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Eliza Jabłońska

ONE–TO–ONE SOLUTIONS OF A GENERALIZED GOŁĄB–SCHINZEL EQUATION

Abstract. Let $\mathbb{K}$ be the field of real or complex numbers and let X be a nontrivial linear space over $\mathbb{K}$. Assume that $f : X \to \mathbb{K}$ is injective, $M : \mathbb{K} \to \mathbb{K}$ and $M \circ f \not\equiv \text{const}$. We give a necessary and sufficient condition for functions f and M to satisfy the equation

$$f(x + M(f(x))y) = f(x)f(y).$$

The functional equation

$$f(x + M(f(x))y) = f(x)f(y) \tag{1}$$

is a generalization of the well–known Gołąb–Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y). \tag{2}$$

The equation (2) has been studied, for the first time, by S. Gołąb and A. Schinzel [7] in the class of real continuous functions, in connection with looking for subgroups of the centroaffine group of the field. It turns out that this equation is useful when dealing with associative operations, subgroups of the semigroup of real affine mappings, classification of quasialgebras, subsemigroups of the group of affine mappings of the field, subgroups of the group L_2^1, differential equations in meteorology and fluid mechanics and classification of near–rings. In this connection the equation (2) and also its generalizations have been studied by many authors under various assumptions about f; for example: continuity [1], [2], [7], [11], boundedness on a set of second category with the Baire property or of the positive Lebesgue measure [5], [9], measurability in the Baire or Christensen sense [5], [4] and others.

An extensive bibliography considering the Gołąb–Schinzel equation can be found in a survey paper of J. Brzdęk [6].

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We consider here the equation (1) in the class of unknown functions $M : \mathbb{K} \to \mathbb{K}$ and one–to–one $f : X \to \mathbb{K}$, where X is a nontrivial linear space over the field $\mathbb{K}$ of real or complex numbers. Our results are analogous to results of J. Brzdęk from his Ph.D. thesis [3] for a special case of the equation (1), namely, the equation

$$f(x + f(x)^n y) = f(x)f(y).$$

Throughout the paper $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of all reals and complex numbers, respectively.

First we recall some results from [8] which will be useful in the sequel.

LEMMA 1 (see [8, Lemma 2 and Lemma 3]). *Let X be a nontrivial linear space over a commutative field $\mathbb{K}$, $f : X \to \mathbb{K}$, $f \not\equiv 0$, $f \not\equiv 1$, $A = f^{-1}(\{1\})$, $W = f(X) \setminus \{0\}$ and $M : \mathbb{K} \to \mathbb{K}$. Suppose f and M satisfy equation* (1). *Then the following assertions hold:*

(i) $(M \circ f)^{-1}(\{0\}) = f^{-1}(\{0\})$;
(ii) $A \setminus \{0\}$ *is the set of periods of* f;
(iii) A *is a subgroup of* $(X, +)$;
(iv) W *is a subgroup of* $(\mathbb{K} \setminus \{0\}, \cdot)$.

PROPOSITION 1 (cf. [8, Proposition 2]). *Let X be a nontrivial linear space over a commutative field $\mathbb{K}$, $f : X \to \mathbb{K}$, $M : \mathbb{K} \to \mathbb{K}$, $M(1) = 1$ and $M \circ f \not\equiv 1$. If f, M satisfy equation* (1), *then* $0 \in f(X)$.

The proof of Proposition 1 runs in exactly the same way as that of Proposition 2 in [8], so we omit it.

LEMMA 2. (see [8, Lemma 4]) *Let X be a nontrivial linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $f : X \to \mathbb{K}$, $M : \mathbb{K} \to \mathbb{K}$, $W = f(X) \setminus \{0\}$ and $M(W) \setminus \{1\} \neq \emptyset$. If f, M satisfy* (1), *then there exists an $x_0 \in X \setminus \{0\}$ such that*

$$\operatorname{supp} f := \{x \in X : f(x) \neq 0\} \subset (M(W) - 1)x_0 + A_0, \tag{3}$$

where A_0 is a linear subspace of X spanned by A over the field

$$\mathbb{K}_0 = \begin{cases} \mathbb{R}, \text{ if } M(W) \subset \mathbb{R}; \\ \mathbb{C}, \text{ otherwise.} \end{cases} \tag{4}$$

If, moreover, $A = A_0$, then $x_0 \notin A$,

$$f(x) = \begin{cases} a, \ x \in (M(a) - 1)x_0 + A \text{ and } a \in W; \\ 0, \text{ otherwise} \end{cases} \tag{5}$$

and $M|_{f(X)}$ is injective.

