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ON LEFT MULTIPLIERS AND THE COMMUTATIVITY OF PRIME RINGS

Abstract. Let R be an associative ring. An additive mapping $H: R \longrightarrow R$ is called a *left multiplier* if $H(xy) = H(x)y$, holds for all $x, y \in R$. In this paper, we investigate commutativity of prime rings satisfying certain identities involving left multiplier. Some related results have also been discussed.

1. Introduction

Throughout the discussion, unless otherwise mentioned, R denotes an associative ring having at least two elements with center $Z(R)$. However, R may not have unity. For any $x, y \in R$, the symbol $[x, y]$(resp. $x \circ y$) will denote the commutator $xy - yx$ (resp. the anti-commutator $xy + yx$). We shall make extensive use of the following basic commutator identities throughout the discussion without any specific mention:

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z;$$
$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].$$
$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z.$$

Recall that R is *prime* if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. An additive mapping $H: R \longrightarrow R$ is called a *left(resp.right) multiplier* if $H(xy) = H(x)y(resp.H(xy) = xH(y))$, holds for all $x, y \in R$. A multiplier is an additive mapping which is both right as well as left multiplier. Considerable work has been done on left(right) multipliers in prime and semiprime rings during the last couple of decades (see [18-20] for a partial bibliography). An additive mapping $d: R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, holds for all $x, y \in R$. An additive mapping $\delta: R \longrightarrow R$ is said to be a left derivation if $\delta(xy) = x\delta(y) + y\delta(x)$ holds for all $x, y \in R$.

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Following [6], an additive mapping $F : R \longrightarrow R$ is said to be a *generalized derivation* on R if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Obviously, generalized derivation with $d = 0$ covers the concept of left multipliers.

Over the last few decades, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of derivations of R. The first result in this direction is due to E. C. Posner [14] who proved that if a prime ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. This result was subsequently refined and extented by a number of algebraists; we refer to ([5],[7],[13]) for a state-of-the art account and a comprehensive bibliography. In [8], Bresar and Vukman showed that a prime ring must be commutative if it admits a nonzero left derivation. Recently, many authors have obtained commutativity theorems for prime and semiprime rings admitting derivation, left derivation and generalized derivation (see for example; [1], [2], [3], [4], [7], [8], [9],[10], [11], [15],[16], and [17] where further references can be found).

In this paper, our attempt is to investigate commutativity of rings satisfying certain identities involving left multipliers on the ring.

2. The conditions $H([x, y]) \pm [x, y] = 0$

In the year 2003, Quadri et al. [15] established that a prime ring R must be commutative if it admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y]) = [x, y]$ (resp. $F([x, y]) + [x, y] = 0$) for all $x, y \in I$. Now, it is natural to ask that what can we say about the commutativity of a prime R if the generalized derivation F in the above conditions is replaced by a left multiplier. In this section, we have investigated this problem and obtained commutativity of R.

THEOREM 2.1. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x) \neq x$, for all $x \in I$. Further, if $H([x, y]) - [x, y] = 0$, for all $x, y \in I$, then R is commutative.*

Proof. We are given that H is left multiplier of R such that

$$H([x, y]) - [x, y] = 0, \text{for all } x, y \in I. \tag{2.1}$$

This can be rewritten as

$$(H(x) - x)y - (H(y) - y)x = 0, \text{for all } x, y \in I. \tag{2.2}$$

Replacing x by xr in (2.2), we find that

$$(H(x) - x)ry - (H(y) - y)xr = 0, \text{for all } x, y \in I, r \in R. \tag{2.3}$$

Using (2.2) in (2.3) to simplify, we obtain

$$(H(x)-x)[r,y]=0 \text{ for all } x,y\in I,\ r\in R. \tag{2.4}$$

Again, replace r by rs in (2.4) and use (2.4), to get $(H(x)-x)r[s,y]=0$, for all $x,y\in I$ and $r,s\in R$ *i.e.*, $(H(x)-x)R[s,y]=\{0\}$, for all $x,y\in I$ and $s\in R$. Since R is prime, the above expression yields that either $[s,y]=0$ or $H(x)-x=0$, for all $x,y\in I$ and $s\in R$. Thus, application of our hypotheses implies that $[s,y]=0$, for all $y\in I$ and $s\in R$ and therefore, $I\subseteq Z(R)$. Hence, R is commutative. ◇

REMARK 2.1. In Theorem 2.1, if a left multiplier is zero, then R is commutative.

