Research Article

Mohammed M.M. Jaradat*, Zead Mustafa, Sami Ullah Khan, Muhammad Arshad, and Jamshaid Ahmad

Some fixed point results on $G$-metric and $G_b$-metric spaces

https://doi.org/10.1515/dema-2017-0018
Received April 16, 2017; accepted June 20, 2017

Abstract: The purpose of this paper is to prove some fixed point results using $JS$-$G$-contraction on $G$-metric spaces, also to prove some fixed point results on $G_b$-complete metric space for a new contraction. Our results extend and improve some results in the literature. Moreover, some examples are presented to illustrate the validity of our results.

Keywords: fixed point, $G$-metric space, $G_b$-metric space, $JS$-$G$-contraction

MSC: Primary 47H10; Secondary 54H25.

1 Introduction

Mustafa and Sims [1] introduced the notion of $G$-metric spaces as a generalization of classical metric spaces and obtained some fixed point theorems for mappings satisfying different generalized contractive conditions. Thereafter, the concept of $G$-metric space has been studied and used to obtain various fixed point theorems by several mathematicians (see ([2–24]).

Definition 1.1. [1] Let $X$ be a non empty and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties

(G1) $G(a, b, c) = 0$ if $a = b = c$,
(G2) $0 < G(a, a, b)$ for all $a, b \in X$ with $a \neq b$,
(G3) $G(a, a, b) \leq G(a, b, c)$ for all $a, b, c \in X$ with $b \neq c$,
(G4) $G(a, b, c) = G(a, c, b) = G(b, c, a) = \cdots$ (symmetry in all three variables),
(G5) $G(a, b, c) \leq G(a, w, w) + G(w, b, c)$ for all $a, b, c, w \in X$ (rectangle inequality).

Then the function $G$ is called a generalized metric, or, a $G$-metric on $X$ and the pair $(X, G)$ is called a $G$-metric space. Throughout this paper we mean by $\mathbb{N}$ the set of all Natural Numbers.

*Corresponding Author: Mohammed M.M. Jaradat: Department of Mathematics, Statistics and Physics, Qatar University, Doha-Qatar, E-mail: mmjst4@qu.edu.qa
Zead Mustafa: Department of Mathematics, Statistics and Physics, Qatar University, Doha-Qatar, E-mail: zead@qu.edu.qa and Department of Mathematics, The Hashemite University, Zarqa- Jordan, E-mail: zmagabih@hu.edu.jo
Sami Ullah Khan: Department of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan and Department of Mathematics, Gomal University D. I. Khan, KPK, Pakistan, E-mail: gomal85@gmail.com
Muhammad Arshad: Department of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan, E-mail: marshad_zia@yahoo.com
Jamshaid Ahmad: Department of Mathematics, University of Jeddah, P.O.Box 80327, Jeddah 21589, Saudi Arabia, E-mail: jamshaid_jasim@yahoo.com

© 2017 Mohammed M.M. Jaradat et al., published by De Gruyter Open. This work is licensed under the Creative Commons Attribution-NonCommercial-NoDerivs 3.0 License.
Definition 1.2. [1] Let $(X, G)$ be a $G$-metric space, and let $(a_n)$ be a sequence of points of $X$. Then we say that 
$(a_n)$ is $G$-convergent to $a \in X$ if $\lim_{n, m \to \infty} G(a, a_n, a_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that 
$G(a, a_n, a_m) < \varepsilon$ for all $n, m \geq N$. We call $a$ the limit of the sequence and write $a_n \to x$ or $\lim_{n \to \infty} a_n = a$.

Proposition 1.3. [1] Let $(X, G)$ be a $G$-metric space. The following statements are equivalent:

1. $(a_n)$ is $G$-convergent to $a$.
2. $G(a_n, a_n, a) \to 0$ as $n \to +\infty$.
3. $G(a_n, a, a) \to 0$ as $n \to +\infty$.
4. $G(a_n, a_m, a) \to 0$ as $n, m \to +\infty$.

Definition 1.4. [1] Let $(X, G)$ be a $G$-metric space. A sequence $(a_n)$ is called a $G$-Cauchy sequence if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(a_n, a_m, a_l) < \varepsilon$ for all $n, m, l \geq N$, that is $G(a_n, a_m, a_l) \to 0$ as $n, m, l \to +\infty$.

Definition 1.5. [1] A $G$-metric space $(X, G)$ is called $G$-complete if every $G$-Cauchy sequence is $G$-convergent in $(X, G)$.

Corollary 1.6. [1] Let $(X, d)$ be a metric space, then $(X, d)$ is complete metric space iff $(X, G_m)$ is complete $G$-metric space where 
$$G_m(a, b, c) = \max\{d(a, b), d(b, c), d(a, c)\}$$

Corollary 1.7. [1] A $G$-metric space $(X, G)$ is continuous on its three variables.

Very recently, Jleli and Samet [25] introduced a new type of contraction which involves the following set of all functions $\psi : (0, \infty) \to (1, \infty)$ satisfying the conditions:

- $(\psi_1)$ $\psi$ is nondecreasing;
- $(\psi_2)$ for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n \to \infty} \psi(t_n) = 1$ if and only if $\lim_{n \to \infty} t_n = 0$;
- $(\psi_3)$ there exist $r \in (0, 1)$ and $L \in (0, \infty]$ such that $\lim_{t \to 0^+} \frac{\psi(t) - 1}{t} = L$.

To be consistent with Jleli and Samet [25], we denote by $Ϝ$ the set of all functions $\psi : (0, \infty) \to (1, \infty)$ satisfying the conditions $(\psi_1 - \psi_3)$.

Also, they established the following result as a generalization of Banach Contraction Principle.

Theorem 1.8. [25, Corollary 2.1] Let $(X, d)$ be a complete metric space and $f : X \to X$ be a mapping. Suppose that there exist $\psi \in \Psi$ and $k \in (0, 1)$ such that 
$$x, y \in X, \quad d(fx, fy) \neq 0 \Rightarrow \psi(d(fx, fy)) \leq \psi(d(x, y))^k.$$ 

Then $f$ has a unique fixed point.

