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## ON THE CATEGORY OF PSEUDO-BCI-ALGEBRAS

**Abstract.** The category  $\mathbf{psBCI}$  of pseudo-BCI-algebras and homomorphisms between them is investigated. It is also shown that the category  $\mathbf{psBCI}_p$  of  $p$ -semisimple pseudo-BCI-algebras and homomorphisms between them is a reflective subcategory of  $\mathbf{psBCI}$ .

### 1. Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are (pseudo-)MV-algebras, (pseudo-)BL-algebras, (pseudo-)BCK-algebras, (pseudo-)BCI-algebras and others. They are strongly connected with logic. For example, BCI-algebras introduced by K. Iséki in 1966 ([7]) have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming.

The notion of pseudo-BCI-algebras has been introduced by W. A. Dudek and Y. B. Jun in [3] as an extension of BCI-algebras and it was investigated by several authors in [4], [5], [8] and [9]. Pseudo-BCI-algebras are algebraic models of some extension of a non-commutative version of the BCI-logic. These algebras have also connections with other algebras of logic such as pseudo-BCK-algebras, pseudo-BL-algebras and pseudo-MV-algebras.

In this paper, the category  $\mathbf{psBCI}$  of pseudo-BCI-algebras and homomorphisms between them is considered. We prove that it has equalizers, coequalizers, products, pullbacks, limits, kernel pairs and it is complete. Moreover, we show that in  $\mathbf{psBCI}$  surjective morphisms and coequalizers coincide. Finally, the category  $\mathbf{psBCI}_p$  of  $p$ -semisimple pseudo-BCI-algebras and homomorphisms between them is studied. We show that it is a reflective subcategory of  $\mathbf{psBCI}$  and it is isomorphic with the category  $\mathbf{Grp}$  of groups and group homomorphisms.

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## 2. Preliminaries

We include some necessary material concerning pseudo-BCI-algebras, needed in the sequel.

A *pseudo-BCI-algebra* is a structure  $(X, \leq, \rightarrow, \rightsquigarrow, 1)$ , where  $\leq$  is a binary relation on a set  $X$ ,  $\rightarrow$  and  $\rightsquigarrow$  are binary operations on  $X$  and  $1$  is an element of  $X$  such that, for all  $x, y, z \in X$ , we have

- (a1)  $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$ ,  $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$ ,
- (a2)  $x \leq (x \rightarrow y) \rightsquigarrow y$ ,  $x \leq (x \rightsquigarrow y) \rightarrow y$ ,
- (a3)  $x \leq x$ ,
- (a4) if  $x \leq y$  and  $y \leq x$ , then  $x = y$ ,
- (a5)  $x \leq y$  iff  $x \rightarrow y = 1$  iff  $x \rightsquigarrow y = 1$ .

It is obvious that any pseudo-BCI-algebra  $(X, \leq, \rightarrow, \rightsquigarrow, 1)$  can be regarded as a universal algebra  $(X, \rightarrow, \rightsquigarrow, 1)$  of type  $(2, 2, 0)$ . Note that every pseudo-BCI-algebra satisfying  $x \rightarrow y = x \rightsquigarrow y$ , for all  $x, y \in X$  is a BCI-algebra.

Every pseudo-BCI-algebra satisfying  $x \leq 1$ , for all  $x \in X$  is a pseudo-BCK-algebra. A pseudo-BCI-algebra which is not a pseudo-BCK-algebra will be called *proper*.

Later in the paper, we will usually use the symbol  $X$  in place of  $(X, \rightarrow, \rightsquigarrow, 1)$ .

Any pseudo-BCI-algebra  $(X, \rightarrow, \rightsquigarrow, 1)$  satisfies the following, for all  $x, y, z \in X$ ,

- (b1) if  $1 \leq x$ , then  $x = 1$ ,
- (b2) if  $x \leq y$ , then  $y \rightarrow z \leq x \rightarrow z$  and  $y \rightsquigarrow z \leq x \rightsquigarrow z$ ,
- (b3) if  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ ,
- (b4)  $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$ ,
- (b5)  $x \leq y \rightarrow z$  iff  $y \leq x \rightsquigarrow z$ ,
- (b6)  $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$ ,  $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$ ,
- (b7) if  $x \leq y$ , then  $z \rightarrow x \leq z \rightarrow y$  and  $z \rightsquigarrow x \leq z \rightsquigarrow y$ ,
- (b8)  $1 \rightarrow x = 1 \rightsquigarrow x = x$ ,
- (b9)  $((x \rightarrow y) \rightsquigarrow y) \rightarrow y = x \rightarrow y$ ,  $((x \rightsquigarrow y) \rightarrow y) \rightsquigarrow y = x \rightsquigarrow y$ ,
- (b10)  $x \rightarrow y \leq (y \rightarrow x) \rightsquigarrow 1$ ,
- (b11)  $x \rightsquigarrow y \leq (y \rightsquigarrow x) \rightarrow 1$ ,
- (b12)  $(x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \rightsquigarrow (y \rightsquigarrow 1)$ ,
- (b13)  $(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \rightarrow (y \rightarrow 1)$ ,
- (b14)  $x \rightarrow 1 = x \rightsquigarrow 1$ .

If  $(X, \leq, \rightarrow, \rightsquigarrow, 1)$  is a pseudo-BCI-algebra, then by (a3), (a4), (b3) and (b1),  $(X, \leq)$  is a poset with  $1$  as a maximal element.

For any pseudo-BCI-algebra  $(X, \rightarrow, \rightsquigarrow, 1)$  the set

$$K(X) = \{x \in X : x \leq 1\}$$

is a subalgebra of  $X$  (called pseudo-BCK-part of  $X$ , see [3]).

