
A SUZUKI TYPE UNIQUE COMMON FIXED POINT THEOREM FOR TWO PAIRS OF HYBRID MAPS UNDER A NEW CONDITION IN PARTIAL METRIC SPACES

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Abstract. In this paper, we introduce a new condition namely: condition (W.C.C.) and utilize the same to prove a Suzuki type unique common fixed point theorem for two hybrid pairs of mappings in partial metric spaces employing the partial Hausdorff metric which generalizes several known results of the existing literature proved in metric and partial metric spaces.

1. Introduction and preliminaries

A subset $A$ of a metric space $(X, d)$ is said to be
1. closed if $A = \overline{A}$ where $\overline{A} = \{ x \in X : d(x, A) = 0 \}$,
2. bounded if $\delta(A) < \infty$ where $\delta(A) = \sup\{d(a, b) : a, b \in A\}$.

If $(X, d)$ is a metric space, then on the lines of Nadler [13], we adopt
1. $CL(X) = \{ A : A$ is a non-empty closed subset of $X \}$,
2. $CB(X) = \{ A : A$ is a non-empty closed and bounded subset of $X \}$,
3. for non-empty closed and bounded subsets $A, B$ of $X$ and $x \in X$,
   $$d(x, A) = \inf\{d(x, a) : a \in A\}$$

and
   $$H(A, B) = \max \{ \{ \sup d(a, B) : a \in A \}, \{ \sup d(A, b) : b \in B \} \}.$$ 

The study of fixed points of multi-valued mappings using the Hausdorff distance was initiated by Nadler [13] wherein author proved the following multivalued analogue of Banach Contraction Principle, which is also sometimes referred as Nadler Contraction Principle:

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Theorem 1.1. Let \((X, d)\) be a complete metric space and \(T : X \to CB(X)\) be a mapping satisfying \(H(Tx, Ty) \leq kd(x, y), \) where \(k \in [0, 1). \) Then there exists some \(x \in X\) such that \(x \in Tx.\)

In recent years, a rich and useful fixed point theory is flourishing wherein various authors extended Theorem 1.1 employing weak and generalized contraction conditions (e.g. [9, 11, 14, 15, 16, 17]). On the other hand, the basic notion of partial metric space was introduced by S. G. Mathews [12] as a part of the study of denotational semantics of data flow network. He presented a modified version of the Banach contraction principle which is more conducive to such setting (e.g. [3, 6]). In fact, the partial metric spaces offer a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory (e.g. [4, 5, 7, 8, 18, 19, 20, 21, 22, 23, 24]). In this direction, Aydi et al. [1] introduced the concept of a partial Hausdorff metric and extended Nadler’s fixed point theorem to the setting of partial metric spaces. Towards the application side, the theory of multi-valued maps has fruitful and established applications in control theory, convex optimization, differential equations and economics (see [2]).

On the lines of [1, 10, 12], the following definitions and results will be needed in the sequel.

Definition 1.2. [12] A partial metric on a nonempty set \(X\) is a function \(p : X \times X \to \mathbb{R}^+\) such that for all \(x, y, z \in X:\)

\(\begin{align*}
(p_1) \quad x = y & \iff p(x, x) = p(x, y) = p(y, y), \\
(p_2) \quad p(x, x) \leq p(x, y), \\
(p_3) \quad p(x, y) = p(y, x), \\
(p_4) \quad p(x, y) \leq p(x, z) + p(z, y) - p(z, z).
\end{align*}\)

In this case, \((X, p)\) is called a partial metric space.

It can be easily shown that \(|p(x, y) - p(y, z)| \leq p(x, z)\) \(\forall x, y, z \in X.\) Also, clearly \(p(x, y) = 0\) implies \(x = y\) from \((p_1)\) and \((p_2)\). But if \(x = y,\) \(p(x, y)\) may not be zero. A basic example of a partial metric space is the pair \((\mathbb{R}^+, p),\) where \(p(x, y) = \max\{x, y\}\) for all \(x, y \in \mathbb{R}^+.\) Each partial metric \(p\) on \(X\) generates \(T_0\) topology \(\tau_p\) on \(X,\) which has a base comprised of the family of open \(p -\) balls \(\{B_p(x, \epsilon) \mid x \in X, \epsilon > 0\}\) for all \(x \in X\) and \(\epsilon > 0,\) where \(B_p(x, \epsilon) = \{y \in X \mid p(x, y) < p(x, x) + \epsilon\}\) for all \(x \in X\) and \(\epsilon > 0.\) If \(p\) is a partial metric on \(X,\) then the function \(p^s : X \times X \to \mathbb{R}^+\) given by \(p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)\) is a metric on \(X.\)
**Definition 1.3.** [12] Let \((X, p)\) be a partial metric space. Then