In 2015, Hussain et al. [26] customized the above family of functions and proved a fixed point theorem as a generalization of [25]. They customized the family of functions $\psi : [0, \infty) \to [1, \infty)$ to be as follows:

- $(\psi_1)$ $\psi$ is nondecreasing and $\psi(t) = 1$ if and only if $t = 0$;
- $(\psi_2)$ for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n \to \infty} \psi(t_n) = 1$ if and only if $\lim_{n \to \infty} t_n = 0$;
- $(\psi_3)$ there exist $r \in (0, 1)$ and $L \in (0, \infty]$ such that $\lim_{t \to 0^+} \frac{\psi(t) - 1}{t} = L$;
- $(\psi_4)$ $\psi(u + v) \leq \psi(u) \psi(v)$ for all $u, v > 0$.

To be consistent with Hussain et al. [26], we denote by $\Psi$ the set of all functions $\psi : [0, \infty) \to [1, \infty)$ satisfying the conditions $(\psi_1 - \psi_4)$. For more details in this direction, we refer the reader to [27–30].

In this paper, we introduce a new contraction called $JS$-$G$-contraction and we prove some fixed point results of such contraction in the setting of $G$-metric spaces, also we prove some fixed point results on $G_b$-complete metric space for a new contraction.
2 Fixed Point Results on $G$-Metric Space

We start this section by introducing the following definition.

**Definition 2.1.** Let $(X, G)$ be a $G$-metric space, and let $g : X \to X$ be a self mapping. Then $g$ is said to be a $JS$-$G$-contraction whenever there exist a function $\psi \in \mathcal{P}$ and positive real numbers $r_1, r_2, r_3, r_4$ with $0 \leq r_4 + 3r_2 + r_3 + 2r_4 < 1$ such that

$$
\psi(G(ga, gb, gc)) \leq [\psi(G(a, b, c))]^{r_1} [\psi(G(a, ga, gc))]^{r_2} [\psi(G(b, gb, gc))]^{r_3} \\
\times [\psi(G(a, gb) + G(b, ga))]^{r_4},
$$

(2.1)

for all $a, b, c \in X$.

**Theorem 2.2.** Let $(X, G)$ be a complete $G$-metric space and $g : X \to X$ be a $JS$-$G$-contraction. Then $g$ has a unique fixed point.

**Proof.** Let $a_0 \in X$ be arbitrary. For $a_0 \in X$, we define the sequence $\{a_n\}$ by $a_n = g^n a_0 = ga_{n-1}$. If there exist $n_0 \in \mathbb{N}$ such that $a_{n_0} = a_{n_0+1}$, then $a_{n_0}$ is a fixed point of $g$, and we have nothing to prove. Thus, we suppose that $a_n \neq a_{n+1}$, i.e., $G(ga_{n-1}, ga_n, ga_n) > 0$ for all $n \in \mathbb{N}$. Now, we will prove that $\lim_{n \to \infty} G(a_n, a_{n-1}, a_{n+1}) = 0$.

Since $g$ is a $JS$-$G$-contraction, by using condition (2.1), we get that

$$
1 < \psi(G(a_n, a_{n+1}, a_{n+1})) = \psi(G(ga_{n-1}, ga_n, ga_n)) \\
\leq [\psi(G(a_{n-1}, a_{n+1}, a_{n+1}))]^{r_1} [\psi(G(ga_{n-1}, ga_n, ga_n))]^{r_2} [\psi(G(a_n, ga_n, ga_n))]^{r_3} \\
\times [\psi(G(ga_n, ga_n) + G(a_n, ga_{n+1}, ga_{n+1}))]^{r_4} \\
= [\psi(G(a_{n-1}, a_{n+1}, a_{n+1}))]^{r_1} [\psi(G(a_{n-1}, a_{n+1}, a_{n+1}))]^{r_2} [\psi(G(a_n, a_{n+1}, a_{n+1})))^{r_3} [\psi(G(ga_{n-1}, ga_n, ga_n))]^{r_4}.
$$

Using (G5) and $\psi_0$, we get

$$
\psi(G(a_{n-1}, a_{n+1}, a_{n+1})) \leq \psi(G(a_{n-1}, a_{n+1}, a_{n+1})) \\
\leq \psi(G(a_{n-1}, a_{n+1}, a_{n+1})) + 2G(a_n, a_{n+1}, a_{n+1}) \\
\leq \psi(G(a_{n-1}, a_{n+1}, a_{n+1})) \psi(2G(a_n, a_{n+1}, a_{n+1})) \\
= \psi(G(a_{n-1}, a_{n+1}, a_{n+1})) \psi(G(a_n, a_{n+1}, a_{n+1}) + G(a_n, a_{n+1}, a_{n+1})) \\
\leq \psi(G(a_{n-1}, a_{n+1}, a_{n+1})) \psi(G(a_n, a_{n+1}, a_{n+1}))^2,
$$

and

$$
\psi(G(a_{n-1}, a_{n+1}, a_{n+1})) \leq \psi(G(a_{n-1}, a_{n+1}, a_{n+1})) \\
\leq \psi(G(a_{n-1}, a_{n+1}, a_{n+1})) \psi(G(a_n, a_{n+1}, a_{n+1})).
$$

Therefore,

$$
1 < \psi(G(a_n, a_{n+1}, a_{n+1})) \\
\leq [\psi(G(a_{n-1}, a_{n+1}, a_{n+1}))]^{r_1} [\psi(G(a_{n-1}, a_{n+1}, a_{n+1}))]^{r_2} [\psi(G(a_n, a_{n+1}, a_{n+1}))]^{r_3} [\psi(G(a_{n-1}, a_{n+1}, a_{n+1}))]^{r_4}.
$$
So, by reordering the product terms of the above inequality, then using the induction, we get that

\[
1 < \psi(G(a_n, a_{n+1}, a_{n+1})) \leq \left[ \psi(G(a_{n-1}, a_n)) \right]^{\frac{r_1 + r_2 + r_3}{2r_2 + r_3 - r_4}} \\
\vdots \\
\leq \left[ \psi(G(a_0, a_1, a_1)) \right]^{\left(\frac{r_1 + r_2 + r_3}{2r_2 + r_3 - r_4}\right)^n}.
\]

(2.2)

Taking limit as \( n \to \infty \), and noting that \( \frac{r_1 + r_2 + r_3}{2r_2 + r_3 - r_4} < 1 \), we get

\[
\lim_{n \to \infty} \psi(G(a_n, a_{n+1}, a_{n+1})) = 1,
\]

(2.3)

which implies by (\( \psi_2 \)) that

\[
\lim_{n \to \infty} G(a_n, a_{n+1}, a_{n+1}) = 0.
\]

