Research Article

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On generalized Baskakov-Durrmeyer-Stancu type operators

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Abstract: In this paper, we study some local approximation properties of generalized Baskakov-Durrmeyer-Stancu operators. First, we establish a recurrence relation for the central moments of these operators, then we obtain a local direct result in terms of the second order modulus of smoothness. Further, we study the rate of convergence in Lipschitz type space and the weighted approximation properties in terms of the modulus of continuity, respectively. Finally, we investigate the statistical approximation property of the new operators with the aid of a Korovkin type statistical approximation theorem.

Keywords: Baskakov-Durrmeyer-Stancu operators, Lipschitz type space, rate of convergence, modulus of smoothness, A-statistical convergence

MSC: 26A15, 26A16, 41A25, 41A36

1 Introduction

Miheşan [1] introduced the following generalized Baskakov operators with a constant $a \geq 0$ independent of $n$:

$$B_n^a(f; x) = \sum_{k=0}^{\infty} W_{n,k}^a(x)f\left(\frac{k}{n}\right),$$

where $W_{n,k}^a(x) = e^{-ax} P_k(n, a) \frac{x^k}{k!} \left(1 + x\right)^n$ such that $\sum_{k=0}^{\infty} W_{n,k}^a(x) = 1$ and $P_k(n, a) = \sum_{i=0}^{k} \binom{k}{i} (n)_i a^{k-i}$ with $(n)_0 = 1$, $(n)_i = n(n+1)...(n+i-1)$ for $i \geq 1$. When $a = 0$, we recover the well-known Baskakov operators [2].

In [3], Waifi and Khatoon studied the rate of convergence of the generalized Baskakov operators in terms of the modulus of continuity, and obtained a Voronovskaja type theorem and a direct estimate of these operators in terms of the Ditzian-Totik modulus of smoothness, respectively. In [4], Erencin and Bascanbaz-Tunca studied the weighted approximation properties and estimated the order of approximation in terms of the usual modulus of continuity for the operators (1). In 2008, Waifi and Khatoon [5] obtained the convergence and a Voronovskaja type theorem for first derivatives of generalized Baskakov operators for functions of one and two variables in exponential and polynomial weighted spaces.

For $f \in C_b[0, \infty)$, the space of all bounded and continuous functions on $[0, \infty)$ equipped with the norm $\|f\| = \sup_{x \in [0, \infty)} |f(x)|$, Erencin [6] introduced the following Durrmeyer type modification of the operators (1):

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where $B(x, y)$ is the beta function given by

$$B(x, y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} \, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x > 0, y > 0.$$ 

For the operators (2), Erencin [6] established certain direct theorems in terms of the second order modulus of continuity, the elements of Lipschitz type space and the usual modulus of continuity. In 2014, Agrawal et al. [7] extended her study and discussed some direct results in simultaneous approximation by these operators obtaining pointwise convergence theorem, Voronovskaja type theorem and the error estimation by the modulus of continuity. They also obtained the error estimation in the approximation of functions having derivatives of bounded variation for the operators introduced in [6].

Let $\mathcal{L}$ denote the class of all Lebesgue measurable functions $f$ on $[0, \infty)$ such that

$$\int_0^\infty \frac{|f(t)|}{(1+t)^m} \, dt < \infty$$

for some positive integer $m$. Obviously $C_0[0, \infty) \subset \mathcal{L}$.

For $f \in \mathcal{L}$ and $n \in \mathbb{N}$, we define the Stancu type generalization of Durrmeyer type modification of the operators (2) as follows:

$$L_{n, a}^{\alpha, \beta}(f; x) = \sum_{k=0}^\infty W_{n,k}^a(x) \frac{1}{B(k+1, n)} \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} f(t) \, dt,$$

where $\alpha$ and $\beta$ are non-negative numbers and $n > m$ with $\int_0^\infty \frac{|f(t)|}{(1+t)^m} \, dt < \infty$.

The aim of this article is to present for the operators defined by (3) some direct local approximation results in terms of the second order modulus of smoothness, the elements of Lipschitz type space, the weighted space and the degree of approximation. We also obtain a statistical approximation result for (3) with the aid of a Korovkin type statistical approximation theorem given in [8].

### 2 Auxiliary results

**Lemma 1.** ([7]) For every $x \in (0, \infty)$, we have

$$x(1+x)^2 \left\{ \frac{d}{dx} W_{n,k}^a(x) \right\} = \left\{ (k-nx)(1+x) - ax \right\} W_{n,k}^a(x).$$

For $m \in \mathbb{N}^0 := \mathbb{N} \cup \{0\}$, the $m$–th order moments of the operators (3) are defined by

$$T_{n,m}^{a,\alpha, \beta}(x) := L_{n,a}^{\alpha, \beta}(t^m; x) = \sum_{k=0}^\infty W_{n,k}^a(x) \frac{1}{B(k+1, n)} \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} \left( \frac{nt + \alpha}{n + \beta} \right)^m \, dt, \quad n > m.$$