(i) a sequence \(\{x_n\}\) in \((X, p)\) is said to converge to a point \(x \in X\) if and only if \(p(x, x) = \lim_{n \to \infty} p(x, x_n)\),

(ii) a sequence \(\{x_n\}\) in \((X, p)\) is said to be Cauchy sequence if \(\lim_{n,m \to \infty} p(x_n, x_m)\) exists and is finite,

(iii) \((X, p)\) is said to be complete if every Cauchy sequence \(\{x_n\}\) in \(X\) converges, w.r.t to \(\tau_p\), to a point \(x \in X\) such that
\[
p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m).
\]

**Lemma 1.4.** [12] Let \((X, p)\) be a partial metric space. Then

(a) \(\{x_n\}\) is a Cauchy sequence in \((X, p)\) if and only if it is a Cauchy sequence in the metric space \((X, p^s)\),

(b) \((X, p)\) is complete iff the metric space \((X, p^s)\) is complete. Further more, \(\lim_{n \to \infty} p^s(x_n, x) = 0\) if and only if
\[
p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m).
\]

**Lemma 1.5.** [10] Let \((X, p)\) be a partial metric space and let \(A\) be any nonempty subset of \(X\). Then \(a \in \overline{A}\) if and only if \(p(a, A) = p(a, a)\), where \(\overline{A}\) denotes the closure of \(A\) with respect to the partial metric \(p\).

Recall that \(A\) is closed in \((X, p)\) if and only if \(A = \overline{A}\).

On the lines of [1], let \((X, p)\) be a partial metric space while \(CB^p(X)\) be the family of all nonempty, closed and bounded subsets of the partial metric space \((X, p)\) induced by the partial metric \(p\). For \(A, B \in CB^p(X)\) and \(x \in X\), define
\[
p(A, B) = \inf \{p(a, b) : a \in A, b \in B\},
p(x, A) = \inf \{p(x, a) : a \in A\},
\delta_p(A, B) = \sup\{p(a, B) : a \in A\},
\delta_p(B, A) = \sup\{p(b, A) : b \in B\}
\]
and
\[
H_p(A, B) = \max \{\delta_p(A, B), \delta_p(B, A)\}
\]
wherein \(H_p\) is called the partial Hausdorff metric induced by partial metric \(p\).

Hassen Aydi et al. [1] proved that any Hausdorff metric is a partial Hausdorff metric but converse need not be true (see Example 2.6 in [1]).

**Lemma 1.6.** [1] Let \((X, p)\) be a partial metric space. If \(A, B, C \in CB^p(X)\), then

(i) \(\delta_p(A, A) = \sup\{p(a, a) : a \in A\}\),

(ii) \(\delta_p(A, A) \leq \delta_p(A, B)\),

(iii) \(\delta_p(A, B) = 0\) implies that \(A \subseteq B\),

(iv) \(\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)\).
Lemma 1.7. [1] Let \((X, p)\) be a partial metric space. If \(A, B, C \in CB^p(X)\), we have

(i) \(H_p(A, A) \leq H_p(A, B)\),
(ii) \(H_p(A, B) = H_p(B, A)\),
(iii) \(H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C^*} p(c, c)\).

Lemma 1.8. [1] Let \((X, p)\) be a partial metric space. For any \(A, B \in CB^p(X)\), the following holds:

\[ H_p(A, B) = 0 \text{ implies that } A = B. \]

In [1], authors have also showed (by dint of an example) that \(H_p(A, A)\) need not be zero.

Lemma 1.9. [1] Let \((X, p)\) be a partial metric space, \(A, B \in CB^p(X)\) and \(h > 1\). For any \(a \in A\), there exists \(b \in B\) such that \(p(a, b) \leq hH_p(A, B)\).

Theorem 1.10. [1] Let \((X, p)\) be a complete partial metric space and let \(T : X \to CB^p(X)\) be a multi-valued mapping such that for all \(x, y \in X\), \(H_p(Tx, Ty) \leq k p(x, y)\) where \(k \in (0, 1)\), then \(T\) has a fixed point.

Very recently Abbas et al. [25] generalized Theorem 1.10, by proving the following Suzuki type theorem.