(2.4)

From the condition (\( \psi_3 \)), there exist \( 0 < r < 1 \) and \( L \in (0, \infty) \) such that

\[
\lim_{n \to \infty} \frac{\psi(G(a_n, a_{n+1}, a_{n+1})) - 1}{[G(a_n, a_{n+1}, a_{n+1})]^r} = L.
\]

Suppose that \( L < \infty \). In this case, let \( B_1 = \frac{1}{2} > 0 \). From the definition of the limit, there exists \( n_0 \in \mathbb{N} \) such that

\[
\left| \frac{\psi(G(a_n, a_{n+1}, a_{n+1})) - 1}{[G(a_n, a_{n+1}, a_{n+1})]^r} - L \right| < B_1,
\]

for all \( n > n_0 \). This implies that

\[
\frac{\psi(G(a_n, a_{n+1}, a_{n+1})) - 1}{[G(a_n, a_{n+1}, a_{n+1})]^r} \geq L - B_1 = \frac{L}{2} = B_1,
\]

for all \( n > n_0 \). Then

\[
n(G(a_n, a_{n+1}, a_{n+1}))^r \leq A_1 n[\psi(G(a_n, a_{n+1}, a_{n+1})) - 1],
\]

where \( A_1 = \frac{1}{B_1} \).

Now for \( L = \infty \), let \( B_2 > 0 \) be an arbitrary number. From the definition of the limit there exist \( n_1 \in \mathbb{N} \) such that

\[
\frac{\psi(G(a_n, a_{n+1}, a_{n+1})) - 1}{[G(a_n, a_{n+1}, a_{n+1})]^r} \geq B_2,
\]

for all \( n \geq n_1 \). Then

\[
n(G(a_n, a_{n+1}, a_{n+1}))^r \leq A_2 n[\psi(G(a_n, a_{n+1}, a_{n+1})) - 1],
\]

where \( A_2 = \frac{1}{B_2} \). Thus, in both cases, there exist \( A = \max\{A_1, A_2\} > 0 \) and \( n^* = \max\{n_0, n_1\} \in \mathbb{N} \) such that

\[
n(G(a_n, a_{n+1}, a_{n+1}))^r \leq A n[\psi(G(a_n, a_{n+1}, a_{n+1})) - 1] \text{ for all } n \geq n^*.
\]

Now, using (2.2) we get

\[
n(G(a_n, a_{n+1}, a_{n+1}))^r \leq A n \left[ [\psi(G(a_0, a_1, a_1))]^{n^*} - 1 \right],
\]

Some fixed point results on \( G \)-metric and \( G_b \)-metric spaces
where, \( a = \frac{r_1 + r_2 + r_3}{1 - r_2 - r_3 - r_4} \). But,

\[
\lim_{n \to \infty} n \left[ \left( \psi(G(a_0, a_1, a_1)) \right)^n - 1 \right] = \lim_{n \to \infty} \frac{\left( \psi(G(a_0, a_1, a_1)) \right)^n - 1}{1/n} \\
= \lim_{n \to \infty} \frac{\alpha^n \ln(a) \ln(\psi(G(a_0, a_1, a_1))) \left[ \psi(G(a_0, a_1, a_1)) \right]^n}{-1/n^2} \\
= \lim_{n \to \infty} \frac{-n^2 \alpha^n \ln(a) \ln(\psi(G(a_0, a_1, a_1))) \left[ \psi(G(a_0, a_1, a_1)) \right]^n}{\alpha_1^n} \\
= \lim_{n \to \infty} \frac{-n^2 \alpha^n \ln(a) \ln(\psi(G(a_0, a_1, a_1))) \left[ \psi(G(a_0, a_1, a_1)) \right]^n}{\alpha_1^n} \\
= 0 \times \ln(a) \ln(\psi(G(a_0, a_1, a_1))) \\
= 0 \ (\text{where } a_1 = 1/a),
\]

which implies that \( \lim_{n \to \infty} n(G(a_n, a_{n+1}, a_{n+1}))^r = 0 \), thus there exists \( n_2 \in \mathbb{N} \) such that

\[
G(a_n, a_{n+1}, a_{n+1}) \leq \frac{1}{n^{1/r}},
\]

for all \( n > n_2 \). Now, for \( m > n > n_2 \), we have

\[
G(a_n, a_m, a_m) \leq \sum_{i=n}^{m-1} G(a_i, a_{i+1}, a_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^r} \leq \sum_{i=1}^{\infty} \frac{1}{i^r}.
\]

Since \( 0 < r < 1 \), then \( \sum_{i=1}^{\infty} \frac{1}{i^r} \) is convergent and hence \( G(a_n, a_m, a_m) \to 0 \) as \( m, n \to \infty \). Thus, we proved that \( \{a_n\} \) is a G-Cauchy sequence. Completeness of \( (X, G) \) ensures that there exists \( a^* \in X \) such that \( a_n \to a^* \) as \( n \to \infty \).

Now we shall show that \( a^* \) is a fixed point of \( g \). Using (G5) we get that

\[
G(a^*, a^*, ga^*) \leq G(a^*, a^*, a^*) + G(a_{n+1}, a_{n+1}, ga^*) \\
= G(a^*, a^*, a^*) + G(ga_n, ga_n, ga^*)
\]

and

\[
G(a_n, a_{n+1}, ga^*) \leq \left( G(a_n, a_{n+1}, a^*) \right) + \left( G(a^*, a^*, ga^*) \right).
\]

Hence, by the properties of \( \psi \) we get that

\[
\psi(G(a^*, a^*, ga^*)) \leq \psi(G(a^*, a^*, a_{n+1})) \psi(G(ga_n, ga_n, ga^*))
\]

and

\[
\psi(G(a_n, a_{n+1}, ga^*)) \leq \psi(G(a_n, a_{n+1}, a^*)) \psi(G(a^*, a^*, ga^*)).
\]