**Lemma 2.** For the functions $T_{n,m}^{a,\alpha, \beta}(x)$, where $x \in (0, \infty)$, we have $T_{n,0}^{a,\alpha, \beta}(x) = 1$,

$$T_{n,1}^{a,\alpha, \beta}(x) = \frac{n}{(n + \beta)(n-1)} \left( nx + \frac{ax}{1+x} + 1 \right) + \frac{\alpha}{n + \beta}, \quad n > 1,$$

and there holds the following recurrence relation for $n > m \geq 1$:

$$x(1+x)(T_{n,m}^{a,\alpha, \beta}(x))' = T_{n,m}^{a,\alpha, \beta}(x) \left( -(m+1) + \frac{\alpha}{n} (2m - n + 1) - \frac{ax}{1+x} - nx \right)$$

$$+ T_{n,m+1}^{a,\alpha, \beta}(x)(n-m-1) \frac{n + \beta}{n} + T_{n,m+1}^{a,\alpha, \beta}(x) \frac{ma}{n + \beta} \left( 1 - \frac{\alpha}{n} \right). \quad (4)$$
Proof. Let \( x \in (0, \infty) \) be given. Then, by (1) (see also [7]), we have \( B_\alpha^n (1; x) = 1 \) and \( B_\alpha^n (t; x) = x + \frac{\alpha}{n(1 + x)} \). Hence

\[
T_{a,n,\beta}^a(x) = \sum_{k=0}^n \frac{W_{a,n,k}(x)}{B(k+1,n)} \int_0^\infty \frac{t^k}{(1 + t)^{n+k+1}} \frac{nt + \alpha}{n + \beta} \, dt
\]

Further, from Lemma 1, we obtain

\[
\left( T_{a,n,\beta}^a(x) \right)' = \sum_{k=0}^\infty \frac{d}{dx} \left( \frac{W_{a,n,k}(x)}{B(k+1,n)} \right) \int_0^\infty \frac{t^k}{(1 + t)^{n+k+1}} \left( \frac{nt + \alpha}{n + \beta} \right)^m \, dt
\]

We may write \( I_1 \) as

\[
I_1 = \frac{1}{x(1 + x)} \sum_{k=0}^\infty \frac{W_{a,n,k}(x)}{B(k+1,n)} \int_0^\infty \frac{t^k}{(1 + t)^{n+k+1}} \frac{t(k - (n + 1)t + (n + 1)tnx)}{(n + \beta)} \left( \frac{nt + \alpha}{n + \beta} \right)^m \, dt
\]

Using the equality

\[
t(1 + t) \frac{d}{dt} \left( \frac{t^k}{(1 + t)^{n+k+1}} \right) = \{ k - (n + 1)t \} \frac{t^k}{(1 + t)^{n+k+1}}
\]

and integrating by parts, then taking into account the identity \( t = \frac{n + \beta}{n} \left( \frac{nt + \alpha}{n + \beta} - \frac{\alpha}{n + \beta} \right) \), we get

\[
I_2 = \frac{-1}{x(1 + x)} \left( \frac{m a}{n + \beta} T_{a,n,\beta}^a(x) - \frac{ma^2}{n(n + \beta)} T_{a,n,\beta}^a(x) + (m + 2) \frac{n + \beta}{n} T_{a,n,\beta}^a(x) \right)
\]

Further, from Lemma 1, we obtain

\[
\left( T_{a,n,\beta}^a(x) \right)' = \sum_{k=0}^\infty \frac{W_{a,n,k}(x)}{B(k+1,n)} \int_0^\infty \frac{t^k}{(1 + t)^{n+k+1}} \left( \frac{nt + \alpha}{n + \beta} \right)^m \, dt
\]

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\]

Using the equality

\[
t(1 + t) \frac{d}{dt} \left( \frac{t^k}{(1 + t)^{n+k+1}} \right) = \{ k - (n + 1)t \} \frac{t^k}{(1 + t)^{n+k+1}}
\]

and integrating by parts, then taking into account the identity \( t = \frac{n + \beta}{n} \left( \frac{nt + \alpha}{n + \beta} - \frac{\alpha}{n + \beta} \right) \), we get

\[
I_2 = \frac{-1}{x(1 + x)} \left( \frac{m a}{n + \beta} T_{a,n,\beta}^a(x) - \frac{ma^2}{n(n + \beta)} T_{a,n,\beta}^a(x) + (m + 2) \frac{n + \beta}{n} T_{a,n,\beta}^a(x) \right)
\]
Analogously for $I_3$, we have

$$I_3 = \frac{n + 1}{x(1 + x)} \sum_{k=0}^{\infty} \frac{W_{n,k}^a(x)}{B(k + 1, n)} \int_0^\infty \frac{t^k}{(1 + t)^{n+k+1}} \left[ \frac{nt + a}{n + \beta} \right]^{m+1} - \frac{a}{n + \beta} \left[ \frac{nt + a}{n + \beta} \right]^m \, dt$$

$$= \frac{n + 1}{x(1 + x)} \left[ \frac{n + \beta}{n} \alpha_n,\beta_m(x) - \frac{a}{n} \alpha_n,\beta_m(x) \right].$$

(8)

Now, combining (5)-(8), we find (4), which was to be proved.

For $m \in \mathbb{N}^0$, the $m$–th order central moments of the operators (3) are defined for $n > m$ as follows:

$$\mu_n^{a,\alpha,\beta}(x) := I_n^{a,\alpha,\beta}((t - x)^m; x) = \sum_{k=0}^{\infty} \frac{W_{n,k}^a(x)}{B(k + 1, n)} \int_0^\infty \frac{t^k}{(1 + t)^{n+k+1}} \left( \frac{nt + a}{n + \beta} - x \right)^m \, dt.$$

Lemma 3. For the functions $\mu_n^{a,\alpha,\beta}(x)$, where $x \in (0, \infty)$, we have $\mu_n^{a,\alpha,\beta}(x) = 1$,

$$\mu_n^{a,\alpha,\beta}(x) = \frac{(1 - \beta)n + \beta}{n} x + \frac{n}{(n + \beta)(n - 1)} \left( \frac{ax}{1 + x} + \frac{a}{n + \beta} \right), \quad n > 1,$$

and the following recurrence relation holds true for $n > m \geq 1$:

$$\frac{n + \beta}{n} (n - m - 1) \mu_{n,m}^{a,\alpha,\beta}(x) = x(1 + x) \left\{ (\mu_{n,m}^{a,\alpha,\beta}(x))' + m \mu_{n,m+1}^{a,\alpha,\beta}(x) \right\}$$

$$+ \mu_{n,m}^{a,\alpha,\beta}(x) \left\{ (m + 1) - 2(m + 1) \frac{n + \beta}{n} \frac{a}{n + \beta} - x \right\}$$

$$\geq m \mu_{n,m-1}^{a,\alpha,\beta}(x) \left( \frac{1}{n + \beta} x + 1 - \frac{n + \beta}{n} \left( \frac{a}{n + \beta} - x \right) \right).$$

