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GENERALIZED PAPPUS’ THEOREM

Abstract. The paper contains a generalization to the $n$-dimensional projective space over a commutative field of a famous theorem of Pappus.

The well-known theorem of Pappus, as one of the most important theorems of the projective geometry, was a subject of many investigations. We may mention here for instance the works [1], [2], [3] and [4], where this theorem was generalized to the $n$-dimensional projective space $P^n$ (projective space over an arbitrary commutative field). In particular, the generalization from [1] concerns two sets of points $A = \{a_0, \ldots, a_n\}$ and $B = \{b_0, \ldots, b_n\}$ on two hyperplanes $H_1$ and $H_2$, respectively. The theorem says that the dimension of the join of subspaces (points in general) $S_0, \ldots, S_n$ is not greater than $n - 1$ ($S_j = \bigcap_{i=0, i\neq j}^{n} S_{ij}$, where $S_{ij} = J(b_i, A \setminus \{a_i, a_j\})$, $i \neq j$ (the symbol $J(P_1, \ldots, P_m)$ denotes the join of subspaces $P_1, \ldots, P_m$). Points $a_0, \ldots, a_n$ as well as $b_0, \ldots, b_n$ are assumed to be in a general position i.e. no $n$ of them are in an $(n-2)$-dimensional subspace. Obviously, when $n = 2$, it is the usual plane Pappus’ theorem. In this work we present a more general theorem than that from [1]. Throughout the paper we investigate two sets of points $A = \{a_0, \ldots, a_n\}$ and $B = \{b_0, \ldots, b_n\}$ such that $\dim J(A) = n - 1$, $\dim J(B) = k$, $1 \leq k \leq n - 1$, and points $a_0, \ldots, a_n$ as well as $b_0, \ldots, b_n$ are in a general position (no $k+1$ points of $b_0, \ldots, b_n$ are in a $(k-1)$-dimensional subspace). Then we shall show that $\dim J(S_0, \ldots, S_n) \leq k$. First we prove

Lemma 1. If points $b_0, \ldots, b_m$ ($0 \leq m \leq k - 1$) are in $H_1 = J(A)$, then $\dim J(S_0, \ldots, S_n) \leq \max(1, m)$.

Proof. First suppose $b_0 \in H_1$ and $b_i \notin H_1$ for $i = 1, \ldots n$. Then [1] $S_j = a_0$ (when $b_0 \notin J(A \setminus \{a_0, a_j\}$) or $S_j = \emptyset$ (when $b_0 \in J(A \setminus \{a_0, a_j\}$) for all $j = 1, \ldots, n$. Since $\dim S_0 = 0$, $\dim J(S_0, \ldots, S_n) = 1$.

2000 Mathematics Subject Classification: 51M04, 51M20.

Key words and phrases: Pappus theorem, projective space, commutativity.
Let now $m \geq 1$. For $j \leq m$, $S_j = A_j \cap B_j$, where $A_j = \bigcap_{i=0, i \neq j}^{n} S_{ij}$, $B_j = \bigcap_{i=m+1}^{n} S_{ij}$. Notice that $\{a_0, \ldots, a_m\} \subseteq B_j$ and $\dim B_j = m$. Hence $B_j \cap H_1 = J(\{a_0, \ldots, a_m\} \setminus \{a_j\})$. Since $A_j \subseteq H_1$, $S_j \subseteq J(a_0, \ldots, a_m)$. On the other hand, for $j > m + 1$, $S_j = C_j \cap D_j$, where $C_j = \bigcap_{i=0}^{m} S_{ij}$, $D_j = \bigcap_{i=m+1}^{n} S_{ij}$. We have $\{a_0, \ldots, a_m\} \subseteq D_j$, $\dim D_j = m + 1$ and, consequently, $\dim (D_j \cap H_1) = m$. It means that $D_j \cap H_1 = J(a_0, \ldots, a_m)$. As in the previous case $C_j \subseteq H_1$, hence $S_j \subseteq J(a_0, \ldots, a_m)$. Thus we see that $J(S_0, \ldots, S_n) \subseteq J(a_0, \ldots, a_m)$. This ends the proof. ■

