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A GENERALIZED OSTROWSKI TYPE INEQUALITY FOR A RANDOM VARIABLE WHOSE PROBABILITY DENSITY FUNCTION BELONGS TO $L_\infty[a,b]$

Abstract. We establish here an inequality of Ostrowski type for a random variable whose probability density function belongs to $L_\infty[a,b]$, in terms of the cumulative distribution function and expectation. The inequality is then applied to generalized beta random variable.

1. Introduction

The following theorem describes an inequality which is known in literature as Ostrowski's inequality [6].

THEOREM 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in I^0 (interior of I), and let $a, b \in I^0$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have*

$$(1.1)\qquad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\,\|f'\|_\infty,$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In [2], N. S. Barnet and S. S. Dragomir established the following version of Ostrowski type inequality for cumulative and probability distribution functions.

THEOREM 2. *Let X be a random variable with probability density function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}^+$ and with cumulative distribution function $F(x) =$*

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$\Pr(X \leq x)$. *If* $f \in L_\infty[a,b]$ *and* $\|f\|_\infty := \sup_{t\in[a,b]} |f(t)| < \infty$, *then we have the inequality:*

$$\left|\Pr(X \leq x) - \frac{b - E(X)}{b-a}\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_\infty, \tag{1.2}$$

for all $x \in [a,b]$.

Equivalently,

$$\left|P_r(X \geq x) - \frac{E(X) - a}{b-a}\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_\infty. \tag{1.3}$$

The constant $\frac{1}{4}$ *in* (1.2) *and* (1.3) *is sharp.*

In this work, we establish a generalized Ostrowski type inequality for cumulative distribution function and expectation of a random variable. Applications for the generalized beta distribution are also given.

2. Main results

The following theorem holds.

THEOREM 3. *Let* X *and* F *be as defined above. Let* $f \in L_\infty[a,b]$ *and put* $\|f\|_\infty = \sup_{t\in[a,b]} f(t) < \infty$. *Then, we have the inequality*

$$\begin{aligned}\left|(1-h)\Pr(X \leq x) + \frac{h}{2} - \frac{b-E(X)}{b-a}\right| \\ \leq \left[\frac{1}{4}\left(h^2 + (1-h)^2\right) + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f\|_\infty,\end{aligned} \tag{2.1}$$

or equivalently,

$$\begin{aligned}\left|(1-h)\Pr(X \geq x) + \frac{h}{2} - \frac{E(X)-a}{b-a}\right| \\ \leq \left[\frac{1}{4}\left(h^2 + (1-h)^2\right) + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f\|_\infty,\end{aligned} \tag{2.2}$$

for all $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$ *and* $h \in [0,1]$.

Proof. As defined in [4], consider the kernel $p : [a,b]^2 \longrightarrow \mathbb{R}$ given by

$$p(x,t) = \begin{cases} t - \left(a + h\dfrac{b-a}{2}\right) & \text{if } t \in [a,x] \\ t - \left(b - h\dfrac{b-a}{2}\right) & \text{if } t \in (x,b]. \end{cases}$$

Then the Riemann-Stieltjes integral $\int_a^b p(x,t)dF(t)$ exists for any $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$ and integration by parts for Riemann-Stieljes integral gives

$$\int_a^b p(x,t)dF(t) = \int_a^x \left[t - \left(a + h\frac{b-a}{2}\right)\right] dF(t) + \int_x^b \left[t - \left(b - h\frac{b-a}{2}\right)\right] dF(t)$$
$$= \left[t - \left(a + h\frac{b-a}{2}\right)\right] F(t) \Big|_a^x - \int_a^x F(t)dt + \left[t - \left(b - h\frac{b-a}{2}\right) F(t)\right] \Big|_x^b - \int_x^b F(t)dt$$
$$= (b-a)\left[(1-h)F(x) + \frac{h}{2}\right] - \int_a^b F(t)dt. \tag{2.3}$$

On the other hand, we have

$$E(X) = \int_a^b tdF(t) = tF(t) \Big|_a^b - \int_a^b F(t)dt = b - \int_a^b F(t)dt,$$

which implies

$$\int_a^b F(t)dt = b - E(X). \tag{2.4}$$

Using (2.3) and (2.4), we get the identity

$$\int_a^b p(x,t)dF(t) = (b-a)\left[(1-h)F(x) + \frac{h}{2}\right] + E(X) - b. \tag{2.5}$$