(9)

Proof. For $x \in (0, \infty)$ and $n > 1$, we have, by Lemma 2, that

$$\mu_n^{a,\alpha,\beta}(x) = L_n^{a,\alpha,\beta}(t - x; x) = T_n^{a,\alpha,\beta}(x) - T_{n,0}^{\alpha,\beta}(x)$$

$$= \frac{n}{(n + \beta)(n - 1)} \left( nx + \frac{ax}{1 + x} + 1 \right) + \frac{a}{n + \beta} - x$$

$$= \frac{(1 - \beta)n + \beta}{n} x + \frac{n}{(n + \beta)(n - 1)} \left( \frac{ax}{1 + x} + 1 \right) + \frac{a}{n + \beta}.$$

Further, by Lemma 1, we may write

$$\left( \mu_n^{a,\alpha,\beta}(x) \right)' = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{W_{n,k}^a(x)}{B(k + 1, n)} \int_0^\infty \frac{t^k}{(1 + t)^{n+k+1}} \left( \frac{nt + a}{n + \beta} - x \right)^m \, dt$$

$$= \sum_{k=0}^{\infty} \frac{d}{dx} \left( W_{n,k}^a(x) \right) \int_0^\infty \frac{t^k}{(1 + t)^{n+k+1}} \left( \frac{nt + a}{n + \beta} - x \right)^m \, dt$$

$$= -m \sum_{k=0}^{\infty} \frac{d}{dx} W_{n,k}^a(x) \int_0^\infty \frac{t^k}{(1 + t)^{n+k+1}} \left( \frac{nt + a}{n + \beta} - x \right)^{m-1} \, dt$$

$$= I_1 - m \mu_{n,m+1}^{a,\alpha,\beta}(x).$$

(10)

In view of Lemma 1, we obtain

$$x(1 + x)^2 I_1 = \sum_{k=0}^{\infty} \frac{k}{x(1 + x)^2} \frac{d}{dx} \left( W_{n,k}^a(x) \right) \int_0^\infty \frac{t^k}{(1 + t)^{n+k+1}} \left( \frac{nt + a}{n + \beta} - x \right)^m \, dt$$

$$= \sum_{k=0}^{\infty} \frac{k}{x(1 + x)^2} \frac{d}{dx} W_{n,k}^a(x) \int_0^\infty \frac{t^k}{(1 + t)^{n+k+1}} \left( \frac{nt + a}{n + \beta} - x \right)^m \, dt$$

$$= (1 + x) I_2 - ax \mu_{n,m}^{a,\alpha,\beta}(x).$$

(11)
We may write \( I_2 \) as follows:

\[
I_2 = \sum_{k=0}^{\infty} \frac{W^a_{n,k}(x)}{B(k+1, n)} \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} \left\{ k - (n+1)t \right\} \left( \frac{nt + \alpha}{n + \beta} - x \right)^m \, dt
\]

\[
+ \sum_{k=0}^{\infty} \frac{W^a_{n,k}(x)}{B(k+1, n)} \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} (n+1)(t-x) \left( \frac{nt + \alpha}{n + \beta} - x \right)^m \, dt
\]

\[
+ \sum_{k=0}^{\infty} \frac{W^a_{n,k}(x)}{B(k+1, n)} \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} (n+1)x - nx \left( \frac{nt + \alpha}{n + \beta} - x \right)^m \, dt
\]

\[
=: I_3 + I_4 + x \mu_{n,m}^a(x). \tag{12}
\]

Now, using the identity \( t - x = \frac{n + \beta}{n} \left( \frac{nt + \alpha}{n + \beta} - x \right) - \frac{\alpha - \beta x}{n + \beta} \), we have

\[
I_4 = (n + 1) \frac{n + \beta}{n} \mu_{n,m}^a(x) - (n + 1) \frac{\alpha - \beta x}{n + \beta} \mu_{n,m}^a(x). \tag{13}
\]

Taking into account that \( t(1+t) \frac{d}{dt} \left( \frac{k^\beta}{(1+t)^{n+k+1}} \right) = (k - (n+1)t) \frac{k^\beta}{(1+t)^{n+k+1}} \) and integrating by parts, we obtain

\[
I_3 = -\sum_{k=0}^{\infty} \frac{W^a_{n,k}(x)}{B(k+1, n)} \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} \frac{d}{dt} \left\{ t(1+t) \left( \frac{nt + \alpha}{n + \beta} - x \right)^m \right\} \, dt.
\]

Again putting \( t - x = \frac{n + \beta}{n} \left( \frac{nt + \alpha}{n + \beta} - x \right) - \frac{\alpha - \beta x}{n + \beta} \), we obtain

\[
I_3 = -(m+1) \mu_{n,m}^a(x) + m \left( \frac{\alpha}{n + \beta} - x \right) \mu_{n,m}^a(x) - m \frac{n + \beta}{n} \left( \frac{\alpha}{n + \beta} - x \right) \mu_{n,m-1}^a(x)
\]

\[
- (m+2) \frac{n + \beta}{n} \mu_{n,m+1}^a(x) + 2(m+1) \frac{n + \beta}{n} \left( \frac{\alpha}{n + \beta} - x \right) \mu_{n,m}^a(x). \tag{14}
\]

Combining (10)-(14), we get the recurrence relation (9).

\[
\square
\]

### 3 Main results

**Theorem 1.** Let \( f \in L \) be continuous on the interval \([0, b]\), \( b > 0 \). Then the sequence \( \{L_{n,a}^t f\} \) converges to \( f \) uniformly on \([0, b]\) as \( n \to \infty \).