Theorem 1.11. Let \((X, p)\) be a complete partial metric space and let \(T : X \to CB^p(X)\) be a multivalued mapping and \(\varphi : [0, 1) \to (0, 1]\) be a non-increasing function defined by

\[
\varphi(r) = \begin{cases} 
1, & \text{if } 0 \leq r < \frac{1}{2}, \\
1 - r, & \text{if } \frac{1}{2} \leq r < 1.
\end{cases}
\]

If there exists \(r \in [0, 1)\) such that \(T\) satisfies the condition

\[ \varphi(r)p(x, Tx) \leq p(x, y) \text{ implies } H_p(Tx, Ty) \leq r \max \left\{ \frac{p(x, y), p(x, Tx), p(y, Ty)}{\frac{1}{2}[p(x, Ty) + p(y, Tx)]} \right\} \]

for all \(x, y \in X\), then \(T\) has a fixed point i.e. there exists a point \(z \in X\) such that \(z \in Tz\).

In this paper, we introduce the condition (W.C.C.) and utilize the same to prove a common fixed point theorem for two hybrid pairs of mappings in partial metric spaces using partial Hausdorff metric.
2. Main results

We begin with the following lemma, which will be utilized to prove our main result.

**Lemma 2.1.** Let $x_n \to x$ as $n \to \infty$ in a partial metric space $(X, p)$ such that $p(x, x) = 0$. Then $\lim_{n \to \infty} p(x_n, B) = p(x, B)$ for any $B \in CB^p(X)$.

**Proof.** Since $x_n \to x$, we have $\lim_{n \to \infty} p(x_n, x) = p(x, x) = 0$.

By triangular inequality for $x_n \in X$ and $y \in B$, we have

$$p(x_n, y) \leq p(x_n, x) + p(x_n, y) - p(x, x),$$

which implies that $p(x_n, B) \leq p(x_n, x) + p(x, B)$.

Therefore $\lim_{n \to \infty} p(x_n, B) \leq p(x, B). \ldots \ldots \ (i)$.

But also $p(x, y) \leq p(x, x_n) + p(x_n, y) - p(x_n, x_n).$

$$p(x, y) \leq p(x, x_n) + p(x_n, y).$$

$$p(x, B) \leq p(x, x_n) + p(x_n, B).$$

Therefore $p(x, B) \leq \lim_{n \to \infty} p(x_n, B). \ldots \ldots \ (ii)$.

From (i) and (ii) we have $\lim_{n \to \infty} p(x_n, B) = p(x, B)$. ■

Now we introduce the following new condition, namely, the condition (W.C.C.) on mappings which are not necessarily continuous and commutative.

**Definition 2.2.** Let $(X, p)$ be a partial metric space with $f, g : X \to X$ and $S : X \to CB^p(X)$. Then the triplet $(f, g; S)$ is said to satisfy condition (W.C.C.) if $p(fx, gy) \leq p(y, Sx)$ for all $x, y \in X$.

The following example illustrates the condition (W.C.C.)

**Example 2.3.** Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}, \forall x, y \in X$. Let $f, g : X \to X$ and $S : X \to CB^p(X)$ be defined by $fx = 0$, for all $x \in X$,

$$gx = \begin{cases} 
0, & \text{if } x \in [0, \frac{1}{2}], \\
\frac{\pi}{32}, & \text{if } x \in (\frac{1}{2}, 1], 
\end{cases}$$

and $Sx = \{[0, \frac{1}{4}]\}, \forall x \in X$. We consider the following two cases:

Case (i). If $x \in X$ and $y \in [0, \frac{1}{2}]$, then

$$p(fx, gy) = 0 = p(y, Sx).$$

Case (ii). If $x \in X$ and $y \in (\frac{1}{2}, 1]$, then

$$p(fx, gy) = \frac{y}{32} < y = p(y, Sx).$$

Thus $(f, g; S)$ satisfy the condition (W.C.C.).
The following example shows that the triplet \((f, g; S)\) satisfying the condition (W.C.C.) need not be continuous even if \(S\) is a single-valued mapping.

**Example 2.4.** Let \(X = [0, 1]\) and \(p(x, y) = \max\{x, y\}, \forall x, y \in X\). In what follows, \(O(X)\) stands for the collections of singleton subsets of the set \(X\). Let \(f, g : X \to X\) and \(S : X \to O(X) \subseteq CB(X)\) be defined by

\[
    f(x) = \begin{cases}
        \frac{x}{12}, & \text{if } x \neq 1, \\
        \frac{1}{24}, & \text{if } x = 1,
    \end{cases}
    \quad
g(x) = \begin{cases}
        \frac{x}{6}, & \text{if } x \neq 1, \\
        \frac{1}{12}, & \text{if } x = 1,
    \end{cases}
\]

and

\[
    S(x) = \begin{cases}
        \left\{ \frac{x}{2} \right\}, & \text{if } x \neq 1, \\
        \left\{ \frac{1}{4} \right\}, & \text{if } x = 1.
    \end{cases}
\]