Let  $(X, \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then  $X$  is *p-semisimple* if it satisfies for all  $x \in X$ ,

$$\text{if } x \leq 1, \text{ then } x = 1.$$

Note that if  $X$  is a p-semisimple pseudo-BCI-algebra, then  $K(X) = \{1\}$ . Hence, if  $X$  is a p-semisimple pseudo-BCK-algebra, then  $X = \{1\}$ . It is proved in [5] that  $(X, \rightarrow, \rightsquigarrow, 1)$  is p-semisimple if and only if for all  $x, y \in X$ ,  $(x \rightarrow 1) \rightsquigarrow y = (y \rightsquigarrow 1) \rightarrow x$ .

Let  $(X, \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. We say that a subset  $D$  of  $X$  is a *deductive system* of  $X$  if it satisfies: (1)  $1 \in D$ , (2) for all  $x, y \in X$ , if  $x \in D$  and  $x \rightarrow y \in D$ , then  $y \in D$ . Under this definition,  $\{1\}$  and  $X$  are the simplest examples of deductive systems. Note that the condition (2) can be replaced by (2') for all  $x, y \in X$ , if  $x \in D$  and  $x \rightsquigarrow y \in D$ , then  $y \in D$ . It can be easily proved that for any  $x, y \in X$ , if  $x \in D$  and  $x \leq y$ , then  $y \in D$ .

A deductive system  $D$  of a pseudo-BCI-algebra  $(X, \rightarrow, \rightsquigarrow, 1)$  is called *closed* if  $D$  is closed under operations  $\rightarrow$  and  $\rightsquigarrow$ , that is, if  $D$  is a subalgebra of  $X$ . It is not difficult to show (see [4]) that a deductive system  $D$  of a pseudo-BCI-algebra  $(X, \rightarrow, \rightsquigarrow, 1)$  is closed if and only if for any  $x \in D$ ,  $x \rightarrow 1 = x \rightsquigarrow 1 \in D$ . Obviously, the pseudo-BCK-part  $K(X)$  is a closed deductive system of  $X$ .

A deductive system  $D$  of a pseudo-BCI-algebra  $(X, \rightarrow, \rightsquigarrow, 1)$  is said to be *compatible* if for all  $x, y \in X$ ,

$$x \rightarrow y \in D \text{ iff } x \rightsquigarrow y \in D.$$

Further, if  $D$  is a compatible deductive system of  $X$ , then the relation  $\Theta_D$  defined by

$$(1) \quad (x, y) \in \Theta_D \text{ iff } x \rightarrow y \in D \text{ and } y \rightarrow x \in D$$

is a congruence. We say that  $\Theta \in \text{Con}(X)$  is a *relative congruence* of  $(X, \rightarrow, \rightsquigarrow, 1)$  if the quotient algebra  $(X/\Theta, \rightarrow, \rightsquigarrow, [1]_\Theta)$  is a pseudo-BCI-algebra. It is proved in [4] that relative congruences of  $X$  correspond one-to-one to closed compatible deductive systems of  $X$ , that is, every relative congruence of  $X$  is given by (1) for some closed compatible deductive system  $D$ . For every relative congruence  $\Theta_D$ , the quotient algebra  $(X/\Theta_D, \rightarrow, \rightsquigarrow, [1]_{\Theta_D})$  will be usually denoted by  $(X/D, \rightarrow, \rightsquigarrow, 1/D)$  and then we will write  $x/D$  instead of  $[x]_{\Theta_D}$ .

We know that pseudo-BCK-part  $K(X)$  of a pseudo-BCI-algebra  $(X, \rightarrow, \rightsquigarrow, 1)$  is a closed deductive system of  $X$ . It is proved in [4] that

it is also compatible and we have that  $(X/K(X), \rightarrow, \rightsquigarrow, 1/K(X))$  is a p-semisimple pseudo-BCI-algebra.

Moreover we will need the following fact.

**LEMMA 2.1.** *Let  $f : X \rightarrow Y$  be a homomorphism of pseudo-BCI-algebras  $X, Y$ . Then  $\text{Ker}(f) = \{x \in X : f(x) = 1\}$  is a closed compatible deductive system of  $X$ .*

**Proof.** Routine. ■

### 3. The category **psBCI**

All notions from the category theory occurring in this section the reader can find in [1] or [11].

If we consider the class of all pseudo-BCI-algebras as the class of objects and the class of all homomorphisms between pseudo-BCI-algebras as the class of morphisms, then we obtain the category of pseudo-BCI-algebras. We denote it by **psBCI**. In the section, we investigate this category.

First, remark that the class of objects in **psBCI** is not a set. Therefore, **psBCI** is not a small category. Moreover, we can define a forgetful functor  $F : \mathbf{psBCI} \rightarrow \mathbf{Set}$  which is faithful. Hence, the category **psBCI** is concrete and embedded in the category **Set** of sets and functions.

Observe yet that in **psBCI**,  $\{1\}$  is a zero object because it is an initial object as well as a terminal object. Indeed, there is a unique morphism  $f : \{1\} \rightarrow X$  for any object  $X$ , so  $\{1\}$  is an initial object. Similarly, there exists a unique morphism  $g : X \rightarrow \{1\}$  for any object  $X$ , so  $\{1\}$  is also a terminal object. Further, note that  $0_{\{1\}} : X \rightarrow \{1\}$  is a zero morphism in **psBCI**, since it is in the same time a constant morphism and coconstant morphism.

