DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 4 2014

Jurancy Ereú, José Giménez, Nelson Merentes

ON BI-DIMENSIONAL SECOND μ-VARIATION

Abstract. In this paper, we present a generalization of the notion of bounded slope variation for functions defined on a rectangle $I_{\mathbf{a}}^{\mathbf{b}}$ in $\mathbb{R}^2$. Given a strictly increasing function μ, defined in a closed real interval, we introduce the class $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$, of functions of *bounded second μ-variation* on $I_{\mathbf{a}}^{\mathbf{b}}$, and show that this class can be equipped with a norm with respect to which it is a Banach space. We also deal with the important case of factorizable functions in $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ and finally we exhibit a relation between this class and the one of double Riemann–Stieltjes integrals of functions of bi-dimensional bounded variation.

1. Introduction

In 1881, Camile Jordan ([11]) introduced the concept of function of bounded variation after a rigorous study of the proof given by Dirichlet ([5]) on the convergence of the Fourier series of a monotone function. In fact, Jordan showed that a function is of bounded variation if and only if it is the difference of two monotone functions. From the spatial point of view, the notion of bounded variation was extended to functions defined on the plane in 1905 by Hardy and Vitali ([1], [8], [19]). In 1908, De La Vallée Poussin ([18]) introduced the notion of second variation of a function, showing that a function is of bounded second variation if and only if it is the difference of two convex functions. The subsequent denomination of this class as functions of bounded convexity, apparently, is due to A. W. Roberts and D. E. Varberg ([15] and [16]). A few years later, in 1911, F. Riesz ([14]) proved that a function F is of bounded second variation on an interval $[a, b]$ if and only if, it is the indefinite Lebesgue integral of a function f of bounded variation.

2010 *Mathematics Subject Classification*: 46B20, 26B30.
Key words and phrases: functions of bounded second variation, functions of bounded variation.
This research has been partly supported by the Central Bank of Venezuela. We want to give thanks to the library staff of B.C.V. for compiling the references.

DOI: 10.2478/dema-2014-0073

The unvarying interest generated by the classical notion of *function of bounded variation* has lead to some generalizations of the concept, mainly, intended to the search of a bigger class of functions whose elements have pointwise convergent Fourier series.

From the PDEs point of view, the first successful generalization of this concept to functions of several variables is due to L. Tonelli ([12]), who introduced the class of continuous functions of bounded variation in 1926. Later on, L. Cesari ([4]), modified the continuity requirement in Tonelli's definition by a less restrictive integrability requirement, obtaining the class of functions of bounded variation of several variables; actually, he applied the concept to solve a problem concerning the convergence of Fourier series of functions of two variables. After that, many authors have considered several generalizations of the concept of function of bounded variation to study Fourier series in several variables. As in the classical case, these generalizations have found many applications in the study of certain (partial) differential and integral equations (see e.g., [3]), in geometric measure theory, calculus of variations and mathematical physics.

In 2011, the authors ([6]) studied the class $BV^2(I_{\mathbf{a}}^{\mathbf{b}})$, of function of bounded second variation on a rectangle of $\mathbb{R}^2$, and proved that, equipped with a suitable norm, this class is a Banach space. We also showed that integrals of functions of first bounded variation (on $I_{\mathbf{a}}^{\mathbf{b}}$) are in $BV^2(I_{\mathbf{a}}^{\mathbf{b}})$. See also [2, 7, 20, 21] for related generalizations of the concept of function of bounded variation to the plane or of the notion of function of second variation.

In this work, we present a *spatial* generalization of the notion of bounded slope variation or μ-variation, as given by F. N. Huggins in [10], for the case of functions defined on a rectangle $I_{\mathbf{a}}^{\mathbf{b}}$ in $\mathbb{R}^2$ and introduce the space $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$, of all functions of bounded second μ-variation on $I_{\mathbf{a}}^{\mathbf{b}}$. We show that this space can be equipped with a norm and prove that the functions in the unit ball of this normed space are uniformly majorized by a *polynomial-like* continuous function. This fact allows us to show that $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ is a Banach space. We also deal with the important case of factorizable functions in $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ and finally, we show that double μ-Riemann–Stieltjes indefinite integrals of functions of bi-dimensional bounded variation are functions of bounded second μ-variation.

2. Preliminaries

In this section, we will expose the basic facts of the notions of function of bounded variation and function of bounded second variation in one variable.

Given an interval $[a, b] \subset \mathbb{R}$, we will use the notation $\Pi([a, b])$ to denote the set of all partitions of $[a, b]$, whereas $\Pi_3([a, b])$ will denote the subset of $\Pi([a, b])$ consisting of partitions of $[a, b]$ with at least three points.

Recall that a function $u : [a, b] \to \mathbb{R}$ is said to be of bounded variation (in the sense of Jordan) if

$$V(u; [a, b]) := \sup_{\xi \in \Pi([a,b])} \sum_{j=1}^{n} |u(t_j) - u(t_{j-1})| < \infty,$$

where the supremum is taken over the set of all partitions $\xi = \{a = t_o < t_1 < \cdots < t_n = b\} \in \Pi([a, b])$.

The notion of bounded second variation in the sense of De La Vallée Poussin is defined as follows:

A function $u : [a, b] \to \mathbb{R}$ is of bounded second variation if and only if

$$V^2(u; [a, b]) := \sup_{\pi \in \Pi_3([a,b])} \sum_{i=0}^{m-2} |u[t_{i+1}, t_{i+2}] - u[t_i, t_{i+1}]| < \infty,$$

where

$$u[t_{i+1}, t_{i+2}] := \frac{u(t_{i+2}) - u(t_{i+1})}{t_{i+2} - t_{i+1}}, \quad i = 0, \ldots, m-2. \tag{2.1}$$

The class of all the functions of bounded second variation (on $[a, b]$), in the sense of De La Vallée Poussin, is denoted by $BV^2([a, b])$.

The following are known properties of the functional $V^2(\cdot; [a, b])$ and of the functions in the class $BV^2([a, b])$ (see e.g., [13], [15] and [17]).

Proposition 2.1. *Let $u \in BV^2([a, b])$.*

(1) *If $v \in BV^2([a, b])$ and λ is any real constant, then*

$$V^2(\lambda u; [a, b]) = |\lambda| V^2(u; [a, b]),$$
$$V^2(u + v; [a, b]) \le V^2(u; [a, b]) + V^2(v; [a, b]).$$

(2) *(Monotonicity) If $a < c < d < b$, then $V^2(u; [c, d]) \le V^2(u; [a, b])$.*
(3) *(Semi-additivity) If $a < c < b$ then $u \in BV^2([a, c])$, $u \in BV^2([c, b])$ and $V^2(u; [a, b]) \ge V^2(u; [a, c]) + V^2(u; [c, b])$.*
(4) *$u[y_0, y_1]$ is bounded for all $y_0, y_1 \in [a, b]$.*
(5) *u is Lipschitz and therefore absolutely continuous on $[a, b]$.*
(6) *$u \in BV^2([a, b])$ if and only if $u = u_1 - u_2$, where u_1, u_2 are convex functions.*
(7) *A necessary and sufficient condition for a function F to be the integral of a function $f \in BV([a, b])$ is that $F \in BV^2([a, b])$.*
(8) *If u is twice differentiable with u'' integrable on $[a, b]$ then $u \in BV^2([a, b])$ and $V^2(u; [a, b]) = \int_a^b |u''(t)| dt$.*

Now, we recall Huggins' notion ([10]) of a function of bounded second μ-variation.

Definition 2.2. Let $\mu : [a,b] \to \mathbb{R}$ be a strictly increasing function. A function $u : [a,b] \to \mathbb{R}$ is said to be of bounded second μ-variation if and only if

$$V^{\mu,2}(u;[a,b]) := \sup_{\pi\in\Pi_3([a,b])} \sum_{i=0}^{m-2} |u_\mu[t_{i+1},t_{i+2}] - u_\mu[t_i,t_{i+1}]| < \infty,$$

where

$$u_\mu[t_{i+1},t_{i+2}] := \frac{u(t_{i+2}) - u(t_{i+1})}{\mu(t_{i+2}) - \mu(t_{i+1})}, \quad i = 0,...,m-2. \tag{2.2}$$

Example 2.3. If $u(x) = x^{\frac{2}{3}}$, then it is of bounded second μ-variation on $[a,b]$, for $\mu(x) := x^{1/3}$. In fact, it readily follows from the definitions that

$$V^{\mu,2}(u;[a,b]) \le 2(b^{1/3} - a^{1/3}) = 2(\mu(b) - \mu(a)) < +\infty.$$

The class of all the functions of bounded second μ-variation (on $[a,b]$) is denoted by $BV^{\mu,2}([a,b])$.

The following proposition shows an interesting relation between functions of bounded second μ-variation and indefinite Riemann–Stieltjes integrals of functions of (ordinary) bounded variation.

Proposition 2.4. *If $f \in BV([a,b])$ is continuous, μ is a strictly increasing function and $F(\tau) := \int_{a_1}^{\tau} f(t)\,d\mu(t)$ then $V^{\mu,2}(F,[a,b]) \le V(f,[a,b])$ and $F \in BV^{\mu,2}([a,b])$.*

Proof. Let $\xi := \{t_i\}_{i=0}^n \in \Pi_3([a,b])$, by the definition of F and the Mean Value Theorem for Riemann–Stieltjes integrals, we have

$$\begin{aligned}\frac{F(t_{i+2}) - F(t_{i+1})}{\mu(t_{i+2} - \mu(t_{i+1})} &= \frac{\int_{t_{i+1}}^{t_{i+2}} f(t)d\mu(t)}{\mu(t_{i+2}) - \mu(t_{i+1})}\\ &= \frac{f(c_{i+2})(\mu(t_{i+2}) - \mu(t_{i+1}))}{\mu(t_{i+2}) - \mu(t_{i+1})} = f(c_{i+2})\end{aligned}$$

with $c_{i+2} \in (t_{i+1}, t_{i+2})$.