Thus,

\[
\left[ \psi(G(a_n, a_{n+1}, ga^*)) \right]^{r_2 + r_3} \leq \left[ \psi(G(a_n, a_{n+1}, a^*)) \right]^{r_2 + r_3} \left[ \psi(G(a^*, a^*, ga^*)) \right]^{r_2 + r_3}.
\]
However, by using (2.1), \((\psi_k)\) and (2.9) we have
\[
\begin{align*}
\psi \left( G \left( a_{n+1}, a_{n+1}, ga^* \right) \right) &= \psi \left( G \left( ga_n, ga_n, ga^* \right) \right) \\
&\leq \left[ \psi \left( G \left( a_n, a_n, a^* \right) \right) \right] r_1 \left[ \psi \left( G \left( a_n, a_{n+1}, ga^* \right) \right) \right] r_2 \\
&\quad \times \left[ \psi \left( G \left( a_n, a_{n+1}, ga^* \right) \right) \right] r_3 \\
&\quad \times \left[ \psi \left( G \left( a_{n+1}, a_{n+1}, a^* \right) \right) \right] r_4 \\
&\quad \times \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_5 \\
&= \left[ \psi \left( G \left( a_n, a_n, a^* \right) \right) \right] r_1 \left[ \psi \left( G \left( a_n, a_{n+1}, ga^* \right) \right) \right] r_2 r_3 \\
&\quad \times \left[ \psi \left( G \left( a_{n+1}, a_{n+1}, a^* \right) \right) \right] r_4 r_5 \\
&\quad \times \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_1 r_2 r_3 \\
&\leq \left[ \psi \left( G \left( a_n, a_n, a^* \right) \right) \right] r_1 \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_2 r_3 \\
&\quad \times \left[ \psi \left( G \left( a_{n+1}, a_{n+1}, a^* \right) \right) \right] r_4 r_5 \\
&\quad \times \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_1 r_2 r_3 \\
&= \left[ \psi \left( G \left( a_n, a_n, a^* \right) \right) \right] r_1 \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_2 r_3 \\
&\quad \times \left[ \psi \left( G \left( a_{n+1}, a_{n+1}, a^* \right) \right) \right] r_4 r_5 \\
&\quad \times \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_1 r_2 r_3 \\
&\leq \left[ \psi \left( G \left( a_n, a_n, a^* \right) \right) \right] r_1 \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_2 r_3 \\
&\quad \times \left[ \psi \left( G \left( a_{n+1}, a_{n+1}, a^* \right) \right) \right] r_4 r_5 \\
&\quad \times \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_1 r_2 r_3.
\end{align*}
\]

(2.10)

Now, substituting (2.10) in (2.7) we get that
\[
\begin{align*}
\psi(G(a^*, a^*; ga^*)) &\leq \psi(G(a^*, a^*; a_{n+1})) \left[ \psi(G \left( a_n, a_n, a^* \right) \right] r_1 \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_2 r_3 \\
&\quad \left[ \psi \left( G \left( a^*, a^*, ga^* \right) \right) \right] r_1 r_2 r_3 \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_1 r_2 r_3,
\end{align*}
\]

(2.11)

Hence,
\[
1 \leq \left[ \psi(G(a^*, a^*; ga^*)) \right]^{1-r_2-r_3} \leq \psi(G(a^*, a^*; a_{n+1})) \left[ \psi \left( G \left( a_n, a_n, a^* \right) \right) \right] r_1 \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_2 r_3 \\
&\quad \left[ \psi \left( G \left( a^*, a^*, ga^* \right) \right) \right] r_1 r_2 r_3 \left[ \psi \left( G \left( a_n, a_{n+1}, a^* \right) \right) \right] r_1 r_2 r_3.
\]

(2.12)

By taking the limit as \(n \to \infty\) and using (2.4), \((\psi_2)\), Proposition 1.3 and the convergence of \(a_n\) to \(a^*\) in the above equation we get that
\[
\psi(G(a^*, a^*; ga^*)) = 1
\]

(2.13)

which implies by \((\psi_1)\) that \(G(a^*, a^*, ga^*) = 0\) and so \(ga^* = a^*\). Thus, \(a^*\) is a fixed point of \(g\).

Finally to show the uniqueness, assume that there exist \(a' \neq a^*\) such that \(a' = ga'\). By (2.2),
\[
G(a', a', a') = G(ga', ga', ga') > 0.
\]

Thus, by (2.1) we get
\[
\sum(G(a', a', a')) = \psi(G(ga', ga', ga')) \leq \left[ \psi(G(a', a', a')) \right] r_1 \left[ \psi(G(ga', ga', ga')) \right] r_2 \\
&\quad \times \left[ \psi(G(a', ga', ga')) \right] r_1 \left[ \psi(G(a', ga', ga') + G(a', ga', ga')) \right] r_3 \\
&= \left[ \psi(G(a', a', a')) \right] r_1 \left[ \psi(G(a', a', a')) \right] r_2 \left[ \psi(G(a', a', a')) \right] r_3 \\
&\quad \times \left[ \psi \left( G \left( a', a', a' \right) + G(a', a', a') \right) \right] r_1 \left[ \psi \left( G \left( a', a', a' \right) + G(a', a', a') \right) \right] r_2 \left[ \psi \left( G \left( a', a', a' \right) + G(a', a', a') \right) \right] r_3 \\
&= \left[ \psi(G(a', a', a')) \right] r_1 \left[ \psi(G(a', a', a')) \right] r_2 \left[ \psi(G(a', a', a')) \right] r_3 \times \left[ \psi \left( G \left( a', a', a' \right) + G(a', a', a') \right) \right] r_1 \left[ \psi \left( G \left( a', a', a' \right) + G(a', a', a') \right) \right] r_2 \left[ \psi \left( G \left( a', a', a' \right) + G(a', a', a') \right) \right] r_3,
\]

which leads to a contradiction because \(r_1 + r_2 + r_3 < 1\). Therefore, \(g\) has a unique fixed point. \(\square\)

The following result is a direct consequence of Theorem 2.2 by taking \(\psi(t) = t^{1/2}\) in (2.1).
Corollary 2.3. Let \((X, G)\) be a complete \(G\)-metric space and \(g : X \to X\) be a mapping. Suppose that there exist positive real numbers \(r_1, r_2, r_3, r_4\) with \(0 \leq r_1 + 3r_2 + r_3 + 2r_4 < 1\) such that
\[
\sqrt{G}(ga, gb, gc) \leq r_1 \sqrt{G}(a, b, c) + r_2 \sqrt{G}(a, ga, gc) + r_3 \sqrt{G}(b, gb, gc) \\
+ r_4 \sqrt{G}(a, gb, gb) + G(b, ga, ga)
\]
(2.14)
for all \(a, b, c \in X\). Then \(g\) has a unique fixed point.