As shown in [2], if $p : \left[a,b\right]^2 \to \mathbb{R}$ is Riemann integrable on $\left[a,b\right]$ and $v : \left[a,b\right] \to R$ is L-Lipschitzian (Lipschitzian with the constant L), then we have

$$\left|\int_a^b p(x)\,dv(x)\right| \le L\int_a^b |p(x)|\,dx. \tag{2.6}$$

Applying (2.6) for the mapping $p(x,.)$ and the function $F(.)$, we get

$$\left|\int_a^b p(x,t)\,dF(t)\right| \le \|f\|_\infty \int_a^b |p(x,t)|\,dt$$

$$= \|f\|_\infty \left[\int_a^x \left|t-\left(a+h\frac{b-a}{2}\right)\right| dt + \int_x^b \left|t-\left(b-h\frac{b-a}{2}\right)\right| dt\right]$$

$$= \|f\|_\infty \left[\int_a^{a+h\frac{b-a}{2}} \left(a+h\frac{b-a}{2}-t\right) dt + \int_{a+h\frac{b-a}{2}}^{x} \left(t-\left(a+h\frac{b-a}{2}\right)\right) dt\right.$$

$$\left. + \int_x^{b-h\frac{b-a}{2}} \left(b-h\frac{b-a}{2}-t\right) dt + \int_{b-h\frac{b-a}{2}}^{b} \left(t-\left(b-h\frac{b-a}{2}\right)\right) dt\right]$$

$$= \|f\|_\infty (b-a)^2 \left[\frac{1}{4}\left(h^2+(1-h)^2\right) + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right],$$

for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$.

Finally, by the identity (2.5), we deduce that for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$,

$$\left|(1-h)F(x)+\frac{h}{2}-\frac{b-E(X)}{b-a}\right|$$
$$\le \left[\frac{1}{4}\left(h^2+(1-h)^2\right)+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f\|_\infty,$$

which proves (2.1).

Also, taking into account the fact that

$$\Pr(X \le x) = 1 - \Pr(X \ge x),$$

the inequality (2.2) is obtained. ∎

REMARK 1. For $h = 0$ in (2.1) and (2.2), we recapture (1.2) and (1.3). Moreover, as

$$\left[h^2+(1-h)^2\right] \le 1, \text{ for all } h \in [0,1],$$

therefore, (2.1) and (2.2) gives better estimates than (1.2) and (1.3).

We now give some corollaries of Theorem 3 for the expectations of the variable X.

COROLLARY 1. *Under the above assumptions, we have the double inequality*

$$(2.7)\qquad b-\frac{h}{2}(b-a)-\frac{1}{2}\Delta(b-a)^2\|f\|_\infty \le E(X) \le a+\frac{h}{2}(b-a)+\frac{1}{2}\Delta(b-a)^2\|f\|_\infty,$$

where

$$(2.8)\qquad \Delta = h^2-h+1,$$

for $h\in[0,1]$.

Proof. It is known that

$$a \le E(X) \le b.$$

If $x=a$ in (2.1), we obtain

$$\left|\frac{h}{2}-\frac{b-E(X)}{b-a}\right| \le \frac{1}{2}\Delta(b-a)\|f\|,$$

where Δ is as defined above and

$$(2.9)\qquad b-\frac{h}{2}(b-a)-\frac{1}{2}\Delta(b-a)^2\|f\| \le E(X) \le b-\frac{h}{2}(b-a)+\frac{1}{2}\Delta(b-a)^2\|f\|.$$

The left hand estimate of the inequality (2.9) is equivalent to first inequality in (2.7).

Also, if $x=b$ in (2.1) then

$$\left|\frac{E(X)-a}{b-a}-\frac{h}{2}\right| \le \frac{1}{2}\Delta(b-a)\|f\|_\infty,$$

which reduces to

$$(2.10)\qquad a+\frac{h}{2}(b-a)-\frac{1}{2}\Delta(b-a)^2\|f\|_\infty \le E(X) \le a+\frac{h}{2}(b-a)+\frac{1}{2}\Delta(b-a)^2\|f\|_\infty.$$

The right hand side of the inequality (2.10) proves the second inequality of (2.7). ■

REMARK 2. As for the probability density function f associated with random variable X

$$1=\int_a^b f(t)\,dt,$$

implies

$$\|f\|_\infty \ge \frac{1}{b-a}.$$

If we suppose that f is not too large and

$$\|f\|_\infty \le \frac{2-h}{\triangle\,(b-a)}, \tag{2.11}$$

where $\triangle$ is defined by (2.8) and $h \in [0,1]$. Then from the double inequality (2.7) it can be verified that

$$a + \frac{h}{2}(b-a) + \frac{1}{2}\triangle\,(b-a)^2\|f\|_\infty \le b$$

and

$$b - \frac{h}{2}(b-a) - \frac{1}{2}\triangle\,(b-a)^2\|f\|_\infty \ge a,$$

when (2.11) holds. It shows that (2.7) gives a much tighter estimate of the expected value of the random variable X.