On the other hand,

$$\begin{aligned}V^{\mu,2}(F;[a,b]) &= \sup_{\pi\in\Pi_3([a,b])} \sum_{i=0}^{n-2} |F_\mu[t_{i+1},t_{i+2}] - F_\mu[t_i,t_{i+1}]|\\ &= \sup_{\pi\in\Pi_3([a,b])} \sum_{j=0}^{n-2} \left| \frac{F(t_{i+2}) - F(t_{i+1})}{\mu(t_{i+2}) - \mu(t_{i+1})} - \frac{F(t_{i+1}) - F(t_i)}{\mu(t_{i+1}) - \mu(t_i)} \right|\\ &= \sup_{\pi\in\Pi_3([a,b])} \sum_{i=0}^{n-2} |f(c_{i+2}) - f(c_{i+1})| \le V(f;[a,b]) < +\infty,\end{aligned}$$

therefore $F \in BV^{\mu,2}([a,b])$. ■

3. Bi-dimensional second μ-variation

We begin by recalling the definition of (bounded) variation for functions defined on rectangles of $\mathbb{R}^2$ (c.f. [9]).

Let $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbb{R}^2$, such that $a_1 < b_1$ and $a_2 < b_2$. In the sequel, we will use the symbol $I_{\mathbf{a}}^{\mathbf{b}}$ to denote the basic rectangle $[a_1, b_1] \times [a_2, b_2]$.

For $\xi := \{t_i\}_{i=0}^n \in \Pi([a_1, b_1])$ and $\eta := \{s_j\}_{j=0}^m \in \Pi([a_2, b_2])$, we will use the following notation:

(i) $\Delta_{10}u(t_i, s) := u(t_i, s) - u(t_{i-1}, s)$ for $s \in [a_2, b_2]$ fixed.
(ii) $\Delta_{01}u(t, s_j) := u(t, s_j) - u(t, s_{j-1})$ for $t \in [a_1, b_1]$ fixed.
(iii) $\Delta_{11}u(t_i, s_j) := u(t_{i-1}, s_{j-1}) + u(t_i\, s_j) - u(t_{i-1}, s_j) - u(t_i, s_{j-1})$.

Definition 3.1. Let $u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$.

- If $s \in [a_2, b_2]$ is fixed, we define *the variation in the sense of Jordan* of u in $[a_1, b_1] \times \{s\}$ by

$$V_{[a_1,b_1]}(u(\cdot, s)) := \sup_{\xi} \sum_{i=1}^{n} |\Delta_{10}u(t_i, s)|,$$

  where the supremum is taken over the set of all partitions $\xi \in \Pi([a_1, b_1])$.
- Similarly for $t \in [a_1, b_1]$, we define *the variation in the sense of Jordan* of u in $\{t\} \times [a_2, b_2]$ by

$$V_{[a_2,b_2]}(u(t, \cdot)) := \sup_{\eta} \sum_{j=1}^{m} |\Delta_{01}u(t, s_j)|,$$

  where the supremum is taken over the set of all partitions $\eta \in \Pi([a_2, b_2])$.
- We define *the variation of u, in the sense of Hardy–Vitali* as

$$V(u, I_{\mathbf{a}}^{\mathbf{b}}) := \sup_{(\xi,\eta)} \sum_{i=1}^{n} \sum_{j=1}^{m} |\Delta_{11}u(t_i\, s_j)|,$$

  where the supremum is taken over the set of all partitions $(\xi, \eta) \in \Pi([a_1, b_1]) \times \Pi([a_2, b_2])$.
- *The total variation of u on $I_{\mathbf{a}}^{\mathbf{b}}$* is defined as

$$TV(u, I_{\mathbf{a}}^{\mathbf{b}}) := V_{[a_1,b_1]}(u(\cdot, s_0)) + V_{[a_2,b_2]}(u(t_0, \cdot)) + V(u, I_{\mathbf{a}}^{\mathbf{b}}),$$

  where (t_0, s_0) is any point in $I_{\mathbf{a}}^{\mathbf{b}}$.
- u is said to be of total bounded variation if $TV(u, I_{\mathbf{a}}^{\mathbf{b}}) < \infty$. The class of all functions $u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$ of total bounded variation is denoted as $BV(I_{\mathbf{a}}^{\mathbf{b}})$.

Now we present a spatial generalization of the notion of second μ-variation.

Assume $u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$. Let $\xi := \{t_i\}_{i=0}^n \in \Pi_3([a_1, b_1])$, $\eta := \{s_j\}_{j=0}^m \in \Pi_3([a_2, b_2])$ and μ is a real-valued strictly increasing function whose domain includes $[a_1, b_1]$ and $[a_2, b_2]$. We will use the following notations:

(i) For each $s \in [a_2, b_2]$ fixed, set

$$u_\mu[t_{i+1}, t_{i+2}; s] := \frac{u(t_{i+2}, s) - u(t_{i+1}, s)}{\mu(t_{i+2}) - \mu(t_{i+1})},$$
$$\Delta_{10} u_\mu[t_{i+1}, t_{i+2}; s] := u_\mu[t_{i+1}, t_{i+2}; s] - u_\mu[t_i, t_{i+1}; s] \quad \text{and}$$
$$V^{\mu,2}_{[a_1,b_1]}(u(\cdot, s)) := \sup_{\xi \in \Pi_3([a_1,b_1])} \sum_{i=0}^{n-2} |\Delta_{10} u_\mu[t_{i+1}, t_{i+2}; s]| .$$

(ii) Similarly, for each fixed $t \in [a_1, b_1]$

$$u_\mu[t; s_{j+1}, s_{j+2}] := \frac{u(t, s_{j+2}) - u(t, s_{j+1})}{\mu(s_{j+2}) - \mu(s_{j+1})},$$
$$\Delta_{01} u_\mu[t; s_{j+1}, s_{j+2}] := u_\mu[t; s_{j+1}, s_{j+2}] - u_\mu[t; s_j, s_{j+1}] \quad \text{and} \tag{3.1}$$
$$V^{\mu,2}_{[a_2,b_2]}(u(t, \cdot)) := \sup_{\eta \in \Pi_3([a_2,b_2])} \sum_{j=0}^{m-2} |\Delta_{01} u_\mu[t; s_{j+1}, s_{j+2}]| .$$

Definition 3.2. Let $u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$ and let μ be a real-valued strictly increasing function whose domain includes $[a_1, b_1]$ and $[a_2, b_2]$. *The second μ-variation of* u on $I_{\mathbf{a}}^{\mathbf{b}}$ is defined as

$$V^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) := \sup_{(\xi,\eta)} V^{\mu,2}_{I_{\mathbf{a}}^{\mathbf{b}}}(u, \xi \times \eta),$$

where $V^{\mu,2}_{I_{\mathbf{a}}^{\mathbf{b}}}(u, \xi \times \eta) := \sum_{i=0}^{n-2} \sum_{j=0}^{m-2} \left|\Delta^2_{11} u_\mu(t_i, s_j)\right|$ with

$$\Delta^2_{11} u_\mu(t_i, s_j) : = \frac{\Delta_{01} u_\mu[t_{i+2}; s_{j+1}, s_{j+2}] - \Delta_{01} u_\mu[t_{i+1}; s_{j+1}, s_{j+2}]}{\mu(t_{i+2}) - \mu(t_{i+1})}$$
$$- \left[\frac{\Delta_{01} u_\mu[t_{i+1}; s_{j+1}, s_{j+2}] - \Delta_{01} u_\mu[t_i; s_{j+1}, s_{j+2}]}{\mu(t_{i+1}) - \mu(t_i)} \right],$$

the supremum being taken over the set of all partitions $(\xi, \eta) \in \Pi_3([a_1, b_1]) \times \Pi_3([a_2, b_2])$.

The total bi-dimensional second μ-variation of u, is defined by

$$TV^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) := V^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) + V^{\mu,2}_{[a_1,b_1]}(u(\cdot, a_2)) + V^{\mu,2}_{[a_1,b_1]}(u(\cdot, b_2)) + V^{\mu,2}_{[a_2,b_2]}(u(a_1, \cdot)) + V^{\mu,2}_{[a_2,b_2]}(u(b_1, \cdot)) \tag{3.2}$$

and a function $u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$ is said to be of bounded (bi-dimensional) second μ-variation if

$$TV^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) < \infty.$$

The class of all functions $u \in \mathbb{R}^{I_{\mathbf{a}}^{\mathbf{b}}}$ of bounded second μ-variation is denoted by $V^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$; that is,

$$V^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}}) := \left\{u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}/TV^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) < \infty\right\}.$$

Remark 3.3. In order to define Δ_{11}^2, we only considered combinations of expressions of type Δ_{01} (see (3.1)). Actually, Δ_{11}^2 could have been also defined in the following way:

For each $j \in \{0, 2, \ldots, m-2\}$ fixed, we define the function $\Delta_{01}^2[u]_j^\mu : [a_1, b_1] \to \mathbb{R}$ as the (second order) difference quotient variation

$$\Delta_{01}^2[u]_j^\mu(t) := \Delta_{01}u_\mu[t; s_j, s_{j+1}]. \tag{3.3}$$

Then, it is readily seen that

$$\Delta_{11}^2 u_\mu(t_i, s_j) = \Delta_{01}^2[u]_{j+1}^\mu[t_{i+1}, t_{i+2}] - \Delta_{01}^2[u]_{j+1}^\mu[t_i, t_{i+1}] \tag{3.4}$$

(we recall that the notation $f[a, b]$ stands for the difference quotient $f_\mu[a, b] := (f(b) - f(a))/(\mu(b) - \mu(a))$ (see (2.1)).