Remark 2.4. Note that condition (2.14) is equivalent to
\[
G(ga, gb, gc) \leq r_1^2 G(a, b, c) + r_2^2 G(a, ga, gc) + r_3^2 G(b, gb, gc) \\
+ r_4^2 [G(a, gb, gb) + G(b, ga, ga)] \\
+ 2r_1 r_2 \sqrt{G}(a, b, c) G(a, ga, gc) + 2r_1 r_3 \sqrt{G}(a, b, c) G(b, gb, gc) \\
+ 2r_1 r_4 \sqrt{G}(a, b, c) [G(a, gb, gb) + G(b, ga, ga)] \\
+ 2r_2 r_3 \sqrt{G}(a, ga, gc) [G(a, gb, gb) + G(b, ga, ga)] \\
+ 2r_3 r_4 \sqrt{G}(b, gb, gc) [G(a, gb, gb) + G(b, ga, ga)].
\]

Next, in view of Remark 2.4 and by taking \(r_2 = r_3 = r_4 = 0\) in Corollary 2.3, we obtain the following corollary.

Corollary 2.5. Let \((X, G)\) be a complete \(G\)-metric space and \(g : X \to X\) be a mapping. Suppose that there exist positive real numbers \(0 \leq r_1 < 1\), such that
\[
G(ga, gb, gc) \leq r_1^2 G(a, b, c)
\]
(2.15)
for all \(a, b, c \in X\). Then \(g\) has a unique fixed point.

Finally, by taking \(\psi(t) = e^{\sqrt{t}}\) in (2.1), we get the following corollary.

Corollary 2.6. Let \((X, G)\) be a complete \(G\)-metric space and \(g : X \to X\) be a mapping. Suppose that there exist positive real numbers \(r_1, r_2, r_3, r_4\) with \(0 \leq r_1 + 3r_2 + r_3 + 2r_4 < 1\), such that
\[
\sqrt{G}(ga, gb, gc) \leq r_1 \sqrt{G}(a, b, c) + r_2 \sqrt{G}(a, ga, gc) + r_3 \sqrt{G}(b, gb, gc) \\
+ r_4 \sqrt{G}(a, gb, gb) + G(b, ga, ga)
\]
for all \(a, b, c \in X\). Then \(g\) has a unique fixed point.

Remark 2.7. By specifying \(r_i = 0\) for some \(i \in \{1, 2, 3, 4\}\) in Remark 2.4 and Corollary 2.6 we can get several results.

Example 2.8. Let \(X = [0, \infty)\) and the \(G\)-metric \(G_m(a, b, c) = \max\{\|a - b\|, |b - c|, |a - c|\}\). Define \(g : X \to X\) by \(g(x) = \frac{x}{3}\) and \(\psi(t) = e^{\sqrt{t}}\). Then clearly all conditions of Theorem 2.2 are satisfied with \(r_i = \frac{1}{\sqrt{3}}\); \(i = 1, 2, 3, 4\), and \(x = 0\) is a unique fixed point of \(g\).

3 Fixed Point Results on \(G_b\)-Metric Spaces

In this section, using the concepts of \(G_b\)-metric space which was introduced by Aghajani et al. [31] we establish some new fixed point results in this setting.

Definition 3.1. [31] Let \(X\) be a nonempty set and \(s \geq 1\) be a given real number. Suppose that \(G_b : X \times X \times X \to [0, \infty)\) be a function satisfying the following properties
Example 3.10. Denote by unique limit point. Indeed, each Definition 3.8.

Proposition 3.7. \( G_2 \) is nondecreasing and \( G_2(\infty, a) \to 0 \) as \( n \to +\infty \).

Definition 3.4. [31] Let \((X, G_b)\) be a \(G_b\)-metric space, and \((a_n)\) be a sequence in \(X\). Then we say that \((a_n)\) is \(G_b\)-convergent to \(a \in X\) if \(\lim_{n,m \to \infty} G_b(a_n, a_m) = 0\), that is, for any \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(G_b(a_n, a_m) < \epsilon\), for all \(n, m \geq N\). We call \(x\) the limit of the sequence and write \(a_n \to a\) or \(\lim_{n \to \infty} a_n = a\).

Proposition 3.5. [31] Let \((X, G_b)\) be a \(G_b\)-metric space. The following statements are equivalent:

1. \((a_n)\) is \(G_b\)-convergent to \(a\).
2. \(G_b(a_n, a_m) \to 0\) as \(n, m \to +\infty\).
3. \(G_b(a_n, a) \to 0\) as \(n \to +\infty\).
4. \(G_b(a_n, a_m) \to 0\) as \(n, m \to +\infty\).

Definition 3.6. [31] Let \(X\) be a \(G_b\)-metric space. A sequence \((a_n)\) is called a \(G_b\)-Cauchy sequence if for any \(\epsilon > 0\), there is \(N \in \mathbb{N}\) such that \(G_b(a_n, a_m, a_l) < \epsilon\) for all \(n, m, l \geq N\), that is \(G_b(a_n, a_m, a_l) \to 0\) as \(n, m, l \to +\infty\).

Proposition 3.7. [31] Let \((X, G_b)\) be a \(G_b\)-metric space. The following statements are equivalent:

1. \((a_n)\) is \(G_b\)-Cauchy sequence.
2. \(G_b(a_n, a_m) \to 0\) as \(n, m \to +\infty\).

Definition 3.8. [31] A \(G_b\)-metric space \(X\) is called \(G_b\)-complete if every \(G_b\)-Cauchy sequence is \(G_b\)-convergent in \(X\).

Lemma 3.9. Let \(X\) be a \(G_b\)-metric space with \(s \geq 1\). If a sequence \((a_n) \subseteq X\) is \(G_b\)-convergent, then it has a unique limit point.

Very recently, Ahmad et al. [27] studied JS-contraction and considered a new set of real functions, say \(\Omega\). They replaced condition \((\psi_1)\) by another condition called \((\Theta_3)\).

Applying this condition we can have a new range of functions. Thus, consistent with Ahmad et al. [27] we denote by \(\Omega\) the set of all functions \(\theta : [0, \infty) \to [1, \infty)\) satisfying the following conditions:

- \((\theta_1)\): \(\theta\) is nondecreasing and \(\theta(t) = 1\) if and only if \(t = 0\);
- \((\theta_2)\): for each sequence \((t_n) \subseteq (0, \infty)\), \(\lim_{n \to \infty} \theta(t_n) = 1\) if and only if \(\lim_{n \to \infty} t_n = 0\);
- \((\theta_3)\): \(\theta\) is continuous.

