) \qquad F^{(m-1)}(\zeta_k + h\zeta) = h\int_0^{\zeta} F^{(m)}(\zeta_k + h\zeta)d\zeta + F^{(m-1)}(\zeta_k),$$
$$m = 1, \ldots, n;\ k = 1, \ldots, p.$$

Furthermore (9) yields

$$(13) \qquad |F^{(n)}(\zeta_k + h\zeta)| \leq \Phi_{n,\delta_n}^{(n)}(r)|F^{(n)}(\zeta_k)|, \qquad k = 1, \ldots, p.$$

Now, we integrate any of the inequalities (13) along the radius connecting the origin to the point ζ. Assuming $|\zeta| = r < 1$ and using (12) for $m = n$, we obtain

$$(14) \quad |F^{(n-1)}(\zeta_k + h\zeta)| \leq h\Phi_{n,\delta_n}^{(n-1)}(r)|F^{(n)}(\zeta_k)| + |F^{(n-1)}(\zeta_k)|,\ k = 1, \ldots, p.$$

Assuming $m = n - 1$ in (12) and using (14), we have

$$|F^{(n-2)}(\zeta_k + h\zeta)| \leq h^2\Phi_{n,\delta_n}^{(n-2)}(r)|F^{(n)}(\zeta_k)| + hr|F^{(n-1)}(\zeta_k)| + |F^{(n-2)}(\zeta_k)|,$$
$$k = 1, \ldots, p.$$

Continuing this reasoning, we establish validity of the following inequalities:

$$|F^{(n-m)}(\zeta_k + h\zeta)| \leq h^m\Phi_{n,\delta_n}^{(n-m)}(r)|F^{(n)}(\zeta_k)| + \sum_{l=1}^{m} \frac{h^{m-l}r^{m-l}}{(m-l)!}|F^{(n-l)}(\zeta_k)|,$$
$$k = 1, \ldots, p;\ m = 1, \ldots, n.$$

Here, writing $h\zeta = \zeta_{k+1} - \zeta_k$ and taking into account that $4|\zeta| < 3$, we obtaint inequalities

$$(15)\quad |F^{(n-m)}(\zeta_{k+1})| \le h^m \Phi^{(n-m)}_{n,\delta_n}(3/4)|F^{(n)}(\zeta_k)| + \sum_{l=1}^{m} \frac{h^{m-l}(3/4)^{m-l}}{(m-l)!}|F^{(n-l)}(\zeta_k)|,$$
$$k = 1, \dots, p-1;\ m = 1, \dots, n.$$

For $m = n$, we have

$$(16)\quad |F(\zeta_{k+1})| \le h^n \Phi_{n,\delta_n}(3/4)|F^{(n)}(\zeta_k)| + \sum_{l=1}^{n} \frac{h^{n-l}(3/4)^{n-l}}{(n-l)!}|F^{(n-l)}(\zeta_k)|,$$
$$k = 1, \dots, p-1.$$

The value $|F^{(s)}(\zeta_k)|$ in (16) can be estimated by using the value $|F^{(s)}(\zeta_{k-1})|$, by replacing k by $k-1$ in (15). Similarly, one can estimate the values $|F^{(s)}(\zeta_{k-1})|$ by using the module $|F^{(s)}(\zeta_{k-2})|$, and so on. As the result, we arrive at the inequalities

$$|F(\zeta_{k+1})| \le |F(z_1)| + \sum_{m=1}^{n} b_m \left|F^{(m)}(z_1)\right|, \qquad k = 1, \dots, p-1,$$

where $b_1, \dots, b_n$ are constants. By substituting $k = p-1$, we will get $\zeta_p = z_2$ thus proving the inequality (11).

References

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