**Proof.** For \( x \in (0, \infty) \) and \( n \geq 2 \), we have, by Lemma 2, that \( L_{n,a}^t(1; x) = \eta_{n,0}^a(x) = 1 \),

\[
L_{n,a}^t(1; x) = \eta_{n,0}^a(x) = \frac{n}{(n+\beta)(n-1)} \left( nx + \frac{ax}{1+x} + 1 \right) + \frac{\alpha}{n+\beta} \tag{15}
\]

and

\[
L_{n,a}^t(t^2; x) = \eta_{n,2}^a(x) = \left( \frac{n}{n+\beta} \right)^2 \frac{1}{(n-1)(n-2)} \left( (n^2+nx^2) + \frac{a^2}{1+x} + \frac{2anx^2}{1+x} + \frac{4ax}{1+x} + 2 \right)
\]

\[
+ \frac{2an}{(n+\beta)^2} \frac{1}{n-1} \left( nx + \frac{ax}{1+x} + 1 \right) + \left( \frac{\alpha}{n+\beta} \right)^2. \tag{16}
\]

Hence

\[
|L_{n,a}^t(t; x) - x| \leq \left| \frac{n^2}{(n+\beta)(n-1)} - 1 \right| x + \frac{an}{(n+\beta)(n-1)} \frac{x}{1+x}
\]

\[
+ \frac{n}{(n+\beta)(n-1)} \frac{\alpha}{n+\beta} = \Theta \left( \frac{1}{n} \right). \tag{17}
\]
uniformly in $x \in (0, b]$, and

$$|L_{n,a}^{a,b}(t^2; x) - x^2| \leq \left| \frac{n^4 + n^3}{(n + \beta)^2(n - 1)(n - 2)} - 1 \right| x^2 + \frac{4n^3}{(n + \beta)^2(n - 1)(n - 2)} x$$

$$+ \frac{n^2}{(n + \beta)^2(n - 1)(n - 2)} (1 + x)^2 + \frac{n^3}{(n + \beta)^2(n - 1)(n - 2)} \frac{2a x^2}{1 + x}$$

$$+ \frac{4a}{(n + \beta)^2(n - 1)(n - 2)} \frac{2n^2}{1 + x} + \frac{n}{(n + \beta)^2(n - 1)} \frac{2an}{1 + x} + \frac{\left( \frac{a}{n + \beta} \right)^2}{\left( \frac{1}{n} \right)}.$$

(18)

uniformly in $x \in (0, b]$, respectively. Further $L_{n,a}^{a,b}(1; 0) = T_{n,0}^{a,a}(0) = 1$,

$$L_{n,a}^{a,b}(t; 0) = T_{n,1}^{a,a}(0) = \frac{1}{B(1, n)} \int_0^\infty \frac{1}{(1 + t)^{n+1}} \frac{nt + \alpha}{n + \beta} \, dt = \frac{n}{(n + \beta)(n - 1)} + \frac{\alpha}{n + \beta}$$

and

$$I_{n,a}^{a,b}(t^2; 0) = T_{n,2}^{a,a}(0) = \frac{1}{B(1, n)} \int_0^\infty \frac{1}{(1 + t)^{n+2}} \left( \frac{nt + \alpha}{n + \beta} \right) \, dt$$

$$= \frac{2n^2}{(n + \beta)^2(n - 1)(n - 2)} + \frac{2an}{(n + \beta)^2(n - 1)} + \frac{\left( \frac{a}{n + \beta} \right)^2}{\left( \frac{1}{n} \right)}.$$

Hence, using (17)-(18), the assertion of our theorem follows from Korovkin's theorem.

With the aid of $C_B[0, \infty)$ we introduce the space $C_B^2[0, \infty] = \{ g \in C_B[0, \infty] : g', g'' \in C_B[0, \infty] \}$. For $f \in C_B[0, \infty]$ and $\delta > 0$, the Peetre’s $K$–functional is defined as follows:

$$K_2(f, \delta) = \inf \{ ||f - g|| + \delta ||g'|| : g \in C_B^2[0, \infty] \}.\quad (19)$$

By [9, p. 177, Theorem 2.4], there exists an absolute constant $C > 0$ such that

$$K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}), \quad \delta > 0,$$

(20)

where the second order modulus of smoothness is defined by

$$\omega_2(f, \delta) = \sup_{0 < h < \delta} \sup_{x \in [0, \infty)} |f(x + 2h) - 2f(x + h) + f(x)|.$$

The usual modulus of continuity for $f \in C_B[0, \infty)$ is given by

$$\omega(f, \delta) = \sup_{0 < h < \delta} \sup_{x \in [0, \infty)} |f(x + h) - f(x)|.$$

**Theorem 2.** Let $f \in C_B[0, \infty)$ and $x \in (0, \infty)$. Then there exists a constant $C > 0$ independent of $n$ and $x$ such that

$$|L_{n,a}^{a,b}(f; x) - f(x)| \leq C \omega_2(f; \sqrt{\eta_{n,a}^{a,b}(x)}) + \omega(f; \eta_{n,a}^{a,b}(x)),$$

where $\eta_{n,a}^{a,b}(x) = \mu_{n,a}^{a,b}(x) + (\varphi_{n,a}^{a,b}(x))^2$.

**Proof.** First, we define the auxiliary operator

$$T_{n,a}^{a,b}(f; x) = L_{n,a}^{a,b}(f; x) + f(x) - f(T_{n,1}^{a,a}(x)).$$

We observe that $T_{n,a}^{a,b}(1; x) = 1$ and $L_{n,a}^{a,b}(t; x) = x$.