Proof. Suppose that $H([x,y])-[x,y]=0$, for any $x,y\in I$. If $H=0$, then $[x,y]=0$, for all $x,y\in I$. Therefore, I is commutative. Hence, R is commutative. ◇

Using similar arguments as used in proof of the above theorem, we can prove the following:

THEOREM 2.2. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x)\neq -x$, for all $x\in I$. Further, if $H([x,y])+[x,y]=0$, for all $x,y\in I$, then R is commutative.*

Conclusion of Theorems 2.1 and 2.2 still hold if we replace the product $[x,y]$ by $x\circ y$. In fact, we obtain the following results:

THEOREM 2.3. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x)\neq x$, for all $x\in I$. Further if $H(x\circ y)=x\circ y$, for all $x,y\in I$, then R is commutative .*

Proof. By the hypotheses, we have

$$H(x\circ y)-x\circ y=0, \text{for all } x,y\in I. \tag{2.5}$$

This imples that

$$(H(x)-x)y+(H(y)-y)x=0, \text{for all } x,y\in I. \tag{2.6}$$

Replacing x by xr in (2.6), we obtain

$$(H(x)-x)ry+(H(x)-x)xr=0, \text{for all } x,y\in I,\ r\in R. \tag{2.7}$$

Application of (2.6) yields that

$$(H(x)-x)ry-(H(x)-x)yr=0, \text{for all } x,y\in I,\ r\in R. \tag{2.8}$$

That is,

$$(H(x)-x)[r,y]=0, \text{ for all } x,y\in I,\ r\in R. \tag{2.9}$$

Thus, equation (2.9) is same as equation (2.4) and henceforth the proof is follows by the last paragraph of the proof of Theorem 2.1. ◇

Proceeding on same lines with necessary variations, we can prove the following:

THEOREM 2.4. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x) \neq -x$, for all $x \in I$. Further, if $H(x \circ y) + (x \circ y) = 0$, for all $x, y \in I$, then R is commutative.*

REMARK 2.2. In Theorems 2.3 and 3.4, if a left multiplier H is zero, then R is commutative.

Proof. For any $x, y \in I$, we have $H(x \circ y) = x \circ y$. If $H = 0$, then $x \circ y = 0$, for all $x, y \in I$. Replacing x by xz and using the fact that $yx = -xy$, we obtain $x[z, y] = 0$, for all $x, y, z \in I$ *i.e.*, $IR[z, y] = \{0\}$, for all $y, z \in R$. Since R is prime and $I \neq \{0\}$, so that $[z, y] = 0$, for all $y, z \in R$. Thus, I is commutative and hence R is commutative. ◇

COROLLARY 2.1. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x) \neq \pm x$, for all $x \in I$. Further, if $H(xy) \pm xy = 0$, for all $x, y \in I$, then R is commutative.*

Proof. For any $x, y \in I$, we have $H(xy) = xy$. This implies that $H([x, y]) - [x, y] = 0$, for all $x, y \in I$, and hence by Theorem 2.1., R is commutative.

On the other hand if R satisfy the condition $H(xy) + xy = 0$, for all $x, y \in I$. Then for any $x, y \in I$, we have $H(xy + yx) = -(xy + yx)$. This implies that $H(x \circ y) + (x \circ y) = 0$, for all $x, y \in I$. Thus, by Theorem 2.4, R is commutative.◇

Similarly, we can prove the following:

COROLLARY 2.2. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x) \neq \pm x$, for all $x \in I$. Further, if $H(xy) \pm yx = 0$, for all $x, y \in I$, then R is commutative.*

THEOREM 2.5. *Let R be a prime ring and I be a nonzero ideal of R. If R admits a nonzero left multiplier H such that $H(x) \neq x$, for all $x \in I$. Then the following conditions are equivalent:*

(*i*) $H([x, y]) - [x, y] = 0$, *for all* $x, y \in I$;
(*ii*) $H([x, y]) + [x, y] = 0$, *for all* $x, y \in I$;
(*iii*) *For all* $x, y \in I$, *either* $H([x, y]) - [x, y] = 0$ *or* $H([x, y]) + [x, y] = 0$;
(*iv*) *R is commutative.*

Proof. Clearly, $(iv) \Longrightarrow (i)$, $(iv) \Longrightarrow (ii)$ and $(iv) \Longrightarrow (iii)$. Now, we will prove that $(i) \Longrightarrow (iv)$. For any $x, y \in I$, we have $H([x,y]) - [x,y] = 0$, for all $x, y \in I$, then by Theorem 3.1, R is commutative. Similarly, we can show that $(ii) \Longrightarrow (iv)$.