We prove the following

LEMMA 3. *Let X be a nontrivial linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $f : X \to \mathbb{K}$ be injective, $M : \mathbb{K} \to \mathbb{K}$, $M(1) = 1$ and $M \circ f \not\equiv 1$. If f and M satisfy* (1), *then there is $x_0 \in X \setminus \{0\}$ such that $X = \mathbb{K}x_0$, $M|_{f(X)}$ is injective and multiplicative and*

$$M(f(\alpha x_0)) = \alpha + 1 \text{ for each } \alpha \in \mathbb{K}. \tag{6}$$

Proof. Let f, M satisfy (1) and f be injective. According to Proposition 1, $f^{-1}(\{0\}) = \{y_0\}$ for some $y_0 \in X \setminus \{0\}$ and, in view of Lemma 1 (ii), (iii), $A = \{0\}$. Denote $F = \operatorname{supp} f$ and $W = f(X) \setminus \{0\}$.

First we prove that $M(W) \setminus \{1\} \neq \emptyset$. For the contrary suppose that $M(W) = \{1\}$. Then, by Lemma 1 (i),

$$f(x + y) = f(x)f(y) \text{ for every } x \in F \text{ and } y \in X.$$

Hence $F + F' \subset F'$, where $F' = X \setminus F = \{y_0\}$. Thus $F = \{0\}$ and, consequently, $X = \{0, y_0\}$. But $y_0 \neq 0$, what contradicts the linearity of X.

Since $A = \{0\}$, by Lemma 2 there exists $x_0 \in X \setminus \{0\}$ such that

$$f(x) = \begin{cases} a, & \text{if } x = (M(a) - 1)x_0 \text{ and } a \in W; \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

and $M|_{f(X)}$ is injective.

Now we show that $M|_{f(X)}$ is multiplicative. Using Lemma 1 (i), (iv), it is easy to see that

$$M(f(x)f(y)) = 0 = M(f(x))M(f(y))$$

for $x, y \in X$ such that $f(x)f(y) = 0$. So take $x, y \in X$ with $f(x)f(y) \neq 0$. Then, by (7),

$$x = (M(f(x)) - 1)x_0 \text{ and } y = (M(f(y)) - 1)x_0.$$

According to (1)

$$\begin{aligned} 0 \neq f(x)f(y) &= f(x + M(f(x))y) \\ &= f((M(f(x)) - 1)x_0 + M(f(x))((M(f(y)) - 1)x_0)) \\ &= f((M(f(x))M(f(y)) - 1)x_0). \end{aligned}$$

Thus, in view of (7),

$$(M(f(x))M(f(y)) - 1)x_0 = (M(f(x)f(y)) - 1)x_0$$

and whence $M(f(x))M(f(y)) = M(f(x)f(y))$ (because $x_0 \neq 0$). This completes the proof of multiplicativity of $M|_{f(X)}$.

Let $f_0 : \mathbb{K} \to \mathbb{K}$ be given by $f_0(a) = f(ax_0)$. Then f_0, M satisfy (1) and f_0 is injective. Since $M|_{f(X)}$ is injective, so $M \circ f_0 \not\equiv 1$. Hence, according to Proposition 1, $0 \in f_0(\mathbb{K}) = f(\mathbb{K}x_0)$. Moreover, for each $x \in X$ with $f(x) \neq 0$, in view of (7), $f(x) = f((M(a)-1)x_0)$ for some $a \in W$. Thus $f(X) = f(\mathbb{K}x_0)$ and consequently, by injectivity of f, $X = \mathbb{K}x_0$. Hence there is some $a_0 \in \mathbb{K}$ such that $y_0 = a_0x_0$ and, in view of (7),

$$\mathbb{K} \setminus \{M(a) - 1 : a \in W\} = \{a_0\}.$$

Thus, by Lemma 1 (i), $-1 \notin \{M(a)-1 : a \in W\}$. In this way we obtain that $a_0 = -1$ and, consequently, $\mathbb{K} \setminus \{M(a) - 1 : a \in W\} = \{-1\}$. This implies $M(W) = \mathbb{K} \setminus \{0\}$ and $M(f(X)) = \mathbb{K}$. Define a functional $h : X = \mathbb{K}x_0 \to \mathbb{K}$ by $h(\alpha x_0) = \alpha$. From (7) $M(f(\alpha x_0)) = \alpha + 1$ for each $\alpha \in \mathbb{K}$. □

Now, we are in position to prove our main result.