Let $g \in C_B^2[0, \infty)$. Using Taylor’s formula

$$g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - v)g''(v) \, dv$$

(21)
and $\mu_{n,1}^{a,\beta}(x) = T_{n,1}^{a,\beta}(x) - x T_{n,0}^{a,\beta}(x) = T_{n,1}^{a,\beta}(x) - x$, we get

$$|L_{n,a}^{\alpha,\beta}(g; x) - g(x)| = \left| L_{n,a}^{\alpha,\beta} \left( \int_x^1 (t - v)g''(v)\, dv; x \right) \right|$$

$$= \left| L_{n,a}^{\alpha,\beta} \left( \int_x^1 (t - v)g''(v)\, dv; x \right) - \int_x^1 T_{n,1}^{a,\beta}(t - x)^2 \, dt \right|$$

$$\leq \left| L_{n,a}^{\alpha,\beta}(t - x)^2; x \right| \|g''\| + \left( T_{n,1}^{a,\beta}(x) - x \right)^2 \|g''\|$$

$$= \eta_{n,a}^{\alpha,\beta}(x) \|g''\|.$$  \hspace{1cm} (22)

From (21), (3) and Lemma 2, we have

$$|T_{n,a}^{a,\beta}(f; x)| \leq |L_{n,a}^{a,\beta}(1; x)||f|| + 2||f|| \leq 3||f||.$$  \hspace{1cm} (23)

Let $g \in C_0^2[0, \infty)$, and using (22) and (23) in (21), we obtain

$$|L_{n,a}^{\alpha,\beta}(f; x) - f(x)| \leq 4||f - g|| + |L_{n,a}^{\alpha,\beta}(g; x) - g(x)| + |f(T_{n,1}^{a,\beta}(x)) - f(x)|$$

$$\leq 4||f - g|| + \eta_{n,a}^{\alpha,\beta}(x)\|g''\| + |f(T_{n,1}^{a,\beta}(x)) - f(x)|$$

$$\leq 4||f - g|| + \eta_{n,a}^{\alpha,\beta}(x)\|g''\| + \omega(f; |T_{n,1}^{a,\beta}(x) - x|).$$

Now, taking the infimum on the right hand side over all $g \in C_0^2[0, \infty)$, we obtain in view of (19), that

$$|L_{n,a}^{\alpha,\beta}(f; x) - f(x)| \leq 4K_2(f; \eta_{n,a}^{\alpha,\beta}(x)) + \omega(f; |T_{n,1}^{a,\beta}(x) - x|).$$

Thus, by (20), we get

$$|L_{n,a}^{\alpha,\beta}(f; x) - f(x)| \leq C\omega_2(f; \sqrt{\eta_{n,a}^{\alpha,\beta}(x)}) + \omega(f; |\mu_{n,1}^{a,\beta}(x)|),$$

which completes the proof. \hspace{1cm} \Box

**Remark 1.** It is worth mentioning that similar estimation to the estimation established in Theorem 2 could be obtained without using $K$--functionals. In this manner we mention the papers [10], [11], [12], [13], [14], [15] referring to the several extensions of the Baskakov/Durrmeyer/Stancu operators and several methods used in proofs.

Now, we consider the Lipschitz type space (see [6]):

$$Lip^*_M(r) := \left\{ f \in C_0[0, \infty) : |f(t) - f(x)| \leq M \frac{|t - x|}{(t + x)^r}, \; t \in [0, \infty), x \in (0, \infty) \right\},$$

where $M$ is a positive constant and $0 < r \leq 1$.

**Theorem 3.** Let $f \in Lip^*_M(r)$ and $0 < r \leq 1$. Then, for all $x \in (0, \infty)$, we have

$$|L_{n,a}^{\alpha,\beta}(f; x) - f(x)| \leq M \left( \mu_{n,2}^{a,\beta}(x) \right)^{\frac{r}{x}}.$$

where $\mu_{n,2}^{a,\beta}(x) = L_{n,a}^{a,\beta}((t - x)^2; x)$.

**Proof.** First, we prove that the result is true for $r = 1$. Then, for $f \in Lip^*_M(1)$, we obtain

$$|L_{n,a}^{\alpha,\beta}(f; x) - f(x)| \leq \sum_{k=0}^{\infty} \frac{W_{n,k}^a(x)}{B(k+1,n)} \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} \left| \left( \frac{nt + a}{n + \beta} \right) - f(x) \right| \, dt$$

$$\leq M \sum_{k=0}^{\infty} \frac{W_{n,k}^a(x)}{B(k+1,n)} \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} \left| \frac{nt + a}{n + \beta} \right| \, dt.$$
Using $\sqrt{x} < \sqrt{\frac{nt + a}{n+\beta}} + x$ and the Cauchy-Schwarz inequality, we get

$$|L_{n,a,b}^\alpha(f; x) - f(x)| \leq M \sqrt{\frac{x}{\sqrt{M}}} \sum_{k=0}^\infty \frac{W_{n,k}^\alpha(x)}{B(k+1, n)} \int_0^\infty \frac{t^k}{(1 + t)^{n+\beta+1}} \left| \frac{nt + a}{n+\beta} - x \right| dt$$

$$= M \sqrt{\frac{x}{\sqrt{M}}} \frac{\mu_{n,2}^\alpha(x)}{\sqrt{x}}.$$  

Therefore the result is true for $r = 1$. Now, we prove that the result is true for $0 < r < 1$. Applying the Hölder’s inequality two times with $p = \frac{1}{r}$ and $q = \frac{1}{1-r}$, we have