**THEOREM 3.1.** *For any morphism  $f : X \rightarrow Y$  in **psBCI** the following are equivalent:*

- (i)  $f$  is injective,
- (ii) for all morphisms  $g, h$ , if  $f \circ g = f \circ h$ , then  $g = h$ ,
- (iii)  $\text{Ker}(f) = \{1\}$ .

**Proof.** (i) $\Rightarrow$ (ii): Assume that  $f$  is an injective morphism between objects  $X, Y$ . Let  $Z$  be another object, and let  $g, h : Z \rightarrow X$  be morphisms such that  $f \circ g = f \circ h$ . Then for all  $z \in Z$ ,  $f(g(z)) = f(h(z))$ . Hence since  $f$  is injective, we get  $g(z) = h(z)$ . Thus  $g = h$ .

(ii) $\Rightarrow$ (iii): Suppose that  $\text{Ker}(f) \neq \{1\}$ . Then there exists  $x \in \text{Ker}(f)$  and  $x \neq 1$ . Let us consider morphisms  $i : \text{Ker}(f) \rightarrow X$  and  $j : \text{Ker}(f) \rightarrow X$  such that  $i(x) = x$  and  $j(x) = 1$ , for all  $x \in \text{Ker}(f)$ . Then  $f \circ i = f \circ j$ . Now, by (ii),  $i = j$ . Thus we get a contradiction. Therefore  $\text{Ker}(f) = \{1\}$ .

(iii) $\Rightarrow$ (i): Let  $Ker(f) = \{1\}$  and  $x_1, x_2 \in X$  be such that  $f(x_1) = f(x_2)$ . Then  $f(x_1 \rightarrow x_2) = f(x_1) \rightarrow f(x_2) = 1$  and  $f(x_2 \rightarrow x_1) = f(x_2) \rightarrow f(x_1) = 1$ . Hence  $x_1 \rightarrow x_2, x_2 \rightarrow x_1 \in Ker(f) = \{1\}$ . Thus  $x_1 \rightarrow x_2 = x_2 \rightarrow x_1 = 1$ , so,  $x_1 \leq x_2$  and  $x_2 \leq x_1$ . Now it is clear that  $x_1 = x_2$  and  $f$  is injective. ■

**COROLLARY 3.2.** *In the category psBCI injective morphisms and monomorphisms coincide.*

**PROPOSITION 3.3.** *Let  $f : X \rightarrow Y$  be a morphism in psBCI. If  $f$  is surjective, then for all morphisms  $g, h$ , if  $g \circ f = h \circ f$ , then  $g = h$ .*

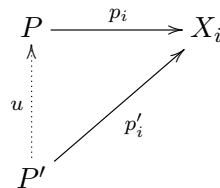
**Proof.** Let  $f : X \rightarrow Y$  be a surjective morphism,  $Z$  be an object and  $g, h : Y \rightarrow Z$  be morphisms such that  $g \circ f = h \circ f$ . Since  $f$  is surjective, for any  $y \in Y$  there exists  $x \in X$  such that  $y = f(x)$ . Then  $g(y) = g(f(x)) = h(f(x)) = h(y)$ , for all  $y \in Y$ . Therefore  $g = h$ . ■

**COROLLARY 3.4.** *A morphism in the category psBCI is an epimorphism if it is surjective.*

**REMARK.** It is well-known that any Hilbert algebra is a pseudo-BCI-algebra (precisely, a BCK-algebra). In [2] there is given an example of an epimorphism between Hilbert algebras (so, pseudo-BCI-algebras) which is not surjective. Thus, in the category psBCI isomorphisms and bismorphisms are not the same.

**COROLLARY 3.5.** *The category psBCI is not balanced.*

Let  $\mathbf{C}$  be a category and  $(X_i)_{i \in I}$  a family of objects in  $\mathbf{C}$ . A *direct product* of a family  $(X_i)_{i \in I}$  is a pair  $(P, (p_i)_{i \in I})$ , where  $P$  is an object in  $\mathbf{C}$  and  $(p_i)_{i \in I}$  is a family of morphisms in  $\mathbf{C}$ ,  $p_i : P \rightarrow X_i$ , such that for any other pair  $(P', (p'_i)_{i \in I})$  composed by an object  $P'$  and a family of morphisms  $(p'_i)_{i \in I}$ ,  $p'_i : P' \rightarrow X_i$ , there is an unique morphism  $u : P' \rightarrow P$  such that  $p_i \circ u = p'_i$  for every  $i \in I$ , so that for every  $i \in I$  the following diagram is commutative:



We say that a category  $\mathbf{C}$  has *products* if there exists a direct product of any family of objects from  $\mathbf{C}$ .

**THEOREM 3.6.** *The category psBCI has products.*

**Proof.** Let  $(X_i)_{i \in I}$  be a family of objects. Consider the set  $P = \prod_{i \in I} X_i$  of all functions  $f : I \rightarrow \bigcup_{i \in I} X_i$  such that  $f(i) \in X_i$  for all  $i \in I$ . A

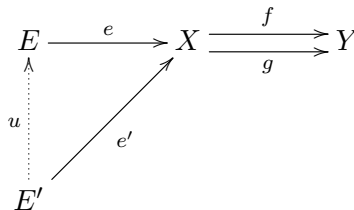
function  $1 : I \rightarrow \bigcup_{i \in I} X_i$  such that  $1(i) = 1$  for all  $i \in I$ , is a special element of  $P$ . Define binary operations  $\rightarrow$  and  $\rightsquigarrow$  on  $P$  as follows: for  $f, g \in P$ ,  $(f \rightarrow g)(i) = f(i) \rightarrow g(i)$  and  $(f \rightsquigarrow g)(i) = f(i) \rightsquigarrow g(i)$  for all  $i \in I$ . We can verify that the structure  $(P, \rightarrow, \rightsquigarrow, 1)$  forms a pseudo-BCI-algebra, that is  $P$  is an object in **psBCI**.