If, instead of proceeding as set forth above, we were chosen to consider the subsequent difference quotients variations of expressions of type Δ_{10}, then an analogous reasoning would lead us to differences of type

$$\Delta_{10}^2[u]_{i+1}^\mu[s_{j+1}, s_{j+2}] - \Delta_{10}^2[u]_{i+1}^\mu[s_j, s_{j+1}] \tag{3.5}$$

but, as it is easy to verify after expanding and regrouping, the difference (3.5) is precisely $\Delta_{11}^2 u_\mu(t_i, s_j)$. Thus our definition of $V^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}})$ is independent of the choice of either Δ_{01} or Δ_{10}, to perform the subsequent difference quotients variations.

Example 3.4. The function $u(x, y) = (xy)^{2/3}$ has bounded second μ-variation on $I_{\mathbf{a}}^{\mathbf{b}} = [a_1, b_1] \times [a_2, b_2]$, for $\mu(t) := t^{1/3}$.

Proof. Indeed, suppose

$$\xi := \{t_i\}_{i=0}^n \in \Pi_3([a_1, b_1]) \quad \text{and} \quad \eta := \{s_j\}_{j=0}^m \in \Pi_3([a_2, b_2]).$$

By Example 2.3 we have

(i) for each $s \in [a_2, b_2]$ fixed:

$$\begin{aligned} V_{[a_1,b_1]}^{\mu,2}(u(\cdot, s)) &= \sup_\xi \sum_{i=0}^{n-2} |\Delta_{10}u_\mu[t_{i+1}, t_{i+2}; s]| \\ &= \sup_\xi \sum_{i=0}^{n-2} |s|^{\frac{2}{3}} \left|(t_{i+2})^{1/3} + (t_{i+1})^{1/3} - \left((t_{i+1})^{1/3} + (t_i)^{1/3}\right)\right| \\ &\leq |s|^{\frac{2}{3}} 2(\mu(b_1) - \mu(a_1)) < +\infty. \end{aligned}$$

(ii) Similarly, for each $t \in [a_1, b_1]$:

$$\begin{aligned} V^{\mu,2}_{[a_2,b_2]}(u(t,\cdot)) &= \sup_{\eta} \sum_{j=0}^{m-2} |\Delta_{01} u_\mu[t; s_{j+1}, s_{j+2}]| \\ &= \sup_{\xi} \sum_{i=0}^{m-2} |t|^{\frac{2}{3}} \left| (s_{j+2})^{1/3} + s_{j+1})^{1/3} - \Big((s_{j+1})^{1/3} + (s_j)^{1/3} \Big) \right| \\ &\leq |t|^{\frac{2}{3}} 2(\mu(b_2) - \mu(a_2)) < +\infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \Delta_{01} u_\mu[t_{i+2}; s_{j+1}, s_{j+2}] &= u_\mu[t_{i+2}; s_{j+1}, s_{j+2}] - u_\mu[t_{i+2}; s_j, s_{j+1}] \\ &= (t_{i+2})^{2/3} \left[(s_{j+2})^{1/3} + (s_{j+1})^{1/3} \right] \\ &\quad - (t_{i+2})^{2/3} \left[(s_{j+1})^{1/3} + (s_j)^{1/3} \right] \\ &= (t_{i+2})^{2/3} [(s_{j+2})^{1/3} - (s_j)^{1/3}]. \end{aligned}$$

Similarly

$$\begin{aligned} \Delta_{01} u_\mu[t_{i+1}; s_{j+1}, s_{j+2}] &= u_\mu[t_{i+1}; s_{j+1}, s_{j+2}] - u_\mu[t_{i+1}; s_j, s_{j+1}] \\ &= (t_{i+1})^{2/3} [(s_{j+2})^{1/3} - (s_j)^{1/3}] \end{aligned}$$

and

$$\begin{aligned} \Delta_{01} u_\mu[t_i; s_{j+1}, s_{j+2}] &= u_\mu[t_i; s_{j+1}, s_{j+2}] - u_\mu[t_i; s_j, s_{j+1}] \\ &= (t_i)^{2/3} [(s_{j+2})^{1/3} - (s_j)^{1/3}]. \end{aligned}$$

Hence

$$\begin{aligned} \Delta^2_{11} u_\mu(t_i, s_j) &= \frac{\Delta_{01} u_\mu[t_{i+2}; s_{j+1}, s_{j+2}] - \Delta_{01} u_\mu[t_{i+1}; s_{j+1}, s_{j+2}]}{\mu(t_{i+2}) - \mu(t_{i+1})} \\ &\quad - \left[\frac{\Delta_{01} u_\mu[t_{i+1}; s_{j+1}, s_{j+2}] - \Delta_{01} u_\mu[t_i; s_{j+1}, s_{j+2}]}{\mu(t_{i+1}) - \mu(t_i)} \right], \\ \Delta^2_{11} u_\mu(t_i, s_j) &= \frac{[(t_{i+2})^{2/3} - (t_{i+1})^{2/3}][(s_{j+2})^{1/3} - (s_j)^{1/3}]}{(t_{i+2})^{1/3} - (t_{i+1})^{1/3}} \\ &\quad - \frac{[(t_{i+1})^{2/3} - (t_i)^{2/3}][(s_{j+2})^{1/3} - (s_j)^{1/3}]}{(t_{i+1})^{1/3} - (t_i)^{1/3}} \\ &= [(t_{i+2})^{1/3} - (t_i)^{1/3}][(s_{j+2})^{1/3} - (s_j)^{1/3}], \end{aligned}$$

and therefore

$$\sum_{i=0}^{n-2}\sum_{j=0}^{m-2} \left|\Delta_{11}^2 u_\mu(t_i,s_j)\right| = \sum_{i=0}^{n-2}\sum_{j=0}^{m-2} [(t_{i+2})^{1/3}-(t_i)^{1/3}][(s_{j+2})^{1/3}-(s_j)^{1/3}]$$
$$= (b_1^{1/3}+(t_{n-1})^{1/3}-(t_1)^{1/3}-a_1^{1/3})(b_2^{1/3}+(s_{m-1})^{1/3}-(s_1)^{1/3}-a_2^{1/3})$$
$$< 2(b_1^{1/3}-a_1^{1/3})2(b_2^{1/3}-a_2^{1/3}) = 4(\mu(b_1)-\mu(a_1))(\mu(b_2)-\mu(a_2)),$$

$$V^{\mu,2}(u,I_{\mathbf{a}}^{\mathbf{b}}) = \sup_{(\xi,\eta)} \sum_{i=0}^{n-2}\sum_{j=0}^{m-2} \left|\Delta_{11}^2 u_\mu(t_i,s_j)\right| \le 4(\mu(b_1)-\mu(a_1))(\mu(b_2)-\mu(a_2)) < +\infty. \tag{3.6}$$

From (i), (ii) and (3.6) it follows that

$$TV^{\mu,2}(u,I_{\mathbf{a}}^{\mathbf{b}}) = V^{\mu,2}(u,I_{\mathbf{a}}^{\mathbf{b}}) + V_{[a_1,b_1]}^{\mu,2}(u(\cdot,a_2)) + V_{[a_1,b_1]}^{\mu,2}(u(\cdot,b_2)) + V_{[a_2,b_2]}^{\mu,2}(u(a_1,\cdot)) + V_{[a_2,b_2]}^{\mu,2}(u(b_1,\cdot)) < +\infty. \blacksquare$$

The proof of the following lemma follows in a straightforward manner from the definition.

Lemma 3.5. *If u and v belong to $V^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ and λ is any real constant then*

$$TV^{\mu,2}(\lambda u, I_{\mathbf{a}}^{\mathbf{b}}) = |\lambda| TV^{\mu,2}(u,I_{\mathbf{a}}^{\mathbf{b}})$$

and

$$TV^{\mu,2}(u+v, I_{\mathbf{a}}^{\mathbf{b}}) \le TV^{\mu,2}(u,I_{\mathbf{a}}^{\mathbf{b}}) + TV^{\mu,2}(v,I_{\mathbf{a}}^{\mathbf{b}}).$$

Lemma 3.6. *Let $u \in V^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$.*
(Monotonicity) If $c=(c_1,c_2)$ and $d=(d_1,d_2)$ with $a_1<c_1<d_1<b_1$ and $a_2<c_2<d_2<b_2$ then

$$TV^{\mu,2}(u,I_{\mathbf{c}}^{\mathbf{d}}) \le TV^{\mu,2}(u,I_{\mathbf{a}}^{\mathbf{b}}).$$

(Semi-additivity) If $a_1<x<b_1$ then

$$V^{\mu,2}\left(u, I_{(\mathbf{a_1},\mathbf{a_2})}^{(\mathbf{x},\mathbf{b_2})}\right) + V^{\mu,2}\left(u, I_{(\mathbf{x},\mathbf{a_2})}^{(\mathbf{b_1},\mathbf{b_2})}\right) \le V^{\mu,2}(u,I_{\mathbf{a}}^{\mathbf{b}}).$$

Proof. For the sake of brevity, we will omit the proof of part (a) which is similar to the (readily seen) correspondent one for functions in $BV(I)$ ($I\subset\mathbb{R}$), taking into account, of course, the specific bi-dimensional setting.