Example 3.10. [27] Let \(\theta_1(t) = e^{\sqrt{t}}, \theta_2(t) = e^{\sqrt[3]{t}}, \theta_3(t) = e^t, \theta_4(t) = \cosh t\) and \(\theta_5(t) = 1 + \ln(1 + t)\) for all \(t > 0\). Then \(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \Omega\).
Remark 3.11. [27] Note that the conditions \((\psi_3)\) and \((\Theta_1)\) are independent of each other. Indeed, for \(p \geq 1\), \(\theta(t) = e^p\) satisfies the conditions \((\psi_1)\) and \((\psi_2)\) but it does not satisfy \((\psi_3)\), while it satisfies the condition \((\Theta_3)\). Therefore \(\Omega \subseteq \Psi\). Again, for \(a > 1\), \(m \in \left(0, \frac{1}{a}\right)\), \(\theta(t) = 1 + t^m(1 + [t])\), where \([t]\) denotes the integral part of \(t\), satisfies the conditions \((\psi_1)\) and \((\psi_2)\) but it does not satisfy \((\Theta_3)\), while it satisfies the condition \((\psi_3)\) for any \(r \in \left(\frac{1}{a}, 1\right)\). Therefore \(\Psi \subseteq \Omega\). Also, if we take \(\theta(t) = e^{\sqrt{t}}\), then \(\theta \in \Psi\) and \(\theta \in \Omega\). Therefore \(\Psi \cap \Omega = \emptyset\).

Definition 3.12. [4] Let \(g : X \to X\) and \(a : X \times X \times X \to [0, \infty)\). Then \(g\) is called \(\alpha\)-admissible if for all \(u, v, w \in X\) with \(a(u, v, w) \geq 1\) implies \(a(gu, gv, gw) \geq 1\).

Definition 3.13. Let \(g : X \to X\) and \(a : X \times X \times X \to [0, \infty)\). Then \(g\) is called rectangular-\(\alpha\)-admissible if

1. \(g\) is \(\alpha\)-admissible,
2. \(a(u, c, c) \geq 1\) and \(a(c, v, w) \geq 1\) implies that \(a(u, v, w) \geq 1\)

where \(u, v, w, c \in X\).

Lemma 3.14. Let \(g\) be a rectangular \(\alpha\)-admissible mapping. Suppose that there exist \(a_0 \in X\) such that \(a(a_0, ga_0, ga_0) \geq 1\). Define the sequence \(a_n = g^n a_0\). Then

\[
a(a_m, a_n, a_n) \geq 1, \text{ for all } m, n \in \mathbb{N} \text{ with } m < n
\]

Proof. Let \(a_n = g^n a_0\) and assume that \(n = m + k\) for some integer \(k \geq 1\). Since \(a(a_0, ga_0, ga_0) \geq 1\) and \(g\) is \(\alpha\)-admissible, then

\[
a(a_1, a_2, a_2) = a(a_1, ga_1, ga_1) = a(ga_0, g^2 a_0, g^2 a_0) \geq 1.
\]

Continuing this process we get that \(a(a_m, a_{m+1}, a_{m+1}) \geq 1\). Similarly we have

\[
a(a_{m+1}, a_{m+2}, a_{m+2}) \geq 1.
\]

Hence, by rectangular \(\alpha\)-admissible we have \(a(a_m, a_{m+2}, a_{m+2}) \geq 1\), now repeating the same process we get that \(a(a_m, a_n, a_n) = a(a_m, a_{m+k}, a_{m+k}) \geq 1\). \(\square\)

Now, we are ready to state our main theorem in this section.

Theorem 3.15. Let \((X, G_b)\) be a \(G_b\)-complete metric space with \(s > 1\). Let \(a : X \times X \times X \to (0, \infty)\) and \(g\) be a rectangular \(\alpha\)-admissible mapping. Suppose that there exist \(\theta \in \Omega\) and \(r \in (0, 1)\) such that

\[
\frac{1}{3s^2} G_b(u, gu, gu) \leq G_b(u, v, w) \Rightarrow a(u, v, w) \theta \left(s^2 G_b(gu, gv, gw)\right) \leq \left[\theta(M(u, v, w))\right]^r \quad (3.1)
\]

for all \(u, v, w \in X\) with at least two of \(gu, gv\) and \(gw\) being not equal, where

\[
M(u, v, w) = \max \left\{ \frac{G_b(u, v, w), G_b(u, gu, gu) G_b(u, gv, gv) + G_b(v, gw, gw) G_b(u, gu, gu)}{1 + s G_b(u, gu, gu) G_b(v, gv, gw) + G_b(u, gu, gu) G_b(u, gv, gv)}, \right.
\]

\[
\left. \frac{G_b(u, gu, gu) G_b(u, gv, gv) - G_b(v, gw, gw) G_b(u, gu, gu)}{1 + s G_b(u, gu, gu) G_b(v, gv, gw) + G_b(u, gu, gu) G_b(u, gv, gv)} \right\}.
\]

Also, suppose that the following assertions hold:

(i) There exists \(a_0 \in X\) such that \(a(a_0, ga_0, ga_0) \geq 1\).

(ii) For any convergence sequence \(\{a_n\}\) to \(a\) with \(a(a_n, a_{n+1}, a_{n+1}) \geq 1\) for all \(n \in \mathbb{N} \cup \{0\}\), we have \(a(a_n, a, a) \geq 1\) for all \(n \in \mathbb{N} \cup \{0\}\). 

Then \(g\) has a fixed point.

(iii) Moreover, if for all \(u, v \in \text{Fix}(g)\) implies \(a(u, v, v) \geq 1\), then the fixed point is unique where \(\text{Fix}(g) = \{u : gu = u\}\).

Proof. Let \(a_0 \in X\) be such that \(a(a_0, ga_0, ga_0) \geq 1\). Define a sequence \(\{a_n\}\) by \(a_n = g^n a_0\) for all \(n \in \mathbb{N}\).

Since \(g\) is an \(\alpha\)-admissible mapping and \(a(a_0, a_1, a_1) = a(a_0, ga_0, ga_0) \geq 1\), we deduce that \(a(a_1, a_2, a_2) = a(ga_0, ga_1, ga_1) \geq 1\). Continuing this process, we get that \(a(a_n, a_{n+1}, a_{n+1}) \geq 1\) for all \(n \in \mathbb{N} \cup \{0\}\). Without
loss of generality, we assume that \( a_n \neq a_{n+1} \) for all \( n \in \mathbb{N} \cup \{0\} \). We shall proceed in proving the theorem using the following two steps.