COROLLARY 2. *With the above assumptions,*

$$\left|E(X) - \frac{a+b}{2}\right| \le \frac{1}{2}(b-a)^2\left[\triangle\,\|f\|_\infty - \frac{1-h}{b-a}\right]. \tag{2.12}$$

Proof. From the inequality (2.7),

$$-\frac{1}{2}(b-a)^2\left[\triangle\,\|f\|_\infty - \frac{1-h}{b-a}\right] \le E(X) - \frac{a+b}{2}$$
$$\le \frac{1}{2}(b-a)^2\left[\triangle\,\|f\|_\infty - \frac{1-h}{b-a}\right],$$

which is exactly (2.12).

This corollary provides the mechanism for finding a sufficient condition, in terms of $\|f\|_\infty$, for the expectation $E(X)$ to be close to the midpoint of the interval, $\frac{a+b}{2}$. ∎

COROLLARY 3. *Let X and f be as above and $\varepsilon > 0$. If*

$$\|f\|_\infty \le \frac{(1-h)}{\triangle\,(b-a)} + \frac{2\varepsilon}{\triangle\,(b-a)^2}, \tag{2.13}$$

then

$$\left|E(X) - \frac{a+b}{2}\right| \le \varepsilon.$$

The following corollary of Theorem 3 also holds.

COROLLARY 4. *Let X and F be as above. Then*

$$\begin{aligned}(2.14)\qquad &\left|(1-h)\Pr\left(X \le \frac{a+b}{2}\right) - \frac{1}{2}(1-h)\right| \\ &\le \frac{1}{4}\left[h^2 + (1-h)^2\right](b-a)\|f\|_\infty + \frac{1}{b-a}\left|E(X) - \frac{a+b}{2}\right| \\ &\le \left(\triangle - \frac{1}{4}\right)(b-a)\|f\|_\infty - \frac{1}{2}(1-h).\end{aligned}$$

Proof. If we choose $x = \frac{a+b}{2}$ in (2.1), then we get

$$\left|(1-h)\Pr\left(X \le \frac{a+b}{2}\right) + \frac{h}{2} - \frac{b-E(X)}{b-a}\right| \le \frac{1}{2}\left(\triangle - \frac{1}{2}\right)(b-a)\|f\|_\infty,$$

the latter may be rewritten in the following form

$$\left|(1-h)\Pr\left(X \le \frac{a+b}{2}\right) + \frac{h}{2} - \frac{1}{2} + \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right)\right| \le \frac{1}{2}\left(\triangle - \frac{1}{2}\right)(b-a)\|f\|_\infty.$$

Using the triangular inequality, we get

$$\begin{aligned}&\left|(1-h)\Pr\left(X \le \frac{a+b}{2}\right) + \frac{h}{2} - \frac{1}{2} + \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right) - \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right)\right| \\ &\le \left|(1-h)\Pr\left(X \le \frac{a+b}{2}\right) + \frac{h}{2} - \frac{1}{2} + \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right)\right| \\ &\quad + \frac{1}{b-a}\left|E(X) - \frac{a+b}{2}\right| \\ &\le \frac{1}{2}\left(\triangle - \frac{1}{2}\right)(b-a)\|f\|_\infty + \frac{1}{b-a}\left|E(X) - \frac{a+b}{2}\right| \\ &\le \left(\triangle - \frac{1}{4}\right)(b-a)\|f\|_\infty - \frac{1}{2}(1-h),\end{aligned}$$

which gives the desired result.

A similar inequality holds for

$$\Pr\left(X \ge \frac{a+b}{2}\right). \qquad \blacksquare$$

COROLLARY 5. *Let X and F be as above. Then*

$$\left|E(X)-\frac{a+b}{2}\right|\leq\frac{1}{2}\left(\Delta-\frac{1}{2}\right)(b-a)^2\|f\|_\infty+(b-a)\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}(1-h)\right|. \tag{2.15}$$

Following the above corollary the proof is obvious and the details are omitted.