To prove (b), let $\xi := \{t_i\}_{i=0}^n \in \Pi_3([a_1,x])$, $\widetilde{\xi} := \{\widetilde{t}_i\}_{i=0}^n \in \Pi_3([x,b_1])$ and $\eta := \{s_j\}_{j=0}^r \in \Pi_3([a_2,b_2])$. Then

$$\sum_{i=0}^{n-2}\sum_{j=0}^{r-2}\left|\Delta_{11}^2 u_\mu(t_i,s_j)\right| + \sum_{i=0}^{n-2}\sum_{j=0}^{r-2}\left|\Delta_{11}^2 u_\mu(\widetilde{t}_i,s_j)\right|$$
$$\leq \sum_{i=0}^{n-2}\sum_{j=0}^{r-2}\left|\Delta_{11}^2 u_\mu(t_i,s_j)\right| + \sum_{j=0}^{r-2}\left|\frac{\Delta_{01}u_\mu[\widetilde{t}_1;s_{j+1},s_{j+2}] - \Delta_{01}u_\mu[\widetilde{t}_0;s_{j+1},s_{j+2}]}{\mu(\widetilde{t}_1)-\mu(\widetilde{t}_0)}\right.$$
$$\left. - \left[\frac{\Delta_{01}u_\mu[t_n;s_{j+1},s_{j+2}] - \Delta_{01}u_\mu[t_{n-1};s_{j+1},s_{j+2}]}{\mu(t_n)-\mu(t_{n-1})}\right]\right|$$
$$+ \sum_{i=0}^{n-2}\sum_{j=0}^{r-2}\left|\Delta_{11}^2 u_\mu(\widetilde{t}_i,s_j)\right| \leq V^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}).$$

Therefore

$$V^{\mu,2}\left(u, I_{(\mathbf{a_1},\mathbf{a_2})}^{(\mathbf{x},\mathbf{b_2})}\right) + V^{\mu,2}\left(u, I_{(\mathbf{x},\mathbf{a_2})}^{(\mathbf{b_1},\mathbf{b_2})}\right) \leq V^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}). \quad \blacksquare$$

Definition 3.7. By Lemma 3.5, $V^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ is a linear space. In the sequel, it will be denoted as $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$.

Definition 3.8. Let $\mu : [a,b] \to \mathbb{R}$ be a strictly increasing function. We say that a function $u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$ is a μ-affine function if there are real constants A, B and C such that

$$u(x,y) = A\mu(x) + B\mu(y) + C.$$

Remark 3.9. Let $\mu : [a,b] \to \mathbb{R}$ be a strictly increasing function. It is readily seen (by just solving, for A, B and C, the system of equations determined by the conditions $u(x_i,y_j) = A\mu(x_i) + B\mu(y_j) + C$) that if $I_{\mathbf{a}}^{\mathbf{b}} \subset \mathbb{R}^2$ is a rectangle and $u : \mathbb{R}^2 \to \mathbb{R}$ is any μ-affine function then the coefficients of u depend linearly on the values of u over the vertices of $I_{\mathbf{a}}^{\mathbf{b}}$ (with coefficients depending on $(\mu(b_i)-\mu(a_i))^{-1}, i = 1,2$) and

$$\Delta_{11}u[\mathbf{a},\mathbf{b}] := u(b_1,b_2) - u(a_1,b_2) - u(b_1,a_2) + u(a_1,a_2) = 0. \tag{3.7}$$

Theorem 3.10. *Let μ be a real-valued strictly increasing function whose domain includes $[a_1,b_1]$ and $[a_2,b_2]$. A function $u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$ satisfies the conditions $\Delta_{11}u[\boldsymbol{a},\boldsymbol{b}] = 0$ and $TV^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) = 0$ then there are constants A, B, C such that $u(x,y) = A\mu(x) + B\mu(y) + C$.*

Proof. If $TV^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) = 0$ then, for all $(x,y) \in (a_1,b_1)\times(a_2,b_2)$, the following implications hold:

$$V_{[a_2,b_2]}^{\mu,2}(u(a_1,\cdot)) = 0 \;\Rightarrow\; \frac{u(a_1,b_2)-u(a_1,y)}{\mu(b_2)-\mu(y)} = \frac{u(a_1,y)-u(a_1,a_2)}{\mu(y)-\mu(a_2)}; \tag{3.8}$$

$$(3.9)\qquad V^{\mu,2}_{[a_2,b_2]}(u(b_1.\cdot)) = 0 \Rightarrow \frac{u(b_1,b_2)-u(b_1,y)}{\mu(b_2)-\mu(y)} = \frac{u(b_1,y)-u(b_1,a_2)}{\mu(y)-\mu(a_2)};$$

$$(3.10)\qquad V^{\mu,2}_{[a_1,b_1]}(u(\cdot,a_2)) = 0 \Rightarrow \frac{u(b_1,a_2)-u(x,a_2)}{\mu(b_1)-\mu(x)} = \frac{u(x,a_2)-u(a_1,a_2)}{\mu(x)-\mu(a_1)};$$

$$(3.11)\qquad V^{\mu,2}_{[a_1,b_1]}(u(\cdot,b_2)) = 0 \Rightarrow \frac{u(b_1,b_2)-u(x,b_2)}{\mu(b_1)-\mu(x)} = \frac{u(x,b_2)-u(a_1,b_2)}{\mu(x)-\mu(a_1)};$$

while $V^{\mu,2}(u, I^{\mathbf{b}}_{\mathbf{a}}) = 0$ implies

$$\begin{aligned} &\frac{1}{\mu(b_1)-\mu(x)}\Bigg[\frac{u(b_1,b_2)-u(b_1,y)}{\mu(b_2)-\mu(y)} - \frac{u(b_1,y)-u(b_1,a_2)}{\mu(y)-\mu(a_2)} \\ &\qquad - \left(\frac{u(x,b_2)-u(x,y)}{\mu(b_2)-\mu(y)} - \frac{u(x,y)-u(x,a_2)}{\mu(y)-\mu(a_2)}\right)\Bigg] \\ &= \frac{1}{\mu(x)-\mu(a_1)}\Bigg[\frac{u(x,b_2)-u(x,y)}{\mu(b_2)-\mu(y)} - \frac{u(x,y)-u(x,a_2)}{\mu(y)-\mu(a_2)} \\ &\qquad - \left(\frac{u(a_1,b_2)-u(a_1,y)}{\mu(b_2)-\mu(y)} - \frac{u(a_1,y)-u(a_1,a_2)}{\mu(y)-\mu(a_2)}\right)\Bigg]. \end{aligned}$$

The first and fourth summands of this last equality vanish, by virtue of (3.8) and (3.9). Hence, we have

$$\begin{aligned} &-\frac{1}{\mu(b_1)-\mu(x)}\left[\frac{u(x,b_2)-u(x,y)}{\mu(b_2)-\mu(y)} - \frac{u(x,y)-u(x,a_2)}{\mu(y)-\mu(a_2)}\right] \\ &\qquad = \frac{1}{\mu(x)-\mu(a_1)}\left[\frac{u(x,b_2)-u(x,y)}{\mu(b_2)-\mu(y)} - \frac{u(x,y)-u(x,a_2)}{\mu(y)-\mu(a_2)}\right] \end{aligned}$$

and solving this equation for $u(x,y)$, we get

$$(3.12)\qquad u(x,y) = \left(\frac{\mu(y)-\mu(a_2)}{\mu(b_2)-\mu(a_2)}\right)u(x,b_2) + \left(\frac{\mu(b_2)-\mu(y)}{\mu(b_2)-\mu(a_2)}\right)u(x,a_2).$$

Moreover, making use of (3.10) and (3.11), we obtain the relations

$$u(x,a_2) = \left(\frac{\mu(x)-\mu(a_1)}{\mu(b_1)-\mu(a_1)}\right)u(b_1,a_2) + \left(\frac{\mu(b_1)-\mu(x)}{\mu(b_1)-\mu(a_1)}\right)u(a_1,a_2)$$

and

$$u(x,b_2) = \left(\frac{\mu(x)-\mu(a_1)}{\mu(b_1)-\mu(a_1)}\right)u(b_1,b_2) + \left(\frac{\mu(b_1)-\mu(x)}{\mu(b_1)-\mu(a_1)}\right)u(a_1,b_2),$$

which, combined with (3.12) yields

$$u(x,y) = \left(\frac{\mu(y)-\mu(a_2)}{\mu(b_2)-\mu(a_2)}\right)\left(\frac{\mu(x)-\mu(a_1)}{\mu(b_1)-\mu(a_1)}\right)u(b_1,b_2) + \left(\frac{\mu(y)-\mu(a_2)}{\mu(b_2)-\mu(a_2)}\right)\left(\frac{\mu(b_1)-\mu(x)}{\mu(b_1)-\mu(a_1)}\right)u(a_1,b_2) + \left(\frac{\mu(b_2)-\mu(y)}{\mu(b_2)-\mu(a_2)}\right)\left(\frac{\mu(x)-\mu(a_1)}{\mu(b_1)-\mu(a_1)}\right)u(b_1,a_2) + \left(\frac{\mu(b_2)-\mu(y)}{\mu(b_2)-\mu(a_2)}\right)\left(\frac{\mu(b_1)-\mu(x)}{\mu(b_1)-\mu(a_1)}\right)u(a_1,a_2). \tag{3.13}$$

Although (3.13) was established assuming that $(x,y) \in (a_1,b_1)\times(a_2,b_2)$, a straightforward computation shows that if we replace in (3.13), (x,y) by any point in the boundary of $I_{\mathbf{a}}^{\mathbf{b}}$, then we get an identity. Thus (3.13) actually holds for all $(x,y) \in I_{\mathbf{a}}^{\mathbf{b}}$. Finally, notice that after expanding and regrouping terms in the right hand side of (3.13), the coefficient of the product $\mu(x)\mu(y)$ is

$$\frac{\Delta_{11}u[\mathbf{a},\mathbf{b}]}{(\mu(b_2)-\mu(a_2))(\mu(b_1)-\mu(a_1))}$$

which is zero, by hypothesis. Thus u must be a μ-affine function. ∎

Based on Theorem 3.10, and in the discussion previous to it, the following definition is now natural.

Definition 3.11. For any $u \in BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ define

$$\|u\| := \Sigma\,|u|[\mathbf{a},\mathbf{b}] + TV^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}), \tag{3.14}$$

where $\Sigma\,|u|[\mathbf{a},\mathbf{b}] := |u(a_1,a_2)| + |u(b_1,b_2)| + |u(a_1,b_2)| + |u(b_1,a_2)|$.