We distinguish the following cases:

Case (i). If \(x \neq 1\) and \(y \neq 1\), then

\[
    p(f(x), g(y)) = \max\left\{ \frac{x}{12}, \frac{y}{6} \right\} = \frac{1}{6} \max\left\{ \frac{x}{2}, y \right\} = \frac{1}{6} p(y, S(x)).
\]

Case (ii). If \(x \neq 1\) and \(y = 1\), then

\[
    p(f(x), g(y)) = \max\left\{ \frac{x}{12}, \frac{1}{12} \right\} < \frac{1}{6} \max\left\{ 1, \frac{x}{2} \right\} = \frac{1}{6} p(y, S(x)).
\]

Case (iii). If \(x = 1\) and \(y \neq 1\), then

\[
    p(f(x), g(y)) = \max\left\{ \frac{1}{24}, \frac{y}{6} \right\} = \frac{1}{6} p(y, S(x)).
\]

Case (iv). If \(x = 1\) and \(y = 1\), then

\[
    p(f(x), g(y)) = \max\left\{ \frac{1}{24}, \frac{1}{12} \right\} = \frac{1}{12} < \max\left\{ 1, \frac{1}{4} \right\} = p(y, S(x)).
\]

Thus \((f, g; S)\) satisfy the condition (W.C.C.). Notice that in this example all mappings are discontinuous.

The following example shows that the triplet \((f, g; S)\) satisfying the condition (W.C.C.) need not be commuting even if \(S\) is a single-valued mapping.

**Example 2.5.** Let \(a\) and \(b\) be non-negative real numbers such that \(b < a\). Let \(X = \{a, b\}\) and \(p(x, y) = \max\{x, y\}, \forall x, y \in X\).

Let \(f, g : X \to X\) and \(S : X \to O(X) \subseteq CB(X)\) be defined by \(fa = fb = b, ga = b, gb = a\) and \(Sa = Sb = \{a\}\).
Clearly the triplet \((f, g; S)\) satisfy the condition (W.C.C.) and the pairs \((f, S)\), \((g, S)\) and \((f, g)\) are not commuting.

Now, we state and prove our main result.

**Theorem 2.6.** Let \((X, p)\) be a complete partial metric space and \(S, T : X \rightarrow CB^p(X)\) and \(f, g : X \rightarrow X\). Assume that there exists \(r \in [0, 1)\) such that for every \(x, y \in X\),

\[
(2.6.1) \quad \varphi(r) \min \{p(x, Sx), p(gy, Ty)\} \leq p(fx, gy)
\]

implies

\[
H_p(Sx, Ty) \leq r \max \left\{ p(fx, gy), \frac{1}{2}[p(fx, Sx) + p(gy, Ty)], \frac{1}{2}[p(fx, Ty) + p(gy, Sx)] \right\}
\]

where \(\varphi\) is a function described by (1),

\[
(2.6.2) \quad \bigcup_{x \in X} Sx \subseteq g(X)\quad \text{and}\quad \bigcup_{x \in X} Tx \subseteq f(X),
\]

and suppose that \(f, g\) are not commuting.

Then \(f, g, S\) and \(T\) have a unique common fixed point in \(X\).

**Proof.** Let \(x_0 \in X\) and suppose that \(h = \frac{1}{\sqrt{r}} > 1\), \(y_0 = fx_0\). Owing to (2.6.2), we have \(Sx_0 \subseteq g(X)\) so that there exists \(x_1 \in X\) such that \(y_1 = gx_1 \in Sx_0\).

In view of Lemma 1.9 with \(h = \frac{1}{\sqrt{r}}\), there exists \(y_2 \in Tx_1\) such that

\[
p(gx_1, y_2) \leq \frac{1}{\sqrt{r}} H_p(Sx_0, Tx_1).
\]

Since \(Tx_1 \subseteq f(X)\), we can find a point \(x_2 \in X\) such that \(y_2 = fx_2 \in Tx_1\).