$$|L_{n,a,b}^\alpha(f; x) - f(x)| \leq \sum_{k=0}^\infty \frac{W_{n,k}^\alpha(x)}{B(k+1, n)} \int_0^\infty \left( \frac{t^k}{(1 + t)^{n+\beta+1}} \right)^{\frac{1}{r}} \left| f\left( \frac{nt + a}{n+\beta} \right) - f(x) \right|^{\frac{1}{r}} dt$$

$$\leq \sum_{k=0}^\infty \frac{W_{n,k}^\alpha(x)}{B(k+1, n)} \int_0^\infty \frac{t^k}{(1 + t)^{n+\beta+1}} \left| f\left( \frac{nt + a}{n+\beta} \right) - f(x) \right| \left( \frac{nt + a}{n+\beta} \right)^{\frac{1}{r}} dt$$

$$\leq \sum_{k=0}^\infty \frac{W_{n,k}^\alpha(x)}{B(k+1, n)} \int_0^\infty \frac{t^k}{(1 + t)^{n+\beta+1}} \left| f\left( \frac{nt + a}{n+\beta} \right) - f(x) \right|^r dt.$$  

Since $f \in Lip^M(r)$, we have

$$|L_{n,a,b}^\alpha(f; x) - f(x)| \leq \frac{M}{x^r} \left( \sum_{k=0}^\infty \frac{W_{n,k}^\alpha(x)}{B(k+1, n)} \int_0^\infty \frac{t^k}{(1 + t)^{n+\beta+1}} \left| \frac{nt + a}{n+\beta} - x \right| dt \right)^r$$

$$= \frac{M}{x^r} \frac{\mu_{n,2}^\alpha(x)}{\sqrt{x}} L_{n,a,b}^{\alpha,\beta}(t-x),$$

which completes the proof.

In what follows we study the weighted approximation properties of (3). Let $B_p[0, \infty)$ be the space of all real valued functions $f$ defined on $[0, \infty)$ satisfying the condition $|f(x)| \leq M_j \rho(x)$, $x \in [0, \infty)$, with some constant $M_j$ depending only on $f$, and $\rho(x) = 1 + x^2$, $x \in [0, \infty)$, is a weight function. Let $C_p[0, \infty)$ be the space of all continuous functions in $B_p[0, \infty)$ with the norm

$$||f||_p = \sup_{x \in [0, \infty)} \frac{|f(x)|}{\rho(x)}$$ and $C_p^0[0, \infty) = \left\{ f \in C_p[0, \infty) : \lim_{x \to \infty} \frac{|f(x)|}{\rho(x)} < \infty \right\}.$

For $f \in C[0, b]$, $b > 0$, the usual modulus of continuity is defined by

$$\omega_b(f, \delta) = \sup_{x \in [0, b]} |f(t) - f(x)|.$$

**Theorem 4.** If $f \in C_p[0, \infty)$, then we have

$$|L_{n,a,b}^{\alpha,\beta}(f, x) - f(x)| \leq 4M_j(1 + b^2)\mu_{n,2}^{\alpha,\beta}(x) + 2\omega_b(f, \sqrt{\mu_{n,2}^{\alpha,\beta}(x)}),$$

where $\mu_{n,2}^{\alpha,\beta}(x) = L_{n,a,b}^{\alpha,\beta}(t-x^2); x$ and $x \in [0, b]$.

**Proof.** Let $x \in [0, b]$ and $t > b + 1$. Then, in view of $t - x > 1$, we have

$$|f(t) - f(x)| \leq M_j(2 + t^2 + x^2) = M_j(2 + 2x^2 + 2x(t-x) + (t-x)^2)$$

$$\leq M_j(t-x)^2(3 + 2x + 2x^2) \leq 4M_j(t-x)^2(1 + x^2)$$

$$\leq 4M_j(t-x)(1 + b^2).$$  

(24)
For \( x \in [0, b] \) and \( t \leq b + 1 \), we have
\[
|f(t) - f(x)| \leq \omega_{b+1}(f, |t - x|) \leq \left(1 + \frac{|t - x|}{\delta}\right)\omega_{b+1}(f, \delta), \quad \delta > 0.
\] 
(25)

In view of (24) and (25), we obtain
\[
|f(t) - f(x)| \leq 4M_f(1 + b^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right)\omega_{b+1}(f, \delta).
\]

Now, using the Cauchy-Schwarz inequality, we have
\[
|L_{n,a}^\alpha(x) - f(x)| \leq 4M_f(1 + b^2)L_{n,a}^\alpha((t - x)^2, x) + \left(1 + \frac{1}{\delta}\right)\omega_{b+1}(f, \delta).
\]

Choosing \( \delta = \sqrt{\mu_{n,2}^\alpha(x)} \), we get the required result. \( \square \)

**Theorem 5.** Let \( f \in C_\rho^0[0, \infty) \). Then we have
\[
\lim_{n \to \infty} \|L_{n,a}^\alpha(f) - f\|_\rho = 0.
\]

**Proof.** Following [16] and [17], it is sufficient to verify the following conditions:
\[
\lim_{n \to \infty} \|L_{n,a}^\alpha(t^i; x) - x^i\|_\rho = 0 \quad \text{for} \quad i \in \{0, 1, 2\}.
\] 
(26)

Since \( L_{n,a}^\alpha(1; x) = 1 \), therefore (26) holds true for \( i = 0 \).

Using (15), we have for \( n > 1 \) that
\[
\|L_{n,a}^\alpha(t; x) - x\|_\rho = \sup_{\rho \in (0, \infty)} \left| \frac{n^2}{(n + \beta)(n - 1)} - 1 \right| + \frac{an}{(n + \beta)(n - 1)} \frac{x}{1 + x} + \frac{n}{(n + \beta)(n - 1)} + \frac{a}{n + \beta} \left| \frac{1}{1 + x^2} \right|
\]
(27)

Applying the limit on both sides as \( n \to \infty \), the condition (26) is satisfied for \( i = 1 \).