Corollary 3.12. *$\|\cdot\|$ is a norm on $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$.*

Proof. Let $u \in BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$. By definition $\|u\| \geq 0$ and clearly $u = 0$ implies $\|u\| = 0$. On the other hand, if $\|u\| = 0$, then $TV^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) = 0$ and $|\Delta_{11}u[\mathbf{a},\mathbf{b}]| \leq \Sigma\,|u|[\mathbf{a},\mathbf{b}] = 0$. It follows, by (3.13), that $u \equiv 0$.

On the other hand, the properties:

(P_2) $\forall\, \alpha \in \mathbb{R} : \|\alpha u\| = |\alpha|\|u\|$ and
(P_3) $\|u+v\| \leq \|u\| + \|v\|, \quad (u, v \in BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}}))$

follow readily from the definition and the properties of the functionals of modulus ($|\ |$) and supremum (sup). ∎

In the following proposition, we present a result which will be fundamental to present $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ as a Banach space. The result asserts that the functions in the *unit ball* of $BV^{\mu,2}$ are (uniformly) majorized by a single function h (a *second degree polynomial-like* function on $\mu(x)$ and $\mu(y)$, whose coefficients depend only on the values of μ at the end points of $I_{\mathbf{a}}^{\mathbf{b}}$). Despite the fact that this result allows us to prove that $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ is a Banach space, the result might be of some interest in itself.

Proposition 3.13. *Let μ be a real-valued strictly increasing continuous function whose domain includes $[a_1, b_1]$ and $[a_2, b_2]$. There is a continuous function $h : \mathbb{R}^2 \to \mathbb{R}$ of the form $\sum_{i=0}^{2} \sum_{j=0}^{2} a_{ij}\, \mu(x)^i \mu(y)^j$, with $a_{ij} \in \mathbb{R}$, such that for all $u \in BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$*

$$|u(x,y)| \le h(x,y)\, \|u\| \quad \textit{for all } (x,y) \in I_{\mathbf{a}}^{\mathbf{b}}.$$

In particular, $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ is a subspace of $\mathcal{B}(I_{\mathbf{a}}^{\mathbf{b}})$, the Banach space of all bounded functions on $I_{\mathbf{a}}^{\mathbf{b}}$ with the sup *norm.*

Proof. Let u be in $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$. Put $\delta_1 := \mu(b_1) - \mu(a_1)$ and $\delta_2 := \mu(b_2) - \mu(a_2)$. Then, by (3.2) and Definition (3.14) for all $(x,y) \in (a_1, b_1) \times (a_2, b_2)$, we have the following inequalities

$$\left| \frac{u(a_1,b_2) - u(a_1,y)}{\mu(b_2) - \mu(y)} - \frac{u(a_1,y) - u(a_1,a_2)}{\mu(y) - \mu(a_2)} \right| \le \|u\|; \tag{3.15}$$

$$\left| \frac{u(b_1,b_2) - u(b_1,y)}{\mu(b_2) - \mu(y)} - \frac{u(b_1,y) - u(b_1,a_2)}{\mu(y) - \mu(a_2)} \right| \le \|u\|; \tag{3.16}$$

$$\left| \frac{u(b_1,a_2) - u(x,a_2)}{\mu(b_1) - \mu(x)} - \frac{u(x,a_2) - u(a_1,a_2)}{\mu(x) - \mu(a_1)} \right| \le \|u\|; \tag{3.17}$$

$$\left| \frac{u(b_1,b_2) - u(x,b_2)}{\mu(b_1) - \mu(x)} - \frac{u(x,b_2) - u(a_1,b_2)}{\mu(x) - \mu(a_1)} \right| \le \|u\|; \tag{3.18}$$

whereas $V^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) \le \|u\|$ implies

$$\begin{aligned} \Bigg| \frac{1}{\mu(b_1) - \mu(x)} \Bigg[& \frac{u(b_1,b_2) - u(b_1,y)}{\mu(b_2) - \mu(y)} - \frac{u(b_1,y) - u(b_1,a_2)}{\mu(y) - \mu(a_2)} \\ & - \left(\frac{u(x,b_2) - u(x,y)}{\mu(b_2) - \mu(y)} - \frac{u(x,y) - u(x,a_2)}{\mu(y) - \mu(a_2)} \right) \Bigg] \\ & - \frac{1}{\mu(x) - \mu(a_1)} \Bigg[\frac{u(x,b_2) - u(x,y)}{\mu(b_2) - \mu(y)} - \frac{u(x,y) - u(x,a_2)}{\mu(y) - \mu(a_2)} \\ & - \left(\frac{u(a_1,b_2) - u(a_1,y)}{\mu(b_2) - \mu(y)} - \frac{u(a_1,y) - u(a_1,a_2)}{\mu(y) - \mu(a_2)} \right) \Bigg] \Bigg| \le \|u\|. \end{aligned} \tag{3.19}$$

Now, (3.17), in turn, implies

$$\begin{aligned} & \left| \frac{\delta_1}{(\mu(b_1) - \mu(x))(\mu(x) - \mu(a_1))} u(x,a_2) \right| \\ & \qquad \le \|u\| + \left| \frac{u(b_1,a_2)}{\mu(b_1) - \mu(x)} \right| + \left| \frac{u(a_1,a_2)}{\mu(x) - \mu(a_1)} \right| \end{aligned} \tag{3.20}$$

or

$$|u(x,a_2)| \leq \frac{(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1)\|u\|}{\delta_1} + \left|\frac{(\mu(x)-\mu(a_1))u(b_1,a_2)}{\delta_1}\right| + \left|\frac{(\mu(b_1)-\mu(x))u(a_1,a_2)}{\delta_1}\right|.$$

Similarly, (3.18) implies

$$|u(x,b_2)| \leq \frac{(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))\|u\|}{\delta_1} + \left|\frac{(\mu(x)-\mu(a_1))u(b_1,b_2)}{\delta_1}\right| + \left|\frac{(\mu(b_1)-\mu(x))u(a_1,b_2)}{\delta_1}\right|. \tag{3.21}$$

On the other hand, from inequality (3.20)

$$\begin{aligned}
&\left|\frac{\delta_1\delta_2 u(x,y)}{(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))(\mu(b_2)-\mu(y))(\mu(y)-\mu(a_2))}\right| \\
&\quad \leq \|u\| + \left|\frac{u(a_1,b_2)-u(a_1,y)}{(\mu(x)-\mu(a_1))(\mu(b_2)-\mu(y))} - \frac{u(a_1,y)-u(a_1,a_2)}{(\mu(x)-\mu(a_1))(\mu(y)-\mu(a_2))}\right| \\
&\quad + \left|\frac{u(b_1,b_2)-u(b_1,y)}{(\mu(b_1)-\mu(x))(\mu(b_2)-\mu(y))} - \frac{u(b_1,y)-u(b_1,a_2)}{(\mu(b_1)-\mu(x))(\mu(y)-\mu(a_2))}\right| \\
&\quad + \left|\frac{\delta_1 u(x,b_2)}{(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))(\mu(b_2)-\mu(y))}\right| \\
&\quad + \left|\frac{\delta_1 u(x,a_2)}{(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))(\mu(y)-\mu(a_2))}\right|,
\end{aligned}$$

and from this, by (3.15) and (3.16), we get

$$\begin{aligned}
&\left|\frac{\delta_1\delta_2 u(x,y)}{(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))(\mu(b_2)-\mu(y))(\mu(y)-\mu(a_2))}\right| \\
&\qquad \leq \|u\| + \frac{\|u\|}{\mu(x)-\mu(a_1)} + \frac{\|u\|}{\mu(b_1)-\mu(x)} \\
&\qquad\quad + \left|\frac{\delta_1 u(x,b_2)}{(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))(\mu(b_2)-\mu(y))}\right| \\
&\qquad\quad + \left|\frac{\delta_1 u(x,a_2)}{(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))(\mu(y)-\mu(a_2))}\right|
\end{aligned}$$

or equivalently

$$
\begin{aligned}
|u(x,y)| \le{}& \frac{(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))(\mu(b_2)-\mu(y))(\mu(y)-\mu(a_2))}{\delta_1\delta_2}\|u\| \\
&+ \frac{(\mu(b_1)-\mu(x))(\mu(b_2)-\mu(y))(\mu(y)-\mu(a_2))}{\delta_1\delta_2}\|u\| \\
&+ \frac{(\mu(x)-\mu(a_1))(\mu(b_2)-\mu(y))(\mu(y)-\mu(a_2))}{\delta_1\delta_2}\|u\| \\
&+ \left|\frac{(\mu(y)-\mu(a_2))u(x,b_2)}{\delta_2}\right| + \left|\frac{(\mu(b_2)-\mu(y))u(x,a_2)}{\delta_2}\right|.
\end{aligned}
$$

Taking into account (3.20), (3.21) and the fact that $\Sigma|u|[\mathbf{a},\mathbf{b}] \le \|u\|$, we obtain

$$
\begin{aligned}
(3.22)\qquad & |u(x,y)| \\
\le{}& \Big[\frac{(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))(\mu(b_2)-\mu(y))(\mu(y)-\mu(a_2))}{\delta_1\delta_2} \\
&+ \frac{(\mu(b_1)-\mu(x))((b_2)-\mu(y))(\mu(y)-\mu(a_2))}{\delta_1\delta_2} \\
&+ \frac{(\mu(x)-\mu(a_1))(\mu(b_2)-\mu(y))(\mu(y)-\mu(a_2))}{\delta_1\delta_2} \\
&+ \frac{(\mu(y)-\mu(a_2))(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))}{\delta_2\delta_1} \\
&+ \frac{(\mu(y)-\mu(a_2))(\mu(x)-\mu(a_1))}{\delta_2\delta_1} \\
&+ \frac{(\mu(y)-\mu(a_2))(\mu(b_1)-\mu(x))}{\delta_1\delta_2} + \frac{(\mu(b_2)-\mu(y))(\mu(x)-\mu(a_1))}{\delta_2\delta_1} \\
&+ \frac{(\mu(b_2)-\mu(y))(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))}{\delta_2\delta_1} \\
&+ \frac{(\mu(b_2)-\mu(y))(\mu(b_1)-\mu(x))}{\delta_2\delta_1}\Big]\|u\|.
\end{aligned}
$$