$(iii) \Longrightarrow (iv)$. For each fixed $y \in I$, we set $I_1 = \{x \in I \mid H([x,y]) - [x,y] = 0\}$ and $I_2 = \{x \in I \mid H([x,y]) + [x,y] = 0\}$. Then, I_1 and I_2 are additive subgroups of I such that $I = I_1 \cup I_2$. Thus, by Brauer's trick, we have either $I_1 = I$ or $I_2 = I$. Further, using similar arguments, we obtain $I = \{y \in I \mid I_1 = I\}$ or $I = \{y \in I \mid I_2 = I\}$. Thus, we find that either $H([x,y]) - [x,y] = 0$, for all $x, y \in I$ or $H([x,y]) + [x,y] = 0$ for all $x, y \in I$. Hence, R is commutative in both the cases by Theorem 2.1.(resp. Theorem 2.2.). This completes the proof of the theorem. $\diamond$

Applications of Theorem 2.1 and 2.3 yields the following results which improve the results proved in [1], [11], [15] and [16].

THEOREM 2.6. *Let R be a prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F([x,y]) - [x,y] = 0$(resp. $F([x,y]) + [x,y] = 0$),for all $x, y \in I$. Moreover, if $F(x) \neq x$, for all $x \in I$, then R is commutative.*

Proof. When the associated derivation $d = 0$, then using Theorem 2.1, we get the required result. On the other hand if $d \neq 0$, then the proof follows from Theorem 2.1 of [15]. $\diamond$

Similarly, in view of Theorem 2.3 & 2.4 above and Theorem 2.3 & 2.4 of [15], we obtain the following result:

THEOREM 2.7. *Let R be a prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F(x \circ y) \pm (x \circ y) = 0$, for all $x, y \in I$. Moreover, if $F(x) \neq x$, for all $x \in I$, then R is commutative.*

3. The conditions $H(xy) \pm xy \in Z(R)$

In the year 2007, the authors together with Asma [3] established that a prime ring R with a nonzero ideal I must be commutative if it admits a nonzero generalized derivation F satisfying any one of the properties: $F(xy) - xy \in Z(R)$, $F(xy) - yx \in Z(R)$, for all $x, y \in I$. If the underlying derivation d is zero, then the problem is still open. In this section, we continue this study and obtain similar results in the setting of left multipliers.

THEOREM 3.1. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x) \neq x$, for all $x \in I$. Further, if $H(xy) - xy \in Z(R)$, for all $x, y \in I$, then R is commutative.*

Proof. For any $x, y \in I$, we have $H(xy) - xy \in Z(R)$. This can be rewritten as

$$H(x)y - xy \in Z(R), \text{for all } x, y \in I. \tag{3.1}$$

That is,

$$[(H(x) - x)y, r] = 0, \text{for all } x, y \in I,\ r \in R. \tag{3.2}$$

This implies that,

$$(H(x) - x)[y, r] + [H(x) - x, r]y = 0, \text{for all } x, y \in I,\ r \in R. \tag{3.3}$$

Replacing x by xz in (3.3), we obtain

$$(H(x) - x)z[y, r] + [(H(x) - x)z, r]y = 0, \text{for all } x, y, z \in I,\ r \in R. \tag{3.4}$$

Combining (3.2) and (3.4), we find that $(H(x)-x)z[y,r] = 0$, for all $x, y, z \in I$ and $r \in R$. This yields that $(H(x) - x)RI[y, r] = \{0\}$, for all $x, y \in I$ and $r \in R$. The primeness of R implies that either $I[x, r] = \{0\}$ or $H(y) - y = 0$, for all $x, y \in I$ and $r \in R$. Since $I \neq \{0\}$ and $H(x) \neq x$, for all $x \in I$, we find that I is central and hence R is commutative. ◇

Similar arguments as above, yield the following:

THEOREM 3.2. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x) \neq -x$, for all $x \in I$. Further, if $H(xy) + xy \in Z(R)$, for all $x, y \in I$, then R is commutative.*

Proof. Suppose H is a nonzero left multiplier satisfying the property $H(xy) + xy \in Z(R)$, for all $x, y \in I$, then the nonzero left multiplier $(-H)$ also satisfies the condition $(-H)(xy) - xy \in Z(R)$, for all $x, y \in I$. Hence by Theorem 3.1, R is commutative. ◇

REMARK 3.1. In Theorem 3.1, if left multiplier is zero, then R is commutative.

THEOREM 3.3. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x) \neq x$, for all $x \in I$. Further, if $H(xy) - yx \in Z(R)$, for all $x, y \in I$, then R is commutative.*