For each  $i \in I$ , there is a natural projection  $p_i : P \rightarrow X_i$  defined by  $p_i(f) = f(i)$  for all  $f \in P$ . Further, for all objects  $P'$  and morphisms  $p'_i : P' \rightarrow X_i$  for  $i \in I$  the map  $u : P' \rightarrow P$  defined by

$$(u(x))(i) = p'_i(x) \text{ for all } x \in P' \text{ and } i \in I$$

is the unique morphism such that  $p_i \circ u = p'_i$ . Thus the category **psBCI** has products. ■

By a couple of morphisms  $(f, g)$  in a category **C** we understand two morphisms  $f, g : X \rightarrow Y$ , where  $X, Y$  are objects in **C**. A pair  $(E, e)$  with  $E$  an object in **C** and  $e : E \rightarrow X$  a morphism in **C**, will be called an *equalizer* of a couple  $(f, g)$  if  $f \circ e = g \circ e$  and for every other pair  $(E', e')$  with  $E'$  an object and  $e' : E' \rightarrow X$  a morphism such that  $f \circ e' = g \circ e'$ , there exists an unique morphism  $u : E' \rightarrow E$  such that  $e' = e \circ u$ :



We say that a category **C** has *equalizers* if there exists an equalizer for any couple of morphisms in **C**.

**THEOREM 3.7.** *The category psBCI has equalizers.*

**Proof.** Let  $(f, g)$  be a couple of morphisms,  $f, g : X \rightarrow Y$ . Then nonempty set  $E = \{x \in X : f(x) = g(x)\}$  is a subalgebra of  $X$  and if we consider the embedding  $e : E \rightarrow X$ , then  $f \circ e = g \circ e$ .

Further, let  $E'$  be other object and let  $e' : E' \rightarrow X$  be a morphism such that  $f \circ e' = g \circ e'$ . We define  $u : E' \rightarrow E$ ,  $u(x) = e'(x)$  for all  $x \in E'$ . Then  $u$  is well defined, since from  $f \circ e' = g \circ e'$  we have  $e'(x) \in E$  for every  $x \in E'$ . It is clear that  $u$  is a morphism and  $e \circ u = e'$ .

The uniqueness of  $u$  follows from the fact that  $e$  is a monomorphism. ■

**COROLLARY 3.8.** *The category psBCI has pullbacks, limits and it is complete.*

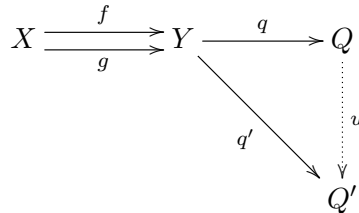
Let  $f : X \rightarrow Y$  be a morphism in  $\mathbf{C}$ . We say that  $f$  is an *equalizer* if there exists a couple of morphisms  $(\alpha, \beta)$  such that  $\alpha, \beta : Y \rightarrow Z$  and  $(X, f)$  is an equalizer of  $(\alpha, \beta)$ . Obviously, every equalizer in  $\mathbf{C}$  is a monomorphism.

Thus by Corollary 3.2, we have the following theorem.

**THEOREM 3.9.** *In the category psBCI every equalizer is injective.*

**REMARK.** The converse of Theorem 3.9 is not true. In [6], there is given an example of an injective morphism between Hilbert algebras (so, pseudo-BCI-algebras) which can not be an equalizer for any couple of morphisms.

Let  $f, g : X \rightarrow Y$ , where  $X, Y$  are objects in a category  $\mathbf{C}$ . A pair  $(Q, q)$  with  $Q$  an object in  $\mathbf{C}$  and  $q : Y \rightarrow Q$  a morphism in  $\mathbf{C}$ , will be called a *coequalizer* of a couple  $(f, g)$  if  $q \circ f = q \circ g$  and for every other pair  $(Q', q')$  with  $Q'$  an object and  $q' : Y \rightarrow Q'$  a morphism such that  $q' \circ f = q' \circ g$ , there exists an unique morphism  $u : Q \rightarrow Q'$  such that  $q' = u \circ q$ :



We say that a category  $\mathbf{C}$  has *coequalizers* if there exists a coequalizer for any couple of morphisms in  $\mathbf{C}$ .

**THEOREM 3.10.** *The category psBCI has coequalizers.*

**Proof.** Let  $(f, g)$  be a couple of morphisms,  $f, g : X \rightarrow Y$ . Put

$$R = \{(f(x), g(x)) \in Y \times Y : x \in X\}.$$

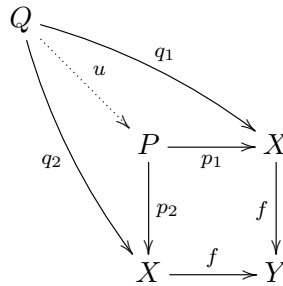
Let  $\Theta$  be the intersection of all relative congruences on  $Y$  (that is, congruences determined by closed compatible deductive systems of  $Y$ ) which contain  $R$ . Then  $Q = Y/\Theta$  is an object in **psBCI**. Let  $q : Y \rightarrow Q$  be the canonical surjection. We show that  $(Q, q)$  is a coequalizer of  $(f, g)$ . Since  $(f(x), g(x)) \in \Theta$  for all  $x \in X$ , we have  $(q \circ f)(x) = q(f(x)) = [f(x)]_\Theta = [g(x)]_\Theta = q(g(x)) = (q \circ g)(x)$  for all  $x \in X$ . Thus  $q \circ f = q \circ g$ .