REMARK 3. *If we assume that f is continuous on* $[a,b]$, *then F is differentiable on* (a,b) , *and we get*

$$\left|(1-h)F(x)+\frac{h}{2}-\frac{1}{b-a}\int_a^b F(t)\,dt\right|\leq\left[\frac{1}{4}\left(h^2+(1-h)^2\right)+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f\|_\infty . \tag{2.16}$$

Using the identity (2.4), *we recapture* (2.1) *and* (2.2) *for random variables whose probability density function are continuous on* $[a,b]$.

3. Applications for beta random variable

If X is a beta random variable with parameters $\beta_3>-1$, $\beta_4>-1$ and for $\beta_2>0$ and any β_1, the generalized beta random variable

$$Y=\beta_1+\beta_2X,$$

is said to have a generalized beta distribution [5] and the probability density function of the generalized beta distribution of beta random variable is,

$$f(x)=\begin{cases}\dfrac{(x-\beta_1)^{\beta_3}(\beta_1+\beta_2-x)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)\,\beta_2^{(\beta_3+\beta_4+1)}}, & \text{for } \beta_1<x<\beta_1+\beta_2\\ 0, & \text{otherwise,}\end{cases}$$

where $\beta(l,m)$ is the beta function with $l,\ m>0$ and is defined as

$$\beta(l,m)=\int_0^1 x^{l-1}(1-x)^{m-1}\,dx.$$

For p, $q>0$ and $h\in[0,1)$, we choose,

$$\beta_1=\frac{h}{2},$$
$$\beta_2=(1-h),$$
$$\beta_3=p-1,$$
$$\beta_4=q-1.$$

Then, the probability density function associated with generalized beta random variable

$$Y = \frac{h}{2} + (1-h)\,X$$

takes the form

$$f(x) = \begin{cases} \dfrac{\left(x-\frac{h}{2}\right)^{p-1}\left(1-\frac{h}{2}-x\right)^{q-1}}{\beta(p,q)\,(1-h)^{p+q-1}}, & \frac{h}{2} < x < 1-\frac{h}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Now,

$$E(Y) = \int_{\frac{h}{2}}^{1-\frac{h}{2}} x f(x)\,dx = (1-h)\,\frac{p}{p+q} + \frac{h}{2}. \tag{3.1}$$

We observe that for $p < 1$

$$\|f(x;p,q)\|_\infty = \sup_{\frac{h}{2}<x<1-\frac{h}{2}} \left[\frac{\left(x-\frac{h}{2}\right)^{p-1}\left(1-\frac{h}{2}-x\right)^{q-1}}{\beta(p,q)\,(1-h)^{p+q-1}}\right].$$

Assume that $p, q > 1$. Then we find that

$$\frac{df(x;p,q)}{dx} = \frac{\left(x-\frac{h}{2}\right)^{p-2}\left(1-\frac{h}{2}-x\right)^{q-2}}{(1-h)^{p+q-1}\,\beta(p,q)} \times\left[(p-1) + \frac{h}{2}(q-p) - (p+q-2)\,x\right].$$

We further observe that for $p, q > 1$, $\frac{df}{dx} = 0$ if and only if $x = x_0 = \frac{(p-1)+\frac{h}{2}(q-p)}{p+q-2}$. We therefore have $\frac{df}{dx} > 0$ on $(\frac{h}{2}, x_0)$ and $\frac{df}{dx} < 0$ on $(x_0, 1-\frac{h}{2})$. Consequently, we see that

$$\begin{aligned} \|f(x;p,q)\|_\infty &= \|f(x_0;p,q)\|_\infty \\ &= \frac{1}{(1-h)\,\beta(p,q)}\left[\frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}\right]. \end{aligned} \tag{3.2}$$

Then, by Theorem 3, we may state the following.

PROPOSITION 1. *Let X be a beta random variable with parameters (p,q). Then for generalized beta random variable*

$$Y = \frac{h}{2} + (1-h)\,X,$$

we have the inequality

$$(3.3)\quad \left|\Pr(Y \le x) - \frac{q}{p+q}\right| \le \frac{1}{(1-h)^2\beta(p,q)}\left[\frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}\right] \times\left[\frac{1}{4}\left(h^2+(1-h)^2\right)+\left(x-\frac{1}{2}\right)^2\right],$$

for all $x \in \left[\frac{h}{2}, 1-\frac{h}{2}\right]$.

In particular

$$(3.4)\quad \left|\Pr\left(Y \le \frac{1}{2}\right) - \frac{q}{p+q}\right| \le \frac{1}{4(1-h)^2\beta(p,q)}\left(h^2+(1-h)^2\right)\left[\frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}\right].$$

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