Proof. We are given that H is a left multiplier of R such that $H(xy) - yx \in Z(R)$ for all $x, y \in I$. This implies that $[H(x)y - yx, r] = 0$, for all $x, y \in I$ and $r \in R$. Replacing y by xy in the last relation and using it, we find that $(H(x)y - yx)[y, r] = 0$, for all $x, y \in I$ and $r \in R$. Again, replace r by rs in the above expression, to get $(H(x)y - yx)r[y, s] + (H(x)y - yx)[y, r]s = 0$, for all $x, y \in I$ and $r, s \in R$. This yields that $(H(x)y - yx)r[y, s] = 0$ *i.e.*, $(H(x)y - yx)R[y, s] = \{0\}$, for all $x, y \in I$ and $s \in R$. The primeness of R implies that either $[y, s] = 0$ or $H(x)y - yx = 0$, for all $x, y \in I$

and $s \in R$. Now, we put $I_1 = \{x \in I \mid [y, s] = 0 \text{ for all } s \in R\}$ and $I_2 = \{x \in I \mid H(x)y - yx = 0, \text{ for all } y \in I\}$. Then, clearly I_1 and I_2 are additive subgroups of I. Moreover, by the discussion given, I is the set-theoretic union of I_1 and I_2. But a group can not be the set-theoretic union of two proper subgroups, hence $I_1 = I$ or $I_2 = I$. If $I_1 = I$, then $[y, s] = 0$, for all $y \in I, s \in R$ and hence R is commutative. On the other hand if $I_2 = I$, then $H(x)y = yx$, for all $x, y \in I$. That is $H(xy) - yx = 0$, for all $x, y \in I$. This implies that $H([x, y]) - [x, y] = 0$, for all $x, y \in I$. Hence, the application of Theorem 2.1 yields the required result. ◇

Using similar arguments as used in the proof of above theorem, we get the following:

THEOREM 3.4. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x) \neq -x$, for all $x \in I$. Further, if $H(xy) + yx \in Z(R)$, for all $x, y \in I$, then R is commutative.*

THEOREM 3.5. *Let R be a prime ring and I be a nonzero ideal of R. If R admits a nonzero left multiplier $H : R \longrightarrow R$ such that $H(x) \neq x$, for all $x \in I$. Then the following conditions are equivalent:*

(*i*) *For all $x, y \in I$, either $H(xy) - xy \in Z(R)$ or $H(xy) + xy \in Z(R)$;*
(*ii*) *For all $x, y \in I$, either $H(xy) - yx \in Z(R)$ or $H(xy) + yx \in Z(R)$ for all $x, y \in I$;*
(*iii*) *R is commutative.*

Proof. Obviously, $(iii) \Longrightarrow (i)$ & (ii). Using similar technics, as used to prove $(iii) \Longrightarrow (iv)$ in case of Theorem 2.5, it can be easily shown that $(i) \Longrightarrow (iii)$ and $(ii) \Longrightarrow (iii)$. ◇

The following theorems improve the results obtained in [2] and [3]:

THEOREM 3.6. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x) \neq x$, for all $x \in I$. Further, if $F(xy) - xy \in Z(R)$, for all $x, y \in I$, then R is commutative.*

Proof. Combining the proof of Theorem 3.1 above and Theorem 2.1 of [3], we get the required result. ◇

Similarly in view of Theorem 3.3 and Theorem 2.3 of [3], we get the following result:

THEOREM 3.7. *Let R be a prime ring and I be a nonzero ideal of R. Suppose that R admits a nonzero left multiplier H such that $H(x) \neq x$, for all $x \in I$. Further, if $F(xy) - yx \in Z(R)$, for all $x, y \in I$, then R is commutative.*

The following example shows that the above results are not true in the case of arbitrary rings:

EXAMPLE 3.1. Suppose that S is any ring. Let $R=\left\{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c\in S\right\}$ and let $I=\left\{\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b \in S\right\}$ be an ideal of R. Define a map H : $R \longrightarrow R$ such that $H\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, it is straightforward to see that R satisfies the properties: (i) $H(xy) \pm xy \in Z(R)$, (ii) $H(xy) \pm yx \in Z(R)$, (iii) $H([x,y]) \pm [x,y] = 0$ and (iv) $H(x \circ y) \pm (y \circ x) = 0$, for all $x, y \in I$. However, R is not commutative.

REMARK 3.2. It can be easily seen that the above results which are obtained for left multipliers are also true in the case of right multipliers.

REMARK 3.3.We conclude our paper with following open questions:

OPEN QUESTIONS 3.4: Let $n \geq 1$ be a fixed positive integer. Suppose R is a prime ring, I a nonzero ideal of R and $H : R \longrightarrow R$ is a nonzero left multiplier on R.

(i) Does the condition $H^n([x,y]) \pm [x,y] = 0$ (or $H^n(x \circ y) \pm (x \circ y) = 0$), for all $x, y \in I$ imply that R is commutative?

(ii) Does the condition $H^n(xy) \pm xy \in Z(R)$ (or $H^n(xy) \pm yx \in Z(R)$), for all $x, y \in I$ imply that R is commutative?

(iii) It is further interesting to explore the commutativity of R satisfying any one of the above properties for positive integer n which is not fixed rather it depends upon the pair x, y for its values.

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