**Lemma 2.** Let $A = \{a_0, \ldots, a_n\}$ and $B = \{b_0, \ldots, b_n\}$ be two sets of points on two hyperplanes $H_1$ and $H_2$, respectively. Points $a_0, \ldots, a_n$ are assumed to be in a general position. If some of points $b_i$ coincide, then $\dim J(S_0, \ldots, S_n) < n - 1$.

**Proof.** In view of Lemma 1 we may assume that $b_i \not\in H_1$ for $i = 0, \ldots, n$. Hence $\dim S_j = 0$, all $j$ (i.e. $S_j$ are points). Suppose e.g. $b_0 = b_1$. We have $S_{01} = J(b_0, a_2, \ldots, a_n) = S_{10} = J(b_1, a_2, \ldots, a_n)$. Hence $S_0, S_1 \in S_{01}$. Observe that for $j \geq 2$, $S_j \in J(b_0, \{a_2, \ldots, a_n\} \setminus \{a_j\}) \subseteq S_{01}$. ■

Let now $A = \{a_0, \ldots, a_n\}$ and $B = \{b_0, \ldots, b_n\}$ be two sets of points in a general position such that $b_i \notin J(A)$ for all $i$, and $\dim J(A) = n - 1$, $\dim J(B) = k$, $1 \leq k \leq n - 2$. There are, among points $a_i$, at least $n - k + 1$ not belonging to $J(B) = H_2$. We choose $n - k - 1$, say $a_0, \ldots, a_{n-k-2}$, from them in such a way that $\dim J(a_0, \ldots, a_n, b_{n-k-2}, H_2) = n - 1$.

**Lemma 3.** There are, in $\overline{H}_2 = J(a_0, \ldots, a_{n-k-2}, B)$, points $c_{k+1}, \ldots, c_n$ such that points $\overline{b}_0, \ldots, \overline{b}_n$ are in a general position, where $\overline{b}_i = b_i$ for $i = 0, \ldots, k$, $\overline{b}_i = c_i$ for $i = k + 1, \ldots, n$, and $\overline{S}_j = S_j$ for $j = n - k - 1, \ldots, n$ ($\overline{S}_j = \bigcap_{i=0, i \neq j}^{n} S_{ij}$, $\overline{S}_j = J(\overline{b}_i, A \setminus \{a_i, a_j\}$).

**Proof.** We choose a point $c_{k+1+i} \neq b_{k+1+i}$, $a_i$ on a line $l_i = J(a_i, b_{k+1+i})$, $i = 0, \ldots, n - k - 2$. Obviously, $c_{k+1} \notin H_2$, $c_{k+2} \notin H_3 = J(H_2, c_{k+1}), \ldots, c_{n-1} \notin H_{n-k-2} = J(H_{n-k-3}, c_{n-2})$. Thus we have $n$ linearly independent points $b_0, \ldots, b_k, c_{k+1}, \ldots, c_{n-1}$ which are vertices of an $(n - 1)$-dimensional simplex $S$ contained in $\overline{H}_2$. Consider the $(n - k - 1)$-dimensional subspace $G$ determined by points $a_0, \ldots, a_{n-k-2}, b_n$. $G$ cuts the faces of $S$ in subspaces $G_i \cap H_i = 1, \ldots, n$. Finally we choose a point $c_n$ in $G$ in such a way that $c_n \notin G_i$, all $i$. We have still to show that $\overline{S}_j = S_j$ for $j = n - k - 1, \ldots, n$. Observe that

$$S_j = \bigcap_{i=0}^{k} S_{ij} \cap \bigcap_{i=k+1, i \neq j}^{n} S_{ij}, \quad \overline{S}_j = \bigcap_{i=0}^{k} S_{ij} \cap \bigcap_{i=k+1, i \neq j}^{n} \overline{S}_{ij}$$