Further, using (16), we obtain for \( n > 2 \) that
\[
\|L_{n,a}^\alpha(t^2; x) - x^2\|_\rho = \sup_{\rho \in (0, \infty)} \left| \frac{n^3}{(n + \beta)^2(n - 1)(n - 2)} - 1 \right| + \frac{an^3}{(n + \beta)^2(n - 1)(n - 2)} \frac{x^2}{1 + x^2} + \frac{2an^2}{(n + \beta)^2(n - 1)(n - 2)} \frac{x^2}{1 + x^2} + \frac{2an}{(n + \beta)^2(n - 1)(n - 2)} \frac{x}{1 + x^2} + \frac{2an}{(n + \beta)^2(n - 1)(n - 2)} \frac{x^2}{1 + x^2} + \frac{2an}{(n + \beta)^2(n - 1)(n - 2)} \frac{x^2}{1 + x^2} + \frac{2an}{(n + \beta)^2(n - 1)(n - 2)} \frac{x^2}{1 + x^2}
\]
which implies that \( \lim_{n \to \infty} \|L_{n,a}^\alpha(t^2; x) - x^2\|_\rho = 0 \). This completes the proof of the theorem. \( \square \)

Next we establish a theorem to approximate all functions in \( C_\rho[0, \infty) \). Such type of results are discussed in [18] for locally integrable functions.
Theorem 6. For each $f \in C_p[0, \infty]$ and $\gamma > 0$, we have

$$\lim_{n \to \infty} \sup_{x \in [0, \infty)} \frac{|L_{n,a}^{\alpha,\beta}(f; x) - f(x)|}{(1 + x^2)^{1+\gamma}} = 0.$$ 

Proof. For any fixed $x_0 > 0$, we may write, in view of $|f(t)| \leq M_f(1 + t^2)$, $t \in [0, \infty)$, that

$$\sup_{x \in [0, \infty)} \frac{|L_{n,a}^{\alpha,\beta}(f; x) - f(x)|}{(1 + x^2)^{1+\gamma}} \leq \sup_{x \leq x_0} \frac{|L_{n,a}^{\alpha,\beta}(f; x) - f(x)|}{(1 + x^2)^{1+\gamma}} + \sup_{x > x_0} \frac{|L_{n,a}^{\alpha,\beta}(f; x) - f(x)|}{(1 + x^2)^{1+\gamma}} \leq M_f \sup_{x \leq x_0} \frac{L_{n,a}^{\alpha,\beta}(1 + t^2; x)}{(1 + x^2)^{1+\gamma}} + \frac{|f(x)|}{(1 + x^2)^{1+\gamma}},$$

(29)

Let $\varepsilon > 0$ be arbitrary. By Theorem 1, for $n$ sufficiently large, we have

$$\sup_{x \leq x_0} |L_{n,a}^{\alpha,\beta}(f; x) - f(x)| < \frac{\varepsilon}{3}.$$ 

(30)

Further, in view of (16), we obtain for $n > 2$ that

$$M_f \sup_{x \leq x_0} \frac{L_{n,a}^{\alpha,\beta}(1 + t^2; x)}{(1 + x^2)^{1+\gamma}} = M_f \sup_{x \leq x_0} \frac{1 + t^2 (t^2; x)}{(1 + x^2)^{1+\gamma}} \leq \frac{M_f}{(1 + x_0^2)^{1+\gamma}} \sup_{x \leq x_0} \frac{1 + x^2}{1 + x^2} \left\{ \frac{n^3 + n^3}{4n^3} + \frac{n^2}{(n + \beta)^2(n - 1)(n - 1) + 1} \frac{x^2}{a^2 x^2} + \frac{n^3}{(n + \beta)^2(n - 1)(n - 1) + 1} \frac{2ax^2}{(n + \beta)^2(n - 1)(n - 1) + 1} \frac{x}{a^2 x} + \frac{2n^2}{(n + \beta)^2(n - 1)(n - 1) + 1} \frac{x^2}{a^2 x} + \frac{2\alpha^2}{(n + \beta)^2(n - 1)(n - 1) + 1} \frac{x}{a^2 x} + \frac{2\alpha^2}{(n + \beta)^2(n - 1)(n - 1) + 1} \right\} \leq \frac{M_f}{(1 + x_0^2)^{1+\gamma}} \left( 5 + 6 + a^2 + 3a + 4a + 2 + 2a + 2aa + 2a + a^2 \right)$$

$$= \frac{M_f}{(1 + x_0^2)^{1+\gamma}} \left( 17 + a^2 + 7a + 4a + 2aa + a^2 \right).$$

We can choose $x_0$ to be so large that

$$\frac{M_f}{(1 + x_0^2)^{1+\gamma}} \left( 17 + a^2 + 7a + 4a + 2aa + a^2 \right) < \frac{\varepsilon}{3}.$$ 

(31)

Thus

$$M_f \sup_{x \leq x_0} \frac{L_{n,a}^{\alpha,\beta}(1 + t^2; x)}{(1 + x^2)^{1+\gamma}} < \frac{\varepsilon}{3}.$$ 

(32)

Finally, since $|f(x)| \leq M_f(1 + x^2)$ for $x \in [0, \infty)$, we get from (31), that

$$\sup_{x \leq x_0} \frac{|f(x)|}{(1 + x^2)^{1+\gamma}} \leq \sup_{x \leq x_0} \frac{M_f}{(1 + x^2)^{1+\gamma}} \leq \frac{M_f}{(1 + x_0^2)^{1+\gamma}} < \frac{\varepsilon}{3}.$$ 

(33)

Combining (29)-(30), (32)-(33), we obtain the required result. 