**Step 1:** We shall show that \( \lim_{n \to \infty} G_b (a_{n+1}, a_n, a_n) = 0 \).

Now,

\[
M (a_{n-1}, a_n, a_n) = \max \left\{ \frac{G_b (a_{n-1}, a_n, a_n) \cdot G_b (a_{n-1}, a_n, a_n)}{1 + G_b (a_{n-1}, a_n, a_n) + G_b (a_{n-1}, a_n, a_n)} \right\}
\]

\[
= \max \left\{ \frac{G_b (a_{n-1}, a_n, a_n) \cdot G_b (a_{n-1}, a_n, a_n)}{1 + G_b (a_{n-1}, a_n, a_n) + G_b (a_{n-1}, a_n, a_n)} \right\}
\]

\[
= \max \left\{ \frac{G_b (a_{n-1}, a_n, a_n), G_b (a_{n-1}, a_n, a_n)}{1 + G_b (a_{n-1}, a_n, a_n) + G_b (a_{n-1}, a_n, a_n)} \right\}
\]

But, from \((G_b)3\), we have \( G_b (a_{n-1}, a_n, a_n) \leq G_b (a_{n-1}, a_n, a_n) \), and so

\[
\frac{G_b (a_{n-1}, a_n, a_n)}{1 + G_b (a_{n-1}, a_n, a_n) + G_b (a_{n-1}, a_n, a_n)} \leq 1
\]

also

\[
\frac{G_b (a_{n-1}, a_n, a_n)}{1 + G_b (a_{n-1}, a_n, a_n) + G_b (a_{n-1}, a_n, a_n)} \leq 1.
\]

Therefore, \( M (a_{n-1}, a_n, a_n) = G_b (a_{n-1}, a_n, a_n) \).

Since \( a (a_{n-1}, a_{n+1}, a_n) \geq 1 \) for each \( n \in \mathbb{N} \cup \{0\} \) and \( \frac{1}{\Gamma (a_{n-1}, a_{n+1}, a_n) \leq G_b (a_{n-1}, a_n, a_n) \), as a result by \((3.1)\) we have

\[
\theta (G_b (a_{n-1}, a_{n+1}, a_n)) = \theta (G_b (g a_{n-1}, g a_n, g a_n)),
\]

\[
\leq a (a_{n-1}, a_n, a_n) \theta \left( s^2 G_b (g a_{n-1}, g a_n, g a_n) \right),
\]

\[
\leq \{\theta (M (a_{n-1}, a_n, a_n))\}^r,
\]

\[
= \{\theta (G_b (a_{n-1}, a_n, a_n))\}^r
\]

Therefore, we have

\[
1 - \theta (G_b (a_{n-1}, a_{n+1}, a_n)) \leq \{\theta (G_b (a_{n-1}, a_n, a_n))\}^r \leq \cdots \leq \{\theta (G_b (a_{0}, a_1, a_1))\}^r.
\]

Taking limit as \( n \to \infty \), we get

\[
\lim_{n \to \infty} \theta (G_b (a_n, a_{n+1}, a_{n+1})) = 1.
\]

This gives us, by \((\theta_2)\),

\[
\lim_{n \to \infty} G_b (a_n, a_{n+1}, a_{n+1}) = 0. \tag{3.4}
\]

But \( G_b (a_{n+1}, a_n, a_n) \leq 2 s G_b (a_n, a_{n+1}, a_{n+1}) \), therefore

\[
\lim_{n \to \infty} G_b (a_{n+1}, a_n, a_n) = 0. \tag{3.5}
\]

**Step 2:** We shall prove that the sequence \( \{a_n\} \) is a \( G_b \)-Cauchy sequence. Suppose on the contrary that \( \{a_n\} \) is not a \( G_b \)-Cauchy sequence. Then there exists \( \varepsilon > 0 \) for which we can find two subsequences \( \{a_{m_i}\} \) and \( \{a_{n_i}\} \) of \( \{a_n\} \) such that \( n_i \) is the smallest index for which

\[
n_i > m_i > i \text{ and } G_b (a_m, a_n, a_n) \geq \varepsilon. \tag{3.6}
\]
This means that
\[ G_b \left( a_m, a_{n-1}, a_{n-1} \right) < \varepsilon. \] (3.7)

By using (3.6) and \((G_b)\), we get
\[ \varepsilon \leq G_b \left( a_m, a_{n_1}, a_{n_1} \right) \leq s G_b \left( a_m, a_{m+1}, a_{m+1} \right) + s G_b \left( a_{m+1}, a_{m+1}, a_{m+1} \right). \]

Taking the upper limit as \( i \to \infty \) and using (3.5) we get
\[ \frac{\varepsilon}{s} \leq \limsup_{i \to \infty} G_b \left( a_{m+1}, a_{n_i}, a_{n_i} \right). \] (3.8)

Notice that from (3.3) and \((\theta)_1\), we get
\[ G_b \left( a_n, a_{n+1}, a_{n+1} \right) \leq G_b \left( a_{n-1}, a_n, a_n \right) \text{ for all } n \in \mathbb{N}, \] (3.9)

Suppose that there exists \( i_0 \in \mathbb{N} \) such that
\[ \frac{1}{3s^2} G_b \left( a_{m_0}, g a_{m_0}, g a_{m_0} \right) \geq G_b \left( a_{m_0}, a_{m_0-1}, a_{m_0-1} \right) \]

and
\[ \frac{1}{3s^2} G_b \left( a_{m_0+1}, g a_{m_0+1}, g a_{m_0+1} \right) \geq G_b \left( a_{m_0+1}, a_{m_0-1}, a_{m_0-1} \right). \]