for $j = n - k - 1, \ldots, n$. Unauthenticated
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Nevertheless, when \( k + 1 \leq i \leq n - 1 \),
\[
S_{ij} = J(a_0, \ldots, a_{n-k-2}, b_i, \{a_{n-k-1}, \ldots, a_n\} \setminus \{a_i, a_j\})
= J(a_0, \ldots, a_{n-k-2}, c_i, \{a_{n-k-1}, \ldots, a_n\} \setminus \{a_i, a_j\}) = \overline{S}_{ij},
\]
since \( c_i \in J(a_{i-k-1}, b_i) \) and points \( c_i, a_{i-k-1}, b_i \) are all distinct. If \( i = n \), then
\[
S_{nj} = J(a_0, \ldots, a_k, \{a_{k+1}, \ldots, a_{n-1}\} \setminus \{a_j\}, b_n)
= J(a_0, \ldots, a_k, \{a_{k+1}, \ldots, a_{n-1}\} \setminus \{a_j\}, c_n) = \overline{S}_{nj},
\]
since \( c_n \in J(a_0, \ldots, a_{n-2}, b_n) = G \) and \( b_n \in J(a_0, \ldots, a_{n-2}, c_n) = G \). This completes the proof. 

**Lemma 4.** As previously, we consider two sets of points, in a general position, \( A = \{a_0, \ldots, a_n\} \) and \( B = \{b_0, \ldots, b_n\} \) such that \( \dim H_1 = \dim H_2 = n - 1 \), where \( H_1 = J(A) \), \( H_2 = J(B) \). If \( a_0, \ldots, a_{k-1} \in H_2 \) and \( b_i \notin H_1 \), all \( i \), then \( \dim J(S_k, \ldots, S_n) \leq n - k - 1 \).

**Proof.** Obviously, without loss of generality, we may assume that \( a_j \notin H_2 \) for \( j \geq k \). Suppose we choose in \( P^n \) an allowable coordinate system in such a way that the \( j \)-th coordinate of point \( a_i \) equals to \( \delta^j_i \), the equation of \( H_2 \) is \( \sum_{i=k}^n x_i = (1,1,\ldots,1,0) \), where \( i = 0, \ldots, n-1, j = 0, \ldots, n \), and \( \delta^j_i \) is the Kronecker \( \delta \). By \( b_{ij} \) we denote the \( j \)-th coordinate of point \( b_i \), \( i = 0, \ldots, n, j = 0, \ldots, n-1 \). Then \( b_{in} \) will equal to \( -\sum_{j=k}^{n-1} b_{ij} \). Notice that \( \sum_{j=k}^{n-1} b_{ij} \neq 0 \) for \( i = 0, \ldots, n \). Let us denote the sums \( \sum_{j=k}^{n-1} b_{ij} \), \( \sum_{j=k}^{n-1} b_{nj} \) by \( M_i \) and \( M \), respectively. One can check easily that the hyperplane \( S_{ni} \) has the equation \( x_i M_i + x_n b_{ni} = 0 \), \( i = 0, \ldots, n - 1 \). Consequently, the \( i \)-th coordinate \( s_{ni} \) of point \( S_n \) is \( b_{ii}/M_i \), \( i = 0, \ldots, n-1 \), while the \( n \)-th coordinate of this point equals to \( -1 \). Similarly, we check that hyperplane \( S_{ji} \) has the equation \( (x_j - x_i) M_i + x_n (b_{ij} - b_{ii}) = 0 \), \( j = k, \ldots, n - 1, i = 0, \ldots, n - 1 \), and the equation of \( S_{jn} \) is \( x_j M + x_n b_{nj} = 0 \), \( j = 0, \ldots, n - 1 \). Hence the \( i \)-th coordinate \( s_{ji} \) of point \( S_j \) is
\[
\frac{b_{ij} - b_{ii}}{M_i} - \frac{b_{nj}}{M}, \quad i = 0, \ldots, n - 1, \quad i \neq j, \quad s_{jj} = -\frac{b_{nj}}{M}, \quad s_{jn} = 1, \quad j = k, \ldots, n - 1.
\]
Observe that
\[
(n - k) s_{ni} + \sum_{j=k}^{n-1} s_{ji} = 0 \quad \text{for} \quad i = 0, \ldots, n.
\]
It means that points \( S_k, \ldots, S_n \) are linearly dependent i.e. \( \dim J(S_k, \ldots, S_n) \leq n - k - 1 \).