Finally, we obtain a Korovkin type statistical approximation theorem for the operators defined by (3). Let $A = (a_{nk})$ be a non-negative infinite summability matrix. For a given sequence $x := \{x_k\}$, the $A$-transform of $x$ denoted by $Ax := (Ax)_n$ is defined as $(Ax)_n = \sum_{k=1}^\infty a_{nk} x_k$ provided that the series converges for each $n$. We say that $A$ is regular, if $\lim_n (Ax)_n = L$ whenever $\lim_n x_n = L$. Then $x = \{x_n\}$ is said to be $A$-statistically
convergent to \( L \), i.e. \( s_{tA} - \lim_n x_n = L \) if, for every \( \varepsilon > 0 \), \( \lim_n \sum_{k \in [1-L,\infty]} a_{nk} = 0 \). If \( A = C_1 \), the Cesàro matrix of order one, then the \( A \)-statistical convergence reduces to the statistical convergence. The statistical convergence of other positive linear operators have been studied intensively by several authors (see e.g. [19], [20], [21], [8]).

**Theorem 7.** Let \( a_{nk} \) be a non-negative regular summability matrix. Then, for all \( f \in C_{0,1} \), we have

\[
st_A - \lim_n \|L_{n,A}^{a,b}(f; x) - f(x)\|_\rho = 0.
\]

**Proof.** Following [8, p. 191, Theorem 3], it is enough to prove that

\[
st_A - \lim_n \|L_{n,A}^{a,b}(t^i; x) - x^i\|_\rho = 0,
\]

where \( x \in [0, 1] \) and \( i \in \{0, 1, 2\} \).

In view of Lemma 2, it is easy to see that

\[
st_A - \lim_n \|L_{n,A}^{a,b}(1; x) - 1\|_\rho = 0. \tag{34}
\]

By (27), we have

\[
\|L_{n,A}^{a,b}(e_1; x) - x\|_\rho \leq \frac{(1 - \beta)n + \beta}{(n + \beta)(n - 1)} + \frac{(1 + a)n}{(n + \beta)(n - 1)} + \frac{\alpha}{n + \beta}.
\]

For each \( \varepsilon > 0 \), we define the following sets:

\[
E_1 := \left\{ n : \|L_{n,A}^{a,b}(t; x) - x\|_\rho \geq \varepsilon \right\},
\]

\[
E_2 := \left\{ n : \frac{(1 - \beta)n + \beta}{(n + \beta)(n - 1)} \geq \frac{\varepsilon}{3} \right\},
\]

\[
E_3 := \left\{ n : \frac{(1 + a)n}{(n + \beta)(n - 1)} \geq \frac{\varepsilon}{3} \right\},
\]

\[
E_4 := \left\{ n : \frac{\alpha}{n + \beta} \geq \frac{\varepsilon}{3} \right\}.
\]

It is clear that \( E_1 \subseteq E_2 \cup E_3 \cup E_4 \), which implies that

\[
\sum_{k \in E_1} a_{nk} \leq \sum_{k \in E_2} a_{nk} + \sum_{k \in E_3} a_{nk} + \sum_{k \in E_4} a_{nk}.
\]

Hence, we have

\[
st_A - \lim_n \|L_{n,A}^{a,b}(t; x) - x\|_\rho = 0. \tag{35}
\]

Using (28), we get

\[
\|L_{n,A}^{a,b}(t^2; x) - x^2\|_\rho \leq \frac{n^4}{(n + \beta)^2(n - 1)(n - 2)} - 1 + \frac{n^3(5 + 2a)}{(n + \beta)^2(n - 1)(n - 2)}
\]

\[
+ \frac{(a^2 + 4a + 2)n^2}{(n + \beta)^2(n - 1)(n - 2)} + \frac{2an^2}{(n + \beta)^2(n - 1)} + \frac{2an(a + 1)}{(n + \beta)^2(n - 1)} + \left( \frac{\alpha}{n + \beta} \right)^2.
\]

Now, we define the following sets:

\[
E_5 := \left\{ n : \|L_{n,A}^{a,b}(t^2; x) - x^2\|_\rho \geq \varepsilon \right\},
\]

\[
E_6 := \left\{ n : \frac{n^4}{(n + \beta)^2(n - 1)(n - 2)} \geq \frac{\varepsilon}{6} \right\},
\]

\[
E_7 := \left\{ n : \frac{n^3(5 + 2a)}{(n + \beta)^2(n - 1)(n - 2)} \geq \frac{\varepsilon}{6} \right\},
\]

\[
E_8 := \left\{ n : \frac{(a^2 + 4a + 2)n^2}{(n + \beta)^2(n - 1)(n - 2)} \geq \frac{\varepsilon}{6} \right\},
\]

\[
E_9 := \left\{ n : \frac{2an^2}{(n + \beta)^2(n - 1)} \geq \frac{\varepsilon}{6} \right\},
\]

\[
E_{10} := \left\{ n : \frac{2an(a + 1)}{(n + \beta)^2(n - 1)} \geq \frac{\varepsilon}{6} \right\},
\]

\[
E_{11} := \left\{ n : \left( \frac{\alpha}{n + \beta} \right)^2 \geq \frac{\varepsilon}{6} \right\}.
\]
Then, we have $E_5 \subseteq E_6 \cup E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11}$, which implies that
\[
\sum_{k \in E_5} a_{nk} \leq \sum_{k \in E_6} a_{nk} + \sum_{k \in E_7} a_{nk} + \sum_{k \in E_8} a_{nk} + \sum_{k \in E_9} a_{nk} + \sum_{k \in E_{10}} a_{nk} + \sum_{k \in E_{11}} a_{nk}.
\]
Hence, we get
\[
\text{st}_A - \lim_n \|L_{n,\alpha}^\beta(t^2; x) - x^2\|_\rho = 0. \tag{36}
\]
Combining (34)-(36), we obtain the statement of our theorem.

References