Therefore

\[
p(gx_1, fx_2) \leq \frac{1}{\sqrt{r}} H_p(Sx_0, Tx_1).
\]

Since \(\varphi(r)p(fx_0, Sx_0) \leq p(fx_0, Sx_0) \leq p(fx_0, gx_1)\), we have

\[
\varphi(r) \min \{p(fx_0, Sx_0), p(gx_1, Tx_1)\} \leq p(fx_0, gx_1).
\]

On using (2.6.1), we have

\[
p(gx_1, fx_2) \leq h H_p(Sx_0, Tx_1) = \frac{1}{\sqrt{r}} H_p(Sx_0, Tx_1)
\]

\[
\leq \sqrt{r} \max \left\{ p(fx_0, gx_1), \frac{1}{2}[p(fx_0, Sx_0) + p(gx_1, Tx_1)], \frac{1}{2}[p(fx_0, Tx_1) + p(gx_1, Sx_0)] \right\}
\]

\[
\leq \sqrt{r} \max \left\{ p(y_0, y_1), \frac{1}{2}[p(y_0, y_1) + p(y_1, y_2)], \frac{1}{2}[p(y_0, y_2) + p(y_1, y_1)] \right\}
\]
so that
\[ p(y_1, y_2) \leq \sqrt{r} \max \left\{ p(y_0, y_1), \frac{1}{2} [p(y_0, y_1) + p(y_1, y_2)] \right\} \] owing to (p4).

Thus, we have
\[ (2) \quad p(y_1, y_2) \leq \beta p(y_0, y_1), \]
where \( \beta = \max \left\{ \sqrt{r}, \frac{\sqrt{r}/2}{1-\sqrt{r}/2} \right\} < 1 \). As \( fx_2 \in Tx_1 \), owing to Lemma 1.9, we can choose \( y_3 \in Sx_2 \) such that
\[ p(fx_2, y_3) \leq \frac{1}{\sqrt{r}} H_p(Sx_2, Tx_1). \]

Since \( Sx_2 \subseteq g(X) \), we can find a point \( x_3 \in X \) such that \( y_3 = gx_3 \in Sx_2 \) so that
\[ p(fx_2, gx_3) \leq \frac{1}{\sqrt{r}} H_p(Sx_2, Tx_1). \]

Since \( \varphi(r)p(gx_1, Tx_1) \leq p(gx_1, Tx_1) \leq p(fx_2, gx_1) \), we have
\[ \varphi(r) \min \{ p(fx_2, Sx_2), p(gx_1, Tx_1) \} \leq p(fx_2, gx_1). \]

Hence, on using (2.6.1), we have
\[ p(fx_2, gx_3) \leq \frac{1}{\sqrt{r}} H_p(Sx_2, Tx_1) \]
\[ \leq \sqrt{r} \max \left\{ p(fx_2, gx_1), \frac{1}{2} [p(fx_2, Sx_2) + p(gx_1, Tx_1)], \frac{1}{2} [p(fx_2, Tx_1) + p(gx_1, Sx_2)] \right\} \]
\[ \leq \sqrt{r} \max \left\{ p(y_2, y_1), \frac{1}{2} [p(y_2, y_3) + p(y_1, y_2)], \frac{1}{2} [p(y_2, y_2) + p(y_1, y_3)] \right\} \]
so that
\[ p(y_2, y_3) \leq \sqrt{r} \max \left\{ p(y_1, y_2), \frac{1}{2} [p(y_1, y_2) + p(y_2, y_3)] \right\} \] owing to (p4).

Thus, we have
\[ (3) \quad p(y_2, y_3) \leq \beta p(y_1, y_2) \leq \beta^2 p(y_0, y_1). \]

Continuing in this way, we are in the receipt of a sequence \( \{y_n\} \) in \( X \) such that for any \( n \in \mathbb{N}, y_{2n+1} = gx_{2n+1} \in Sx_{2n} \), \( y_{2n+2} = fx_{2n+2} \in Tx_{2n+1} \) and
\[ (4) \quad p(y_n, y_{n+1}) \leq \beta^n p(y_0, y_1). \]

Clearly,
\[ (5) \quad p(y_{n+1}, y_n) \to 0 \quad \text{as} \quad n \to \infty. \]
For \( m > n \), we have

\[
(6) \quad p(y_n, y_m) \leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m),
\]

\[
\leq (\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}) p(y_1, y_0), \quad \text{(owing to (4))}
\]

\[
\leq \frac{\beta^n}{1 - \beta} p(y_1, y_0) \to 0 \quad \text{as } n \to \infty.
\]

Thus \( \{y_n\} \) is a Cauchy sequence in \( X \). Hence, in view of Lemma 1.4, \( \{y_n\} \) is a Cauchy sequence in \((X, p^s)\). Since \((X, p)\) is complete, therefore in view of Lemma 1.4, it follows that \((X, p^s)\) is complete so that \( \{y_n\} \) converges to some \( z \in X \) i.e.

\[
\lim_{n \to \infty} p^s(y_n, z) = 0.
\]