THEOREM 1. *Let X be a nontrivial linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $f : X \to \mathbb{K}$ be injective, $M : \mathbb{K} \to \mathbb{K}$ and $M \circ f \not\equiv \mathrm{const}$. Then functions f and M satisfy* (1) *if and only if there is $x_0 \in X \setminus \{0\}$ such that $X = \mathbb{K}x_0$, $M|_{f(X)}$ is injective and multiplicative, and $M \circ f$ is given by* (6).

Proof. Let f and M satisfy (1) and $M \circ f \not\equiv \mathrm{const}$. Then, by Lemma 1 (i), (iii) $f(0) = 1$ and $M(1) \neq 0$. Put $x = 0$ in (1). Then $f(M(1)y) = f(y)$ for $y \in X$. Hence, replacing y by $\frac{z}{M(1)}$, we obtain $f\left(\frac{z}{M(1)}\right) = f(z)$ for each $z \in X$. Consequently, for every $x, z \in X$,

$$f\left(x + \frac{M(f(x))}{M(1)}z\right) = f(x)f\left(\frac{z}{M(1)}\right) = f(x)f(z).$$

In this way we obtain that f and $\widetilde{M}$ also satisfy (1), $\widetilde{M} \circ f \not\equiv 1$ and $\widetilde{M}(1) = 1$, where

$$\widetilde{M}(a) = \frac{M(a)}{M(1)} \quad \text{for} \quad a \in \mathbb{K}.$$

Hence, by Lemma 3, $X = \mathbb{K}x_0$ for some $x_0 \in X \setminus \{0\}$, $\widetilde{M}|_{f(X)}$ is injective and multiplicative and $\widetilde{M}(f(\alpha x_0)) = \alpha + 1$ for each $\alpha \in \mathbb{K}$. Thus M is injective on $f(X)$,

$$M(ab) = M(1)M(a)M(b) \quad \text{for every} \quad a, b \in f(X)$$

and

$$M(f(\alpha x_0)) = M(1)(\alpha + 1) \quad \text{for each} \quad \alpha \in \mathbb{K}.$$

To end the proof we must show that $M(1) = 1$. We have

$$M(f(\alpha x_0)f(\beta x_0)) = M(1)M(f(\alpha x_0))M(f(\beta x_0)) = M(1)(\alpha + 1)(\beta + 1)$$

and

$$M(f(\alpha x_0 + M(f(\alpha x_0))\beta x_0)) = M(f(\alpha x_0 + (\alpha+1)\beta x_0)) \\ = (\alpha + (\alpha+1)\beta) + 1 = (\alpha+1)(\beta+1).$$

Then, by (1), $M(1) = 1$.

Now we prove that the converse statement is also true. Since $X = \mathbb{K}x_0$ and (6) holds, we have

$$M(f(x + M(f(x))y) = M(f(\alpha x_0 + M(f(\alpha x_0))\beta x_0)) \\ = M(f(\alpha x_0 + (\alpha+1)\beta x_0)) = (\alpha+1)(\beta+1) = M(f(\alpha x_0))M(f(\beta x_0)).$$

This yields, thanks to the multiplicativity of $M|_{f(X)}$,

$$M(f(x) + M(f(x))y) = M(f(\alpha x_0)f(\beta x_0)) = M(f(x)f(y))$$

and, in view of injectivity of $M|_{f(X)}$, f and M satisfy (1). This ends the proof. □

REMARK 1. It can bee easily seen that the theorem does not hold if we would require the injectivity of $f|_{\operatorname{supp} f}$ only. For example, functions $f, M : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^x & \text{for } x \in \mathbb{Q}; \\ 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

and

$$M(a) = \begin{cases} 1 & \text{for } a \in \{e^x : x \in \mathbb{Q}\}; \\ 0 & \text{for } a = 0 \end{cases}$$

satisfy (1), $M \circ f = \chi_{\mathbb{Q}}$ and (6) fails to hold.

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DEPARTMENT OF MATHEMATICS
RZESZÓW UNIVERSITY OF TECHNOLOGY
W. Pola 2
35-959 RZESZÓW, POLAND
e-mail: elizapie@prz.rzeszow.pl

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