Let  $Q'$  be another object and let  $q' : Y \rightarrow Q'$  be a morphism such that  $q' \circ f = q' \circ g$ . Let  $\Theta' = \{(y_1, y_2) \in Y \times Y : q'(y_1) = q'(y_2)\} = \{(y_1, y_2) \in Y \times Y : y_1 \rightarrow y_2, y_2 \rightarrow y_1 \in Ker(q')\}$ . Then  $\Theta'$  is a relative congruence determined by a closed compatible deductive system  $Ker(q')$ . Since for every  $x \in X$  we have  $q'(f(x)) = q'(g(x))$ , we obtain  $(f(x), g(x)) \in \Theta'$  for every  $x \in X$ . Hence  $R \subset \Theta'$ . Thus  $\Theta \subset \Theta'$ . We can define now a morphism  $u : Q \rightarrow Q'$  such that  $u([y]_\Theta) = q'(y)$ . Then  $u$  is well defined because for

$[y_1]_{\Theta} = [y_2]_{\Theta}$  we have  $(y_1, y_2) \in \Theta \subset \Theta'$  whence  $q'(y_1) = q'(y_2)$ . Clearly,  $u \circ q = q'$ .

The uniqueness of  $u$  follows from the fact that  $q$  is an epimorphism. This completes the proof. ■

Let  $\mathbf{C}$  be a category and  $f : X \rightarrow Y$  a morphism in  $\mathbf{C}$ . A system  $(P; p_1, p_2)$  formed by an object  $P$  and two morphisms  $p_1, p_2 : P \rightarrow X$ , is called a *kernel pair* of  $f$  if  $f \circ p_1 = f \circ p_2$  and for any other system  $(Q; q_1, q_2)$  with an object  $Q$  and morphisms  $q_1, q_2 : Q \rightarrow X$  such that  $f \circ q_1 = f \circ q_2$ , there exists an unique morphism  $u : Q \rightarrow P$  such that  $p_1 \circ u = q_1$  and  $p_2 \circ u = q_2$ :



We say that a category  $\mathbf{C}$  has *kernel pairs* if every morphism in it has a kernel pair.

**THEOREM 3.11.** *The category psBCI has kernel pairs.*

**Proof.** Let  $f : X \rightarrow Y$  be a morphism. Let us put

$$P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}.$$

Obviously,  $P$  is a subalgebra of the product algebra  $X \times X$ . Let  $p_1, p_2 : P \rightarrow X$  be the canonical projections, that is,  $p_i(x_1, x_2) = x_i$  for  $i = 1, 2$  and all  $(x_1, x_2) \in P$ . We show that  $(P; p_1, p_2)$  is a kernel pair of  $f$ . Clearly,  $f \circ p_1 = f \circ p_2$ . Let  $(Q; q_1, q_2)$  with an object  $Q$  and morphisms  $q_1, q_2 : Q \rightarrow X$  be another system such that  $f \circ q_1 = f \circ q_2$ . We take  $u : Q \rightarrow P$  as

$$u(x) = (q_1(x), q_2(x)) \text{ for all } x \in Q.$$

Now,  $u$  is well defined because  $f \circ q_1 = f \circ q_2$  implies  $f(q_1(x)) = f(q_2(x))$  whence  $(q_1(x), q_2(x)) \in P$ . Further,

$$\begin{aligned} u(x_1 \rightarrow x_2) &= (q_1(x_1 \rightarrow x_2), q_2(x_1 \rightarrow x_2)) \\ &= (q_1(x_1) \rightarrow q_1(x_2), q_2(x_1) \rightarrow q_2(x_2)) \\ &= (q_1(x_1), q_2(x_1)) \rightarrow (q_1(x_2), q_2(x_2)) \\ &= u(x_1) \rightarrow u(x_2) \end{aligned}$$



and

$$\begin{aligned} u(x_1 \rightsquigarrow x_2) &= (q_1(x_1 \rightsquigarrow x_2), q_2(x_1 \rightsquigarrow x_2)) \\ &= (q_1(x_1) \rightsquigarrow q_1(x_2), q_2(x_1) \rightsquigarrow q_2(x_2)) \\ &= (q_1(x_1), q_2(x_1)) \rightsquigarrow (q_1(x_2), q_2(x_2)) \\ &= u(x_1) \rightsquigarrow u(x_2). \end{aligned}$$

Thus  $u$  is a morphism in **psBCI**. Moreover, it is easy to see that  $p_1 \circ u = q_1$  and  $p_2 \circ u = q_2$ .

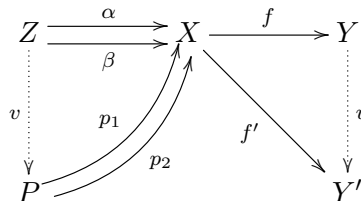
Now, let  $u' : Q \rightarrow P$  be another morphism such that  $p_1 \circ u' = q_1$  and  $p_2 \circ u' = q_2$ . Let  $u'(x) = (x', x'')$ . Then  $p_1 \circ u' = p_1 \circ u$  gives  $p_1(x', x'') = p_1(q_1(x), q_2(x))$  whence  $x' = q_1(x)$  and  $x'' = q_2(x)$ . Thus  $u'(x) = (x', x'') = (q_1(x), q_2(x)) = u(x)$  for all  $x \in Q$ . Hence  $u$  is unique. Therefore, the system  $(P; p_1, p_2)$  is a kernel pair of  $f$ . ■

Let  $f : X \rightarrow Y$  be a morphism in **C**. We say that  $f$  is a *coequalizer* if there exists a couple of morphisms  $(\alpha, \beta)$  such that  $\alpha, \beta : Z \rightarrow X$  and  $(Y, f)$  is a coequalizer of  $(\alpha, \beta)$ . Clearly, every coequalizer in **C** is an epimorphism.