Finally, by regrouping the right hand side of this inequality, we may define

$$
\begin{aligned}
&h(x,y) \\
&:= 1 + \delta_1^{-1}\delta_2^{-1}(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1))(\mu(b_2)-\mu(y))(\mu(y)-\mu(a_2)) \\
&\quad + \delta_1^{-1}(\mu(b_1)-\mu(x))(\mu(x)-\mu(a_1)) + \delta_2^{-1}(\mu(b_2)-\mu(y))(\mu(y)-\mu(a_2)).
\end{aligned}
$$

Now, any point in the boundary of $I_{\mathbf{a}}^{\mathbf{b}}$ which is not a vertex must satisfy, respectively, one of the inequalities (3.15), (3.16), (3.17) or (3.18), and so (3.22) holds also for those points. On the other hand, if (x_0, y_0) is a vertex then $h(x_0, y_0) = 1$ and since $|u(x_0, y_0)| \le \Sigma|u|[\mathbf{a}, \mathbf{b}] \le \|u\|$, the condition (3.22) actually holds for every $(x, y) \in [a_1, b_1] \times [a_2, b_2]$. This finishes the proof. ∎

Corollary 3.14. *If m is a real-valued strictly increasing bounded function whose domain includes $[a_1, b_1]$ and $[a_2, b_2]$ then $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ is a Banach space.*

Proof. Suppose that $\{u_r\}_{r\ge1}$ is a Cauchy sequence in $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ and let h be the function given by Proposition 3.13. Then, for all $(x, y) \in I_{\mathbf{a}}^{\mathbf{b}}$ and all $r, s \in \mathbb{N}$, we have

$$|(u_r - u_s)(x, y)| \le \sup_{(x,y)\in I_{\mathbf{a}}^{\mathbf{b}}} h(x, y)\, \|u_r - u_s\|.$$

Thus, $\{u_r\}_{r\ge1}$ is a Cauchy sequence in $\mathcal{B}(I_{\mathbf{a}}^{\mathbf{b}})$ and therefore there is $u \in \mathcal{B}(I_{\mathbf{a}}^{\mathbf{b}})$ such that $\|u_r - u\|_\infty \to 0$. Fix $\epsilon > 0$. Since $\{u_r\}_{r\ge1}$ is a Cauchy sequence in $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$, there is $\rho \in \mathbb{N}$ such that for all $r, s > \rho$ and all $\xi_0 = \{t_i^0\}_1^{n_0} \in \Pi_3[a_1, b_1]$, $\eta_0 = \{s_j^0\}_1^{m_0} \in \Pi_3[a_2, b_2]$:

$$\begin{aligned}\epsilon &> TV^{\mu,2}(u_r - u_s, I_{\mathbf{a}}^{\mathbf{b}}) \ge V^{\mu,2}(u_r - u_s, I_{\mathbf{a}}^{\mathbf{b}})\\ &\ge \sup\Bigg\{ \sum_{i=0}^{n-2}\sum_{j=0}^{k-2} \left|\Delta_{11}^2 (u_r - u_s)_\mu (t_i, s_j)\right| :\\ &\qquad\qquad \xi = \{t_i\}_1^n \in \Pi_3[a_1, b_1],\ \eta = \{s_j\}_1^k \in \Pi_3[a_2, b_2]\Bigg\}\\ &\ge \sum_{i=0}^{n_0-2}\sum_{j=0}^{m_0-2} \left|\Delta_{11}^2 (u_r - u_s)_\mu (t_i^0, s_j^0)\right|.\end{aligned}$$

It follows that, for all $r > \rho$ and all $\xi_0 = \{t_i^0\}_1^{n_0} \in \Pi_3[a_1, b_1]$, $\eta_0 = \{s_j^0\}_1^{m_0} \in \Pi_3[a_2, b_2]$:

$$\begin{aligned}\epsilon &\ge \lim_{s\to\infty} \sum_{i=0}^{n_0-2}\sum_{j=0}^{m_0-2} \left|\Delta_{11}^2 (u_r - u_s)_\mu (t_i^0, s_j^0)\right|\\ &= \sum_{i=0}^{n_0-2}\sum_{j=0}^{m_0-2} \left|\Delta_{11}^2 (u_r - u)_\mu (t_i^0, s_j^0)\right|.\end{aligned} \tag{3.23}$$

Consequently, for all $r > \rho$

$$V^{\mu,2}(u_r - u, I_{\mathbf{a}}^{\mathbf{b}}) = \sup_{(\xi,\eta)} \sum_{i=0}^{n-2}\sum_{j=0}^{k-2} \left|\Delta_{11}^2(u_r - u)_\mu(t_i, s_j)\right| \le \epsilon.$$

A similar procedure, applied to the rest of the summands of $TV^{\mu,2}(u_r - u_s)$, implies that for all $r > \rho$

$$TV^{\mu,2}(u_r - u, I_{\mathbf{a}}^{\mathbf{b}}) \le 5\epsilon, \tag{3.24}$$

which, in turn, implies $u \in BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ and (since $u_r \to u$ pointwise in $I_{\mathbf{a}}^{\mathbf{b}}$)

$$\lim_{r\to\infty} \|u_r - u\| = 0.$$

We conclude that $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$ is a Banach space. ■

4. Factorizable functions in $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$

In this section, we will study a distinguished subfamily of $BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$.

Definition 4.1. A function $u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$ is said to be factorizable if it can be expressed as the product of two non-zero functions $g : [a_1, b_1] \to \mathbb{R}$ and $h : [a_2, b_2] \to \mathbb{R}$; i.e.,

$$u(t,s) = g(t)h(s); \quad g \neq 0 \;\text{ y }\; h \neq 0.$$

Remark 4.2. Based on the definitions, it is readily seen that a factorizable function $u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$ is of bounded variation if and only if each factor is of (*one-dimensional*) bounded variation.

Lemma 4.3. *If $u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$ with $u(t,s) = g(t)h(s)$ and μ is a real-valued strictly increasing function whose domain includes $[a_1, b_1]$ and $[a_2, b_2]$ then for any partitions $\{t_i\}_{i=0}^n \in \Pi_3([a_1, b_1])$ and $\{s_j\}_{j=0}^m \in \Pi_3([a_2, b_2])$, we have*

$$\Delta_{11}^2 u_\mu(t_i, s_j) = \left(g_\mu[t_{i+1}, t_{i+2}] - g_\mu[t_i, t_{i+1}]\right)\left(h_\mu[s_{j+1}, s_{j+2}] - h_\mu[s_j, s_{j+1}]\right).$$

Proof. Indeed:

$$\begin{aligned}
&\Delta_{11}^2 u_\mu(t_i, s_j) = \Delta_{11}(gh)_\mu(t_i, s_j)\\
&= \frac{\Delta_{01}(gh)_\mu[t_{i+2}; s_{j+1}, s_{j+2}] - \Delta_{01}(gh)_\mu[t_{i+1}; s_{j+1}, s_{j+2}]}{\mu(t_{i+2}) - \mu(t_{i+1})}\\
&\qquad - \left[\frac{\Delta_{01}(gh)_\mu[t_{i+1}; s_{j+1}, s_{j+2}] - \Delta_{01}(gh)_\mu[t_i; s_{j+1}, s_{j+2}]}{\mu(t_{i+1}) - \mu(t_i)}\right]\\
&= \left[\frac{g(t_{i+2})h(s_{j+2}) - g(t_{i+2})h(s_{j+1})}{(\mu(t_{i+2}) - \mu(t_{i+1}))(\mu(s_{j+2}) - \mu(s_{j+1}))}\right.\\
&\qquad\qquad \left. - \frac{g(t_{i+2})h(s_{j+1}) - g(t_{i+2})h(s_j)}{(\mu(t_{i+2}) - \mu(t_{i+1}))(\mu(s_{j+1}) - \mu(s_j))}\right]\\
&\qquad - \left[\frac{g(t_{i+1})h(s_{j+2}) - g(t_{i+1})h(s_{j+1})}{(\mu(t_{i+2}) - \mu(t_{i+1}))(\mu(s_{j+2}) - \mu(s_{j+1}))}\right.\\
&\qquad\qquad \left. - \frac{g(t_{i+1})h(s_{j+1}) - g(t_{i+1})h(s_j)}{(\mu(t_{i+2}) - \mu(t_{i+1}))(\mu(s_{j+1}) - \mu(s_j))}\right]
\end{aligned}$$

$$-\Bigg[\frac{g(t_{i+1})h(s_{j+2})-g(t_{i+1})h(s_{j+1})}{(\mu(t_{i+1})-\mu(t_i))(\mu(s_{j+2})-\mu(s_{j+1}))} - \frac{g(t_{i+1})h(s_{j+1})-g(t_{i+1})h(s_j)}{(\mu(t_{i+1})-\mu(t_i))(\mu(s_{j+1})-\mu(s_j))}\Bigg]$$
$$+\Bigg[\frac{g(t_i)h(s_{j+2})-g(t_i)h(s_{j+1})}{(\mu(t_{i+1})-\mu(t_i))(\mu(s_{j+2})-\mu(s_{j+1}))} - \frac{g(t_i)h(s_{j+1})-g(t_i)h(s_j)}{(\mu(t_{i+1})-\mu(t_i))(\mu(s_{j+1})-\mu(s_j))}\Bigg]$$
$$= (g_\mu[t_{i+1},t_{i+2}]-g_\mu[t_i,t_{i+1}])\,(h_\mu[s_{j+1},s_{j+2}]-h_\mu[s_j,s_{j+1}])$$

and the result follows. ■

Now we present the main theorem of this section.