Now, using Lemma 1.4(b) and (6), we have

\[
(7) \quad p(z, z) = \lim_{n \to \infty} p(y_n, z) = \lim_{n \to \infty} p(y_n, y_m) = 0.
\]

Suppose that the triplet \((f, g; S)\) satisfy the condition \((W.C.C)\), therefore

\[
(8) \quad p(fx, gy) \leq p(y, Sx) \quad \text{for all } x, y \in X.
\]

Owing to (8), we have

\[
p(fx_{2n}, gz) \leq p(z, Sx_{2n}) \leq p(z, gx_{2n+1}),
\]

which on letting \( n \to \infty \) and using Lemma 2.1 and (7) gives

\[
(9) \quad p(z, gz) \leq 0 \quad \text{so that } \quad gz = z.
\]

(10) Now, we assert that \( p(gz, Sx) \leq rp(gz, fx) \) for any \( fx \in X - \{gz\} \).

To accomplish this, let \( fx \in X - \{gz\} \). Since \( y_{2n+1} \to z = gz, \ y_{2n+2} \to z = gz \) and \( p(z, z) = \lim_{n \to \infty} p(y_n, z) = 0 \), there exists a positive integer \( n_0 \) such that for all \( n \geq n_0 \), we have

\[
p(gz, gx_{2n+1}) \leq \frac{1}{3} p(gz, fx) \quad \text{and} \quad p(gz, fx_{2n+2}) \leq \frac{1}{3} p(gz, fx).
\]

For any \( n \geq n_0 \), we have

\[
\varphi(r)p(gx_{2n+1}, Tx_{2n+1}) \leq p(gx_{2n+1}, Tx_{2n+1})
\]

\[
\leq p(fx_{2n+2}, gx_{2n+1})
\]

\[
\leq p(fx_{2n+2}, gz) + p(gz, gx_{2n+1})
\]

\[
\leq \frac{2}{3} p(gz, fx)
\]

\[
= p(fx, gz) - \frac{1}{3} p(fx, gz)
\]

\[
\leq p(fx, gz) - p(gz, gx_{2n+1})
\]

\[
\leq p(fx, gx_{2n+1}).
\]
Hence
\[ \varphi(r) \min \{p(f, Sx), p(gx_{2n+1}, Tx_{2n+1})\} \leq p(f, gx_{2n+1}) \]
which implies (due to (2.6.1)) that
\[ p(fx_{2n+2}, Sx) \leq H_p(Sx, Tx_{2n+1}) \]
\[ \leq r \max \left\{ \frac{1}{2}p(fx, Sx) + p(gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}p(fx, Sx) + p(gx_{2n+1}, Sx) \right\}. \]
Now, letting \( n \to \infty \), we get
\[ p(gz, Sx) \leq r \max \left\{ \frac{1}{2}p(fx, gz) + p(gz, Sx), \frac{1}{2}p(fx, gz) + p(gz, Sx) \right\}, \]
since \( gz = z \)
\[ \leq r \max \left\{ p(fx, gz), \frac{1}{2}p(fx, gz) + p(gz, Sx) \right\}, \]
from (p4).
If \( \max \{p(fx, gz), \frac{1}{2}p(fx, gz) + p(gz, Sx)\} \) = \( p(fx, gz) \),
then
\[ p(gz, Sx) \leq rp(fx, gz). \]
If \( \max \{p(fx, gz), \frac{1}{2}p(fx, gz) + p(gz, Sx)\} \) = \( \frac{1}{2}p(fx, gz) + p(gz, Sx) \),
then
\[ p(gz, Sx) \leq \frac{r}{2}[p(fx, gz) + p(gz, Sx)], \]
which implies that
\[ \left(1 - \frac{r}{2}\right)p(gz, Sx) \leq \frac{r}{2}p(fx, gz), \]
so that
\[ p(gz, Sx) \leq \frac{r}{2-r}p(fx, gz) \leq rp(fx, gz). \]
Thus \( p(gz, Sx) \leq rp(gz, fx) \). Hence, the claim is established.