**PROPOSITION 3.12.** *Let  $f : X \rightarrow Y$  be a coequalizer in **psBCI**. Then  $f$  is a coequalizer of its kernel pair.*

**Proof.** Let  $\alpha, \beta : Z \rightarrow X$  be such that  $f$  is a coequalizer of  $(\alpha, \beta)$  and let  $(P; p_1, p_2)$  be a kernel pair of  $f$ . Since  $f \circ p_1 = f \circ p_2$ , it is sufficient to prove that for any other morphism  $f' : X \rightarrow Y'$  such that  $f' \circ p_1 = f' \circ p_2$ , there exists a unique morphism  $u : Y \rightarrow Y'$  such that  $f' = u \circ f$ .

Since  $f \circ \alpha = f \circ \beta$  and  $(P; p_1, p_2)$  is a kernel pair of  $f$ , we get the existence of a unique morphism  $v : Z \rightarrow P$  such that  $\alpha = p_1 \circ v$  and  $\beta = p_2 \circ v$ :



Hence  $f' \circ \alpha = (f' \circ p_1) \circ v = (f' \circ p_2) \circ v = f' \circ \beta$ . Thus since  $f$  is a coequalizer of  $(\alpha, \beta)$ , we obtain the existence of a unique  $u : Y \rightarrow Y'$  such that  $f' = u \circ f$ . This completes the proof. ■

**THEOREM 3.13.** *Every surjective morphism in **psBCI** is a coequalizer.*

**Proof.** Let  $(P; p_1, p_2)$  be a kernel pair of a surjective morphism  $f : X \rightarrow Y$ . Then as we know  $P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$  and  $p_1, p_2 : P \rightarrow X$  are the canonical projections. It is sufficient to prove that  $(Y, f)$  is a coequalizer of  $(p_1, p_2)$ . Clearly,  $f \circ p_1 = f \circ p_2$ . Let  $f' : X \rightarrow Y'$  be a

morphism such that  $f' \circ p_1 = f' \circ p_2$ . Since  $f$  is surjective, for every  $y \in Y$  there exists  $x \in X$  such that  $f(x) = y$ . Now, take  $u : Y \rightarrow Y'$  as follows:  $u(y) = f'(x)$ . It is well defined because if  $f(x_1) = f(x_2) = y$ , then  $(x_1, x_2) \in P$  and  $u(y) = f'(x_1) = (f' \circ p_1)(x_1, x_2) = (f' \circ p_2)(x_1, x_2) = f'(x_2)$ . Next, let  $y_1, y_2 \in Y$ . Then there exist  $x_1, x_2 \in X$  such that  $f(x_1) = y_1$  and  $f(x_2) = y_2$ , and hence  $f'(x_1) = u(y_1)$  and  $f'(x_2) = u(y_2)$ . Further, we have  $y_1 \rightarrow y_2 = f(x_1) \rightarrow f(x_2) = f(x_1 \rightarrow x_2)$  and  $y_1 \rightsquigarrow y_2 = f(x_1) \rightsquigarrow f(x_2) = f(x_1 \rightsquigarrow x_2)$ . Hence  $u(y_1 \rightarrow y_2) = f'(x_1 \rightarrow x_2) = f'(x_1) \rightarrow f'(x_2) = u(y_1) \rightarrow u(y_2)$  and  $u(y_1 \rightsquigarrow y_2) = f'(x_1 \rightsquigarrow x_2) = f'(x_1) \rightsquigarrow f'(x_2) = u(y_1) \rightsquigarrow u(y_2)$ . Thus  $u$  is a morphism and obviously,  $u \circ f = f'$ . The uniqueness of  $u$  follows from the fact that  $f$  is an epimorphism. Therefore  $f$  is a coequalizer. ■

**PROPOSITION 3.14.** *Let  $X, Y, Z$  be objects in **psBCI** and  $f : X \rightarrow Y$  and  $g : X \rightarrow Z$  be morphisms in **psBCI** such that  $f$  is surjective and  $\text{Ker}(f) \subset \text{Ker}(g)$ . Then there exists a unique morphism  $h : Y \rightarrow Z$  such that  $h \circ f = g$ .*

**Proof.** Let  $(P; p_1, p_2)$  be a kernel pair of  $f$ , that is,  $P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$  and  $p_1, p_2 : P \rightarrow X$  are the canonical projections. Since  $f$  is surjective, by Theorem 3.13, we have that  $f$  is a coequalizer, that is,  $(Y, f)$  is a coequalizer of  $(p_1, p_2)$ . Let  $(x_1, x_2) \in P$ . Then  $f(x_1) = f(x_2)$  which gives that  $x_1 \rightarrow x_2, x_2 \rightarrow x_1 \in \text{Ker}(f) \subset \text{Ker}(g)$ , so  $g(x_1) = g(x_2)$ . Hence, there exists a unique morphism  $h : Y \rightarrow Z$  such that  $h \circ f = g$ :