Theorem 4.4. *A factorizable function $u : I_{\mathbf{a}}^{\mathbf{b}} \to \mathbb{R}$ is of bounded second μ-variation if and only if each factor is of* (*one-dimensional*) *bounded second μ-variation.*

Proof. Let $\xi := \{t_i\}_{i=0}^n \in \Pi_3([a_1,b_1])$, $\eta := \{s_j\}_{j=0}^m \in \Pi_3([a_2,b_2])$ and let μ be a real-valued strictly increasing function whose domain includes $[a_1,b_1]$ and $[a_2,b_2]$. Then if $u = g\cdot h$, we have for $s \in [a_2,b_2]$

$$|\Delta_{10}u_\mu[t_{i+1},t_{i+2};s]| = \left|\frac{g(t_{i+2})h(s)-g(t_{i+1})h(s)}{\mu(t_{i+2})-\mu(t_{i+1})} - \frac{g(t_{i+1})h(s)-g(t_i)h(s)}{\mu(t_{i+1})-\mu(t_i)}\right|$$
$$= |h(s)|\left|\frac{g(t_{i+2})-g(t_{i+1})}{\mu(t_{i+2})-\mu(t_{i+1})} - \frac{g(t_{i+1})-g(t_i)}{\mu(t_{i+1})-\mu(t_i)}\right|,$$

and for $t \in [a_1,b_1]$

$$|\Delta_{01}u_\mu[t;s_{j+1},s_{j+2}]| = \left|\frac{g(t)h(s_{j+2})-g(t)h(s_{j+1})}{\mu(s_{j+2})-\mu(s_{j+1})} - \frac{g(t)h(s_{j+1})-g(t)h(s_j)}{\mu(s_{j+1})-\mu(s_j)}\right|$$
$$= |g(t)|\left|\frac{h(s_{j+2})-h(s_{j+1})}{\mu(s_{j+2})-\mu(s_{j+1})} - \frac{h(s_{j+1})-h(s_j)}{\mu(s_{j+1})-\mu(s_j)}\right|.$$

If $u = g\cdot h \in BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$, then by definition of $TV^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}})$

$$\sup_\xi \sum_{i=0}^{n-2} |\Delta_{10}u_\mu[t_{i+1},t_{i+2};a_2]| = \sup_\xi \sum_{i=0}^{n-2} |h(a_2)|\,|g_\mu[t_{i+1},t_{i+2}]-g_\mu[t_i,t_{i+1}]|$$
$$\le TV^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) < \infty,$$
$$\sup_\eta \sum_{j=0}^{m-2}!|\Delta_{01}u_\mu[a_1;s_{j+1},s_{j+2}]| = \sup_\eta \sum_{j=0}^{m-2} |g(a_1)||h_\mu[s_{j+1},s_{j+2}]-h_\mu[s_j,s_{j+1}]|$$
$$\le TV^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) < \infty.$$

Therefore $g \in BV^{\mu,2}([a_1,b_1])$ and $h \in BV^{\mu,2}([a_2,b_2])$.

Conversely, if $u = g \cdot h$ with $g \in BV^{\mu,2}([a_1, b_1])$ and $h \in BV^{\mu,2}([a_2, b_2])$ then

$$\begin{aligned}
&TV^{\mu,2}(u, I_{\mathbf{a}}^{\mathbf{b}}) \\
&\quad = \sup_{(\xi,\eta)} \sum_{i=0}^{n-2} \sum_{j=0}^{m-2} |g_\mu[t_{i+1,i+2}] - g_\mu[t_i, t_{i+1}]|\, |h_\mu[s_{j+1}, s_{j+2}] - h_\mu[s_j, s_{j+1}]| \\
&\qquad + \sup_{\xi \in \Pi_3([a_1,b_1])} \sum_{i=0}^{n-2} |h(a_2)|\, |g_\mu[t_{i+1}, t_{i+2}] - g_\mu[t_i, t_{i+1}]| \\
&\qquad + \sup_{\xi \in \Pi_3([a_1,b_1])} \sum_{i=0}^{n-2} |h(b_2)|\, |g_\mu[t_{i+1}, t_{i+2}] - g_\mu[t_i, t_{i+1}]| \\
&\qquad + \sup_{\eta \in \Pi_3([a_2,b_2])} \sum_{j=0}^{m-2} |g(a_1)|\, |h_\mu[s_{j+1}, s_{j+2}] - h_\mu[s_j, s_{j+1}]| \\
&\qquad + \sup_{\eta \in \Pi_3([a_2,b_2])} \sum_{j=0}^{m-2} |g(b_1)|\, |h_\mu[s_{j+1}, s_{j+2}] - h_\mu[s_j, s_{j+1}]| \\
&\quad = V^{\mu,2}(g; [a_1, b_1]) V^{\mu,2}(h; [a_2, b_2]) + 2 \max\{|h(a_2)|, |h(b_2)|\} V^{\mu,2}(g; [a_1, b_1]) \\
&\qquad + 2 \max\{|g(a_1)|, |g(b_1)|\} V^{\mu,2}(h; [a_2, b_2]) < \infty. \blacksquare
\end{aligned}$$

5. Relation with Riemann–Stieltjes integrals

In this section, we will present a result that relates the notion of bounded second μ-variation and (double) indefinite Riemann–Stieltjes integrals of functions in $BV(I_{\mathbf{a}}^{\mathbf{b}})$. This generalizes Proposition 2.4.

Remark 5.1. If $f \in BV(I_{\mathbf{a}}^{\mathbf{b}})$ is a continuous function and μ is a real-valued strictly increasing bounded function whose domain includes $[a_1, b_1]$ and $[a_2, b_2]$, then for all $(\tau, \sigma) \in I_{\mathbf{a}}^{\mathbf{b}}$, f is integrable over $I_{\tau\sigma} := [a_1, \tau] \times [a_2, \sigma]$ and its integral over $I_{\tau\sigma}$ is equal to the iterate integral $\int_{a_1}^{\tau} \int_{a_2}^{\sigma} f(t, s)\, d\mu(s)\, d\mu(t) = \int_{a_2}^{\sigma} \int_{a_1}^{\tau} f(t, s)\, d\mu(t)\, d\mu(s)$ ([9, Chapter III]). In this case, if we set

$$F(\tau, \sigma) := \int_{a_1}^{\tau} \int_{a_2}^{\sigma} f(t, s)\, d\mu(s)\, d\mu(t)$$

then, for all $a_1 \le \tau \le b_1$ and $a_2 \le \sigma_1 < \sigma_2 \le b_2$, the (uniform) continuity of f implies that

$$\begin{aligned}
(5.1) \qquad F(\tau, \sigma_2) - F(\tau, \sigma_1) &= \int_{a_1}^{\tau} \int_{\sigma_1}^{\sigma_2} f(t, s)\, d\mu(s)\, d\mu(t) \\
&= (\mu(\sigma_2) - \mu(\sigma_1)) \int_{a_1}^{\tau} f(t, c^*)\, d\mu(t)
\end{aligned}$$

for some $c^* \in (\sigma_1, \sigma_2)$.

We will use identity (5.1) reiteratively in the proof of the next proposition.

Theorem 5.2. *If $f \in BV(I_{\mathbf{a}}^{\mathbf{b}})$ is a continuous function, μ is a real-valued strictly increasing function whose domain includes $[a_1, b_1]$ and $[a_2, b_2]$, and $F(\tau,\sigma) := \int_{a_1}^{\tau}\int_{a_2}^{\sigma} f(t,s)\, d\mu(s)\, d\mu(t)$ then $V^{\mu,2}(F, I_{\mathbf{a}}^{\mathbf{b}}) \le V(f, I_{\mathbf{a}}^{\mathbf{b}})$ and $F \in BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$.*

Proof. Let $\xi := \{t_i\}_{i=0}^{n} \in \Pi_3([a_1,b_1])$, $\eta := \{s_j\}_{j=0}^{m} \in \Pi_3([a_2,b_2])$. By (5.1)

$$\begin{aligned}\Delta_{01}^2[F]_{j+1}^{\mu}[t_{i+1},t_{i+2}] &= \Bigg[\frac{F(t_{i+2},s_{j+2})-F(t_{i+2},s_{j+1})}{[\mu(t_{i+2})-\mu(t_{i+1})][\mu(s_{j+2})-\mu(s_{j+1})]} \\ &\quad - \frac{F(t_{i+2},s_{j+1})-F(t_{i+2},s_{j})}{[\mu(t_{i+2})-\mu(t_{i+1})][\mu(s_{j+1})-\mu(s_{j})]}\Bigg] \\ &= \frac{1}{\mu(t_{i+2})-\mu(t_{i+1})}\int_{a_1}^{t_{i+2}} \frac{\int_{s_{j+1}}^{s_{j+2}} f(t,s)\,d\mu(s)}{\mu(s_{j+2})-\mu(s_{j+1})}d\mu(t) \\ &\quad - \frac{1}{\mu(t_{i+2})-\mu(t_{i+1})}\int_{a_1}^{t_{i+2}} \frac{\int_{s_{j}}^{s_{j+1}} f(t,s)\,d\mu(s)}{\mu(s_{j+1})-\mu(s_{j})}d\mu(t).\end{aligned}$$

Similarly

$$\begin{aligned}\Delta_{01}^2[F]_{j+1}^{\mu}[t_{i},t_{i+1}] &= \Bigg[\frac{F(t_{i+1},s_{j+2})-F(t_{i+1},s_{j+1})}{[\mu(t_{i+1})-\mu(t_{i})][\mu(s_{j+2})-\mu(s_{j+1})]} \\ &\quad - \frac{F(t_{i+1},s_{j+1})-F(t_{i+1},s_{j})}{[\mu(t_{i+1})-\mu(t_{i})][\mu(s_{j+1})-\mu(s_{j})]}\Bigg] \\ &= \frac{1}{\mu(t_{i+1})-\mu(t_{i})}\int_{a_1}^{t_{i+1}} \frac{\int_{s_{j+1}}^{s_{j+2}} f(t,s)\,d\mu(s)}{\mu(s_{j+2})-\mu(s_{j+1})}d\mu(t) \\ &\quad - \frac{1}{\mu(t_{i+1})-\mu(t_{i})}\int_{a_1}^{t_{i+1}} \frac{\int_{s_{j}}^{s_{j+1}} f(t,s)\,d\mu(s)}{\mu(s_{j+1})-\mu(s_{j})}d\mu(t).\end{aligned}$$