Now, we show that \( gz \in Tz \). Firstly, we consider the case \( 0 \leq r < 1 \). On the contrary, let us assume that \( gz \notin Tz = T_\perp z \) as \( Tz \) is closed. Hence, owing to Lemma 1.5, (9) and (7), we have
\[ p(gz, Tz) \neq p(gz, gz) = p(z, z) = 0. \]
In view of (2.6.2) and (11), we can choose \( fa \in Tz \) such that
\[ 2rp(fa, gz) < p(gz, Tz). \]
Since $fa \in Tz$ and $gz \notin Tz$ imply $fa \neq gz$, therefore on using (10), we have
\begin{equation}
 p(gz, Sa) \leq rp(gz, fa).
\end{equation}
Since, $\varphi(r)p(gz, Tz) \leq p(gz, Tz) \leq p(fa, gz)$, it follows that
\begin{equation}
 \varphi(r) \min \{p(fa, Sa), p(gz, Tz)\} \leq p(fa, gz).
\end{equation}
Now, on using (2.6.1), we have
\begin{equation}
 p(fa, Sa) \leq H_p(Sa, Tz)
 \leq r \max \left\{ p(fa, gz), \frac{1}{2} \left[ p(fa, Sa) + p(gz, Tz) \right], \frac{1}{2} \left[ p(fa, Tz) + p(gz, Sa) \right] \right\}
 \leq r \max \left\{ p(fa, gz), \frac{1}{2} \left[ p(fa, Sa) + p(gz, fa) \right], \frac{1}{2} \left[ p(fa, fa) + p(gz, fa) + p(fa, Sa) - p(fa, fa) \right] \right\}
 \leq r \max \left\{ p(fa, gz), \frac{1}{2} \left[ p(fa, Sa) + p(gz, fa) \right] \right\}.
\end{equation}
If $\max \{p(fa, gz), \frac{1}{2} \left[ p(fa, Sa) + p(gz, fa) \right]\} = p(fa, gz)$, then
\begin{equation}
 H_p(Sa, Tz) \leq rp(fa, gz).
\end{equation}
If $\max \{p(fa, gz), \frac{1}{2} \left[ p(fa, Sa) + p(gz, fa) \right]\} = \frac{1}{2} \left[ p(fa, Sa) + p(gz, fa) \right]$, then
\begin{equation}
 H_p(Sa, Tz) \leq \frac{r}{2} \left[ p(fa, Sa) + p(gz, fa) \right].
\end{equation}
But, from (14), we have
\begin{equation}
 d(fa, Sa) \leq \frac{r}{2} \left[ p(fa, Sa) + p(gz, fa) \right],
\end{equation}
which implies that
\begin{equation}
 d(fa, Sa) \leq \frac{r}{2 - r} d(fa, gz) < d(fa, gz).
\end{equation}
So, from (15) and (16), we get $H_p(Sa, Tz) \leq rp(fa, gz)$.

Thus in all the cases, we have
\begin{equation}
 H_p(Sa, Tz) \leq rp(fa, gz).
\end{equation}
Now
\begin{equation}
 p(Sa, Tz) = \inf \{p(x, y) : x \in Sa, y \in Tz\}
 \leq \inf \{p(x, fa) : x \in Sa\}, \text{ since } fa \in Tz
 = p(fa, Sa)
 \leq H_p(Sa, Tz)
\end{equation}
and by (13) and (17), we have
\[ p(gz, Tz) \leq p(gz, Sa) + p(Sa, Tz) \]
\[ \leq p(gz, Sa) + H_p(Sa, Tz) \]
\[ \leq rp(fa, gz) + rp(fa, gz) = 2rp(fa, gz) \]
\[ < p(gz, Tz). \]

This is a contradiction and hence \( gz \in Tz. \)

Since \( gz \in Tz \) and by using (9), we have
\[ (18) \]
\[ z = gz \in Tz. \]

Now, from (8), we have
\[ (19) \]
\[ p(fz, z) = p(fz, gz) \leq p(z, Sz). \]

Since \( gz \in Tz \) and by using \((p_2)\), we have
\[ \varphi(r)p(gz, Tz) \leq p(gz, Tz) \leq p(gz, gz) \leq p(fz, gz). \]

Hence, we have
\[ \varphi(r) \min \{p(fz, Sz), p(gz, Tz)\} \leq p(fz, gz). \]

Now by using (2.6.1), we have
\[ p(z, Sz) \leq H_p(Sz, Tz) \]
\[ \leq r \max \left\{ \frac{1}{2} [p(fz, Sz) + p(gz, Sz)], \frac{1}{2} p(fz, Tz) + p(gz, Sz) \right\} \]
\[ \leq r \max \{p(z, Sz), p(z, Sz), p(z, Sz)\} \]
\[ \quad \text{(from (p_4), (19), (18), Lemma 1.5, (7))} \]
\[ = rp(z, Sz), \]

which in turn yields that \( p(z, Sz) = 0 \). From Lemma 1.5 and (7), we have \( z \in \overline{Sz} = Sz. \)

Now from (19), we get \( p(fz, z) \leq 0 \) so that \( fz = z \). Thus
\[ (20) \]
\[ z = fz \in Sz. \]

Now from (18) and (20), we have that \( z \) is a common fixed point of \( f, g, S \) and \( T \).