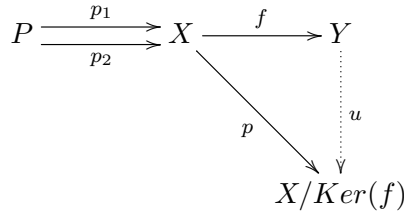
$$\begin{array}{ccccc}
 P & \xrightarrow{p_1} & X & \xrightarrow{f} & Y \\
 & \xrightarrow{p_2} & & & \vdots \\
 & & & \searrow g & \downarrow h \\
 & & & & Z
 \end{array}$$

This completes the proof. ■

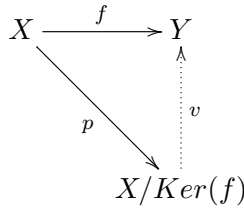
**THEOREM 3.15.** *Every coequalizer in **psBCI** is surjective.*

**Proof.** Let  $f : X \rightarrow Y$  be a coequalizer. By Proposition 3.12,  $f$  is a coequalizer of its kernel pair  $(P; p_1, p_2)$ , where  $P = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$  and  $p_1, p_2 : P \rightarrow X$  are the canonical projections. Note that  $P = \{(x_1, x_2) \in X \times X : x_1 \rightarrow x_2, x_2 \rightarrow x_1 \in \text{Ker}(f)\}$ . Hence  $P$  is a relative congruence determined by the closed compatible deductive system  $\text{Ker}(f)$ . Let  $X/\text{Ker}(f)$  be the corresponding quotient pseudo-BCI-algebra and let  $p : X \rightarrow X/\text{Ker}(f)$  be the canonical surjection. Notice that  $p \circ p_1 = p \circ p_2$ . Indeed, for every  $(x_1, x_2) \in P$  we have  $(p \circ p_1)(x_1, x_2) = x_1/\text{Ker}(f) = x_2/\text{Ker}(f) = (p \circ p_2)(x_1, x_2)$ . Since  $(Y, f)$  is a coequalizer of  $(p_1, p_2)$ , there

exists an unique morphism  $u : Y \rightarrow X/Ker(f)$  such that  $u \circ f = p$ :



Let  $x \in Ker(p)$ . Then  $p(x) = 1/Ker(f)$ . Since  $p(x) = x/Ker(f)$ , we get  $(x, 1) \in P$ , so  $x \in Ker(f)$ . This means that  $Ker(p) \subset Ker(f)$ . Thus by Proposition 3.14, there exists an unique morphism  $v : X/Ker(f) \rightarrow Y$  such that  $v \circ p = f$ :



Now,

$$(u \circ v) \circ p = u \circ f = p = 1_{X/Ker(f)} \circ p$$

and

$$(v \circ u) \circ f = v \circ p = f = 1_Y \circ f.$$

Since  $p$  is surjective and  $f$  is a coequalizer, both are epimorphisms. Hence

$$u \circ v = 1_{X/Ker(f)} \quad \text{and} \quad v \circ u = 1_Y.$$

Thus  $u$  and  $v$  are isomorphisms, one the inverse of the other. Now, we get that  $f = v \circ p$  is surjective, because both  $v$  and  $p$  are surjective. ■

**COROLLARY 3.16.** *In the category psBCI surjective morphisms and coequalizers coincide.*

**REMARK.** In the category psBCI not every epimorphism is a coequalizer. Indeed, in [6] there is given an example of an epimorphism (not a surjective one) between Hilbert algebras (so, pseudo-BCI-algebras) which is not a coequalizer.

#### 4. The category psBCI<sub>p</sub>

The category formed by taking the class of objects as the class of all p-semisimple pseudo-BCI-algebras and the class of morphisms as the class of all homomorphisms between them is called the category of p-semisimple pseudo-BCI-algebras. We denote this category by psBCI<sub>p</sub>. We have an inclusion functor  $I : \text{psBCI}_p \hookrightarrow \text{psBCI}$ , which is faithful and full. Hence

$\mathbf{psBCI}_p$  is a full subcategory of the category  $\mathbf{psBCI}$ . Like  $\mathbf{psBCI}$ , the category  $\mathbf{psBCI}_p$  is not a small category, it is concrete and embedded in the category  $\mathbf{Set}$ ; it also has zero objects ( $\{1\}$  is so) and zero morphisms ( $0_{\{1\}} : X \rightarrow \{1\}$  is the one).

For  $p$ -semisimple pseudo-BCI-algebras we have the following nice fact from [5] (compare with [10] for  $p$ -semisimple BCI-algebras).

**THEOREM 4.1.** *A pseudo-BCI-algebra  $(X, \rightarrow, \rightsquigarrow, 1)$  is  $p$ -semisimple if and only if  $(X, \cdot, ^{-1}, e)$  is a group, where, for any  $x, y \in X$ ,  $x \cdot y = (x \rightarrow 1) \rightsquigarrow y = (y \rightsquigarrow 1) \rightarrow x$ ,  $x^{-1} = x \rightarrow 1 = x \rightsquigarrow 1$  and  $e = 1$ . In this case,  $x \rightarrow y = y \cdot x^{-1}$  and  $x \rightsquigarrow y = x^{-1} \cdot y$  for any  $x, y \in X$ .*

Moreover, it is not difficult to prove that  $f$  is a morphism in the category  $\mathbf{psBCI}_p$  if and only if it is a morphism in the category  $\mathbf{Grp}$  of groups and group homomorphisms. Thus we have the following theorem.

**THEOREM 4.2.** *The category  $\mathbf{psBCI}_p$  is isomorphic with the category  $\mathbf{Grp}$ .*

**REMARK.** From Theorem 4.2, it follows that the category  $\mathbf{psBCI}_p$  has the same properties as the category  $\mathbf{Grp}$ . For example, it has coproducts and it is balanced and cocomplete.