Hence, using the notation as indicated in Remark 5.1, we have

$$\begin{aligned}\Delta_{01}^2[F]_{j+1}^{\mu}[t_{i+1},t_{i+2}] &= \frac{1}{\mu(t_{i+2})-\mu(t_{i+1})}\int_{a_1}^{t_{i+2}} f(t,s_{j+2}^{*})d\mu(t) \\ &\quad - \frac{1}{\mu(t_{i+2})-\mu(t_{i+1})}\int_{a_1}^{t_{i+2}} f(t,s_{j+1}^{*})d\mu(t).\end{aligned}$$

Analogously

$$\begin{aligned}\Delta_{01}^2[F]_{j+1}^{\mu}[t_{i},t_{i+1}] &= \frac{1}{\mu(t_{i+1})-\mu(t_{i})}\int_{a_1}^{t_{i+1}} f(t,{s'}_{j+2}^{*})d\mu(t) \\ &\quad - \frac{1}{\mu(t_{i+1})-\mu(t_{i})}\int_{a_1}^{t_{i+1}} f(t,{s'}_{j+1}^{*})d\mu(t).\end{aligned}$$

Therefore

$$\begin{aligned}
V^{\mu,2}(F,I_{\mathbf{a}}^{\mathbf{b}}) &= \sup_{(\xi,\eta)}\sum_{i=0}^{n-2}\sum_{j=0}^{m-2}\left|\Delta_{11}^{2}F_{\mu}(t_i,s_j)\right|\\
&= \sup_{(\xi,\eta)}\sum_{i=0}^{n-2}\sum_{j=0}^{m-2}\left|\Delta_{01}^{2}[F]_{j+1}^{\mu}[t_{i+1},t_{i+2}]-\Delta_{01}^{2}[F]_{j+1}^{\mu}[t_i,t_{i+1}]\right|\\
&= \sup_{(\xi,\eta)}\sum_{i=0}^{n-2}\sum_{j=0}^{m-2}\left|\frac{1}{\mu(t_{i+2})-\mu(t_{i+1})}\int_{t_{i+1}}^{t_{i+2}} f(t,s_{j+2}^{*})-f(t,s_{j+1}^{*})d\mu(t)\right.\\
&\quad\left.-\frac{1}{\mu(t_{i+1})-\mu(t_i)}\int_{t_i}^{t_{i+1}} f(t,{s'}_{j+2}^{*})-f(t,{s'}_{j+1}^{*})d\mu(t)\right|\\
&= \sup_{(\xi,\eta)}\sum_{i=0}^{n-2}\sum_{j=0}^{m-2}|f(t_{i+2}^{*},s_{j+2}^{*})-f(t_{i+2}^{*},s_{j+1}^{*})\\
&\quad - f(t_{i+1}^{*},{s'}_{j+2}^{*})+f(t_{i+1}^{*},{s'}_{j+1}^{*})| \le V(f,I_{\mathbf{a}}^{\mathbf{b}}).
\end{aligned}$$

We conclude that $V^{\mu,2}(F,I_{\mathbf{a}}^{\mathbf{b}}) \le V(f,I_{\mathbf{a}}^{\mathbf{b}}) < +\infty$.

Now, since by definition $F(a_1,s) = F(t,a_2) = 0$ for all $(t,s) \in I_{\mathbf{a}}^{\mathbf{b}}$, we must have $V_{[a_2,b_2]}^{\mu,2}(F(a_1,\cdot)) = V_{[a_1,b_1]}^{\mu,2}(F(\cdot,a_2)) = 0$. On the other hand, using (5.1) and the mean value theorem, we get

$$\begin{aligned}
V_{[a_2,b_2]}^{\mu,2}(F(b_1,\cdot)) &= \sup_{\eta}\sum_{j=0}^{m-2}|\Delta_{01}F_{\mu}[b_1;s_{j+1},s_{j+2}]|\\
&= \sup_{\eta}\sum_{j=0}^{m-2}\left|\frac{F(b_1,s_{j+2})-F(b_1,s_{j+1})}{\mu(s_{j+2})-\mu(s_{j+1})}-\frac{F(b_1,s_{j+1})-F(b_1,s_j)}{\mu(s_{j+1})-\mu(s_j)}\right|\\
&= \sup_{\eta}\sum_{j=0}^{m-2}\left|\int_{a_1}^{b_1}[f(t,s_{j+2}^{*})-f(t,s_{j+1}^{*})]\,d\mu(t)\right|\\
&= (\mu(b_1)-\mu(a_1))\sup_{\eta}\sum_{j=0}^{m-2}\left|f(c,s_{j+2}^{*})-f(c,s_{j+1}^{*})\right|\\
&\le (\mu(b_1)-\mu(a_1))V_{[a_2,b_2]}(f(c,\cdot)) \le (\mu(b_1)-\mu(a_1))TV(f,I_{\mathbf{a}}^{\mathbf{b}}) < +\infty
\end{aligned}$$

by ([6, Proposition 2.3]). A similar estimate holds for $V_{[a_1,b_1]}^{\mu,2}(F(\cdot,b_2))$. We conclude that $F \in BV^{\mu,2}(I_{\mathbf{a}}^{\mathbf{b}})$. ∎

We would like to thank the reviewer of our initial manuscript by his/her work and valuable comments that have improved the quality of this paper.

References

[1] R. Adams, J. A. Clarkson, *Properties of functions f(x,y) of bounded variation*, Trans. Amer. Math. Soc. 36 (1934), 711–730.

[2] W. Azíz, *Algunas extensiones a* $\mathbb{R}^2$ *de la noción de funciones con j-variación acotada en el sentido de Riesz y controlabilidad de las RNC*, Doctoral Dissertation, Universidad Central de Venezuela, Caracas, 2009.

[3] D. Bugajewska, *On the superposition operator in the space of functions of bounded variation*, Math. Comput. Modelling 52(5–6) (2010), 791–796.

[4] L. Cesari, *Sulle funzioni a variazione limitata*, Ann. Scuola Norm. Sup. Pisa II 5(3–4) (1936), 299–313.

[5] P. L. Dirichelt, *Sur la convergence des séries trigonemétriques que servent á représenter une function arbitraire entre des limites donnés*, J. Reine Angew. Math. 4 (1826), 157–159.

[6] J. Ereu, J. Giménez, N. Merentes, *On Bi-dimensional second variation*, Comment. Math. 52(1) (2012), 39–59.

[7] T. Ereú, N. Merentes, B. Rzepka, J. L. Sánchez, *On composition operator in the algebra of functions of two variables with bounded total Φ-variation in Schramm sense*, JMA 33 (2010), 35–50.

[8] G. H. Hardy, *On double Fourier series, and especially those which represent the double zeta-function width real and inconmesurable parameters*, Quart. J. Math. Oxford. 37 (1905/06), 53–79.

[9] T. H. Hildebrandt, *Introduction to the Theory of Integration*, Academic Press, New York, 1963.

[10] F. N. Huggins, *Bounded slope variation and generalized convexity*, Proc. Amer. Math. Soc. 65 (1977), 65–69.

[11] C. Jordan, *Sur la série de Fourier*, C. R. Acad. Sci. Paris 2 (1881), 228–230.

[12] L. Tonelli, *Sulla cuadratura délie superficie*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. 3(6) (1926), 357–362.

[13] N. Merentes, S. Rivas, *El Operador de Composición en Espacios de Funciones con algún tipo de Variación Acotada*, IX Escuela Venezolana de Matemáticas, Facultad de Ciencias-ULA, Mérida-Venezuela, 1996.

[14] F. Riesz, *Sur certains systems singuliers d'equations integrates*, Ann. Sci. Ecole Norm. Sup. Paris 3(28) (1911), 33–68.

[15] A. W. Roberts, D. E. Varberg, *Functions of bounded convexity*, Bull. Amer. Math. Soc. 75(3) (1969), 568–572.

[16] A. W. Roberts, D. E. Varberg, *Convex Functions*, Academic Press, New York–London, 1973.

[17] A. M. Russell, C. J. F. Upton, *A generalization of a theorem by F. Riesz*, Anal. Math. 9 (1983), 69–77.

[18] Ch. J. De La Vallée Poussin, *Sur l'integrale de Lebesgue*, Trans. Amer. Math. Soc. 16 (1915), 435–501.

[19] G. Vitali, *Sulle funzioni integrali*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. 40 (1904/05), 1021–1034.

[20] M. Wróbel, *On functions of bounded n-th variation*, Ann. Math. Sil. 15 (2001), 79–86.

[21] M. Wróbel, *Uniformly bounded Nemytskij operators beteween the Banach spaces of functions of bounded n-th variation*, J. Math. Anal. Appl. 391(2) (2012), 451–456.

J. Ereú
DEPARTAMENTO DE MATEMÁTICAS
DECANATO DE CIENCIAS Y TECNOLOGÍA
UNIVERSIDAD CENTROCCIDENTAL LISANDRO ALVARADO
BARQUISIMETO, VENEZUELA
E-mail: jereu@ucla.edu.ve

J. Giménez
DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS
UNIVERSIDAD DE LOS ANDES
MÉRIDA, VENEZUELA
E-mail: jgimenez@ula.ve

N. Merentes
ESCUELA DE MATEMÁTICAS
UNIVERSIDAD CENTRAL DE VENEZUELA
CARACAS, VENEZUELA
E-mail: nmerucv@gmail.com

Received Agust 6, 2013; revised version February 12, 2014.

Communicated by M. Bożejko.