Now consider the case when \( \frac{1}{2} \leq r < 1 \). Firstly, we prove that
\[ (21) \]
\[ H_p(Sx, Tz) \leq r \max \left\{ \frac{1}{2} [p(fx, gz) + p(gz, Sz)], \frac{1}{2} [p(fx, Sz) + p(gz, Tz)] \right\} \]

for all \( x, z \in X \) such that \( fx \neq gz \).
Assume that $fx \neq gz$. Then for every $n \in \mathbb{N}$, there exists $z_n \in Sx$ such that

$$p(gz, z_n) \leq p(gz, Sx) + \frac{1}{n}p(fx, gz).$$

Therefore

$$p(fx, Sx) \leq p(fx, z_n)$$
$$\leq p(fx, gz) + p(gz, z_n)$$
$$\leq p(fx, gz) + p(gz, Sx) + \frac{1}{n}p(fx, gz)$$
$$\leq p(fx, gz) + rp(gz, fx) + \frac{1}{n}p(fx, gz) \text{ from (10)}$$
$$\leq \left(1 + r + \frac{1}{n}\right)p(fx, gz).$$

Letting $n \to \infty$, we get

$$p(fx, Sx) \leq (1 + r)p(fx, gz).$$

Thus $\varphi(r)p(fx, Sx) = (1 - r)p(fx, Sx) \leq \frac{1}{1 + r}p(fx, Sx) \leq p(fx, gz)$.

Hence, we have

$$\varphi(r)\min\{p(fx, Sx), p(gz, Tz)\} \leq p(fx, gz).$$

Now using (2.6.1), with $y = z$, we get (21). Since $y_n \to z$, we may assume that $y_n \neq z$ for infinitely many $n$. Taking $x = x_{2n}$ in (21), we get

$$(22) \quad H_p(Sx_{2n}, Tz) \leq r \max \left\{ \frac{1}{2}[p(fx_{2n}, Sx_{2n}) + p(gz, Tz)], \frac{1}{2}[p(fx_{2n}, Tz) + p(gz, Sx_{2n})] \right\}.$$

Now

$$p(gx_{2n+1}, Tz) \leq H_p(Sx_{2n}, Tz)$$
$$\leq r \max \left\{ \frac{1}{2}[p(fx_{2n}, gz), \frac{1}{2}[p(fx_{2n}, Sx_{2n}) + p(gz, Tz)], \frac{1}{2}[p(fx_{2n}, Tz) + p(gz, Sx_{2n})] \right\}.$$

Letting $n \to \infty$ besides using Lemma 2., (7) and (5), we get

$$(23) \quad p(z, Tz) \leq r \max \left\{ \frac{1}{2}[p(z, gz) + p(gz, Tz)], \frac{1}{2}[p(z, Tz) + p(gz, z)] \right\}.$$

By using (9) and $(p_2)$, we have

$$p(z, Tz) \leq rp(z, Tz),$$

which in turn yields that $p(z, Tz) = 0$. From Lemma 1.5 and (7), we have
shown that $z \in Tz = Tz$. Thus

(24) $z = gz \in Tz.$

Now, proceeding as in case $0 \leq r < \frac{1}{2}$, from (18) – (20), we obtain

(25) $z = fz \in Sz.$

Thus in view of (24) and (25), we have that $z$ is a common fixed point of $f, g, S$ and $T$.

Suppose that $z'$ is another common fixed point of $f, g, S$ and $T$. From (8), we have

(26) $p(z, z') = p(fz, gz') \leq p(z', Sz) \leq H_p(Sz, Tz').$

Using $p_2$, we have

$\varphi(r) \min \{p(fz, Sz), p(gz', Tz')\} \leq p(fz, gz').$

Hence by (2.6.1), we have

\[
H_p(Sz, Tz') \leq r \max \left\{ \frac{1}{2}[p(fz, Tz') + p(gz', Sz)], \frac{1}{2}[p(fz, Sz) + p(gz', Tz')], \frac{1}{2}[H_p(Sz, Tz') + H_p(Tz', Sz)] \right\},
\]

from (26)

Thus $H_p(Sz, Tz') = 0$, so that from (26), $z = z'$. Hence $z$ is the unique common fixed point of $f, g, S$ and $T$. Similarly, we can prove the theorem when $(f, g; T)$ satisfy the condition (W.C.C). This concludes the proof. 

References

Suzuki type common fixed point theorem for two pairs of hybrid maps...


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