A subcategory  $\mathbf{C}'$  of a category  $\mathbf{C}$  is called *reflective* if there is a covariant functor  $R : \mathbf{C} \rightarrow \mathbf{C}'$ , called *reflector*, such that for every object  $X$  from  $\mathbf{C}$  there is a morphism  $\phi_R(X) : X \rightarrow R(X)$  in  $\mathbf{C}$  with the properties:

- (i) if  $f : X \rightarrow Y$  is a morphism in  $\mathbf{C}$ , then the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \phi_R(X) \downarrow & & \downarrow \phi_R(Y) \\
 R(X) & \xrightarrow{R(f)} & R(Y)
 \end{array}$$

is commutative, that is,  $\phi_R(Y) \circ f = R(f) \circ \phi_R(X)$ ,

- (ii) if  $X'$  is an object in  $\mathbf{C}'$  and  $f : X \rightarrow X'$  is a morphism in  $\mathbf{C}$ , then there is an unique morphism  $f' : R(X) \rightarrow X'$  in  $\mathbf{C}'$  such that the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f} & X' \\
 \phi_R(X) \searrow & & \nearrow f' \\
 & R(X) &
 \end{array}$$

is commutative, that is,  $f' \circ \phi_R(X) = f$ .

**REMARK.** It is a well known fact that  $\mathbf{C}'$  is a reflective subcategory of a category  $\mathbf{C}$  if and only if there exist a function which assigns to every object  $X$  in  $\mathbf{C}$ , an object  $R(X)$  in  $\mathbf{C}'$  and a function which assigns to every  $X$  in  $\mathbf{C}$ , a morphism  $\phi_R(X) : X \rightarrow R(X)$  in  $\mathbf{C}$  such that for every object  $X'$  in  $\mathbf{C}'$  and every morphism  $f : X \rightarrow X'$  in  $\mathbf{C}$  there is an unique morphism  $f' : R(X) \rightarrow X'$  in  $\mathbf{C}'$  such that  $f' \circ \phi_R(X) = f$ .

**THEOREM 4.3.** *The category  $\mathbf{psBCI}_p$  is a reflective subcategory of the category  $\mathbf{psBCI}$ .*

**Proof.** Let  $X$  be an object in  $\mathbf{psBCI}$ . Then as we know  $X/K(X)$  is an object in  $\mathbf{psBCI}_p$ . Thus, we put  $R(X) = X/K(X)$ . We define  $\phi_R(X) : X \rightarrow R(X)$  as follows

$$(\phi_R(X))(x) = x/K(X), \text{ for all } x \in X,$$

that is,  $\phi_R(X)$  is the canonical surjection.

Now, take a morphism  $f : X \rightarrow Y$ , where  $Y$  is an object in  $\mathbf{psBCI}_p$ . First, note that  $f(x) = 1$ , for all  $x \in K(X)$ . Indeed,  $x \rightarrow 1 = 1$  gives  $1 = f(1) = f(x \rightarrow 1) = f(x) \rightarrow f(1) = f(x) \rightarrow 1$ , that is,  $f(x) \in K(Y) = \{1\}$  whence  $f(x) = 1$ . We define  $f' : R(X) \rightarrow Y$  as follows

$$f'(x/K(X)) = f(x), \text{ for all } x \in X.$$

First of all, we prove that  $f'$  is well defined. Let  $x_1/K(X) = x_2/K(X)$ . Then  $x_1 \rightarrow x_2 \in K(X)$  and  $x_2 \rightarrow x_1 \in K(X)$ , which gives  $f(x_1 \rightarrow x_2) = 1$  and  $f(x_2 \rightarrow x_1) = 1$ , that is,  $f(x_1) = f(x_2)$ . This proves that  $f'$  is well defined. Further, it is easy to show that  $f'$  is a morphism in  $\mathbf{psBCI}_p$  and  $f' \circ \phi_R(X) = f$ .

The uniqueness of  $f'$  follows from the fact that  $\phi_R(X)$  is an epimorphism. This completes the proof. ■

**REMARK.** The reflector  $R : \mathbf{psBCI} \rightarrow \mathbf{psBCI}_p$  is defined in the following way. If for  $X$  from  $\mathbf{psBCI}$  we put

$$R(X) = X/K(X),$$

then we obtain the definition of  $R$  on objects. Now, let  $f : X \rightarrow Y$  be a morphism in  $\mathbf{psBCI}$ . If we define  $R(f) : R(X) \rightarrow R(Y)$  by

$$(R(f))(x/K(X)) = f(x)/K(Y), \text{ for all } x \in X,$$

then we obtain the definition of  $R$  on morphisms. Obviously,  $R$  is a left adjoint for the inclusion functor  $I : \mathbf{psBCI}_p \hookrightarrow \mathbf{psBCI}$ . Moreover,  $R$  is faithful.

### 5. Conclusions

In the category  $\mathbf{psBCI}$ , monomorphisms and injective morphisms coincide, but epimorphisms and surjective morphisms not. These imply that

**psBCI** is not balanced. Since in **psBCI**, not every monomorphism is an equalizer and not every epimorphism is a coequalizer, they are not normal, that is, **psBCI** is not abelian. In the same time, since it has arbitrary limits, it is complete. It is an open problem if it is cocomplete.

The category **psBCI<sub>p</sub>** is a full and reflective subcategory of **psBCI** and it is isomorphic with the category **Grp**. This means that **psBCI<sub>p</sub>** is among other things balanced and cocomplete.

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