Let now \( A = \{a_0, \ldots, a_n\}, B = \{b_0, \ldots, b_n\} \) be two sets of points like those described in the introduction.
THEOREM. If \( \dim J(A) = n - 1 \) and \( \dim J(B) = k \), \( 1 \leq k \leq n - 1 \) and \( J(B) \not\subset J(A) \), then \( \dim J(S_0, \ldots, S_n) \leq k \).

Proof. Of course, we may consider \( k < n - 2 \). In view of Lemma 1, we may assume that \( b_i \not\in H_1 \) for \( i = 0, \ldots, n \). In fact, from \( b_0, \ldots, b_m \in H_1 \) it follows that \( \dim J(S_0, \ldots, S_n) \leq \max(1, m) \), but \( m \leq k - 1 \). Thus [1] the subspaces \( S_0, \ldots, S_n \) are points. Suppose \( \dim J(S_0, \ldots, S_n) > k \). Hence there exist \( k + 2 \) points, among \( S_0, \ldots, S_n \), say \( S_{n-k-1}, \ldots, S_n \), such that \( \dim J(S_{n-k-1}, \ldots, S_n) = k + 1 \). Take into account points \( a_0, \ldots, a_{n-k-2} \). Denote \( J(B, a_0, \ldots, a_{n-k-2}) \) by \( H_2 \). If \( \dim H_2 = n - 1 \), then by Lemma 3, there are points \( c_{k+1}, \ldots, c_n \) in \( H_2 \) such that the respective points \( S_j \) are equal to \( S_{j+k} \) for \( j = n - k - 1, \ldots, n \). According to Lemma 4, \( \dim J(S_{n-k-1}, \ldots, S_n) \leq k \), a contradiction. If \( \dim H_2 < n - 1 \), we add points \( a_{n-k-1}, \ldots, a_m \) to the points \( a_0, \ldots, a_{n-k-2} \) in such a way that \( \dim J(B, a_0, \ldots, a_m) = n - 1 \) and \( \dim J(B, a_0, \ldots, a_{m-1}) < n - 1 \). There is, among points \( a_0, \ldots, a_m \), a subset of \( n - k - 1 \) points, say \( a_{i_1}, \ldots, a_{i_{n-k-1}} \) such that \( \dim J(a_{i_1}, \ldots, a_{i_{n-k-1}}, B) = n - 1 \). Hence, Lemma 3, there exist points \( c_{k+1}, \ldots, c_n \) such that the respective points \( S_j \) are equal to \( S_j \) for \( j \not\in \{i_1, \ldots, i_{n-k-1}\} \). In particular, it has place when \( j = m+1, \ldots, n \). According to Lemma 4, \( \dim J(S_{m+1}, \ldots, S_n) \leq n - m - 2 \). It implies \( \dim J(S_m, \ldots, S_n) \leq n - m - 1 \), \( \dim J(S_{m-1}, \ldots, S_n) \leq n - m \) and so on. Finally, we obtain \( \dim J(S_{n-k-1}, \ldots, S_n) \leq k \) which contradicts with the supposition \( \dim J(S_0, \ldots, S_{n+1}) = k + 1 \). This ends the proof.

References


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Received May 27, 2008.