Then from \((G_b)\), (3.9) we have
\[
\begin{align*}
G_b \left( a_{m_0}, a_{m_0+1}, a_{m_0+1} \right) &\leq s \left[ G_b \left( a_{m_0}, a_{m_0-1}, a_{m_0-1} \right) + G_b \left( a_{m_0-1}, a_{m_0+1}, a_{m_0+1} \right) \right] \\
&\leq s \left[ G_b \left( a_{m_0}, a_{m_0-1}, a_{m_0-1} \right) + 2s G_b \left( a_{m_0+1}, a_{m_0-1}, a_{m_0-1} \right) \right] \\
&\leq s \left[ \frac{1}{3s^2} G_b \left( a_{m_0}, g a_{m_0}, g a_{m_0} \right) + \frac{2s}{3s^2} G_b \left( a_{m_0+1}, g a_{m_0+1}, g a_{m_0+1} \right) \right] \\
&= \left[ \frac{1}{3s^2} G_b \left( a_{m_0}, a_{m_0+1}, a_{m_0+1} \right) + \frac{2}{3} G_b \left( a_{m_0+1}, a_{m_0+2}, a_{m_0+2} \right) \right] \\
&\leq \left( \frac{1}{3s} + \frac{2}{3} \right) G_b \left( a_{m_0}, a_{m_0+1}, a_{m_0+1} \right), \text{ (since } s > 1),
\end{align*}
\] (3.10)

which is a contradiction. Hence, either
\[ \frac{1}{3s^2} G_b \left( a_m, g a_m, g a_m \right) \leq G_b \left( a_m, a_{n-1}, a_{n-1} \right) \]

or
\[ \frac{1}{3s^2} G_b \left( a_{m+1}, g a_{m+1}, g a_{m+1} \right) \leq G_b \left( a_{m+1}, a_{n-1}, a_{n-1} \right), \]

holds for all \( i \in \mathbb{N} \). First suppose that
\[ \frac{1}{3s^2} G_b \left( a_m, g a_m, g a_m \right) \leq G_b \left( a_m, a_{n-1}, a_{n-1} \right). \] (3.11)

From the definition of \( M(u, v, w) \) and using (3.5) and (3.7) we have
\[
\begin{align*}
\limsup_{i \to \infty} M(a_m, a_{n-1}, a_{n-1}) &= \limsup_{i \to \infty} \left\{ G_b(a_m, a_{n-1}, a_{n-1}) , \\
&\quad G_b \left( a_{m_1}, a_{n_{1-1}}, a_{n_{1-1}} \right) G_b(a_{m_1}, a_{n_{1-1}}, a_{n_{1-1}}) G_b(a_{n_{1-1}}, a_{n_{1-1}}, a_{n_{1-1}}) G_b(a_{n_{1-1}}, a_{n_{1-1}}, a_{n_{1-1}}) \right\} \\
&\quad \leq \varepsilon.
\end{align*}
\]
Note that, \( m_i \neq n_i - 1 \), as otherwise \( G_b(\alpha m_i, a_{m_i - 1}, a_{n_i - 1}) = 0 \) and so, by (3.11)

\[ G_b(\alpha m_i, a_{m_i + 1}, a_{m_i + 1}) = G_b(\alpha m_i, g a m_i, g a m_n) = 0 \]

which contradicts our assumption that \( a_n \neq a_{n+1} \) for all \( n \in N \). Hence, \( a(\alpha m_i, a_{m_i - 1}, a_{n_i - 1}) \geq 1 \). Based on the assumption (3.11), (\( \theta_1 \)), \( a(\alpha m_i, a_{m_i - 1}, a_{n_i - 1}) \geq 1 \), (3.8), (3.1) and the above inequality, we obtain that

\[
\theta \left( s^2 \frac{E}{S} \right) \leq a(\alpha m_i, a_{m_i - 1}, a_{n_i - 1}) \theta \left( s^2 \lim_{i \to \infty} G_b(\alpha m_i + 1, a_{n_i}, a_{n_i}) \right)
\]

\[
= a(\alpha m_i, a_{m_i - 1}, a_{n_i - 1}) \theta \left( s^2 \lim_{i \to \infty} G_b(ga m_i, ga m_{n - 1}, ga m_{n - 1}) \right)
\]

\[
\leq \left[ \theta \left( \lim_{i \to \infty} M(\alpha m_i, a_{m_i - 1}, a_{n_i - 1}) \right) \right]^r \leq [\theta (E)]^r,
\]

which implies that \( \theta (sE) \leq [\theta (E)]^r \), a contradiction. Now suppose that

\[
\frac{1}{3s^2} G_b(\alpha m_i + 1, ga m_{i + 1}, ga m_{i + 1}) \leq G_b(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1})
\]

(3.12)

holds for all \( i \in N \). Further, from (3.6) and using (G5), we get

\[
E \leq G_b(\alpha m_i, a_{m_i}, a_{n_i}) \leq sG_b(\alpha m_i, a_{m_i + 1}, a_{m_i + 1}) + sG_b(\alpha m_i, a_{m_i}, a_{n_i}) \leq s^2 G_b(\alpha m_i, a_{m_i + 1}, a_{m_i + 1}) + s^2 G_b(\alpha m_i, a_{m_i + 2}, a_{m_i + 2}) + sG_b(\alpha m_i, a_{m_i + 2}, a_{n_i}).
\]

Taking the upper limit as \( i \to \infty \), and using (3.5) we get

\[
E \leq \lim_{i \to \infty} \sup G_b(\alpha m_i + 2, a_{n_i}, a_{n_i}).
\]

(3.13)

Also, from (G5), we get

\[
G_b(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}) \leq sG_b(\alpha m_i + 1, a_{n_i}, a_{n_i}) + sG_b(\alpha m_i, a_{n_i - 1}, a_{n_i - 1}).
\]

Taking the upper limit as \( i \to \infty \), and using (3.5) and (3.7) we get

\[
\lim_{i \to \infty} \sup G_b(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}) \leq sE.
\]

(3.14)

From the definition of \( M(\alpha, \nu, \omega) \) and using (3.5) and (3.14), we have

\[
\lim_{i \to \infty} \sup \max_{i \to \infty} \left\{ \begin{array}{l}
G_b(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}),
G_b(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}),
G_b(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}),
G_b(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}),
G_b(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}),
\end{array} \right\} \leq sE.
\]

Note that, \( m_i + 1 \neq n_i - 1 \), as otherwise

\[
G_b(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}) = 0
\]

and so, by (3.12) \( G_b(\alpha m_i + 1, a_{m_i + 2}, a_{m_i + 2}) = G_b(\alpha m_i + 1, ga m_{i + 1}, ga m_{i + 1}) = 0 \), which contradicts our assumption that \( a_n \neq a_{n + 1} \) for all \( n \in N \). Hence, \( a(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}) \geq 1 \).

Based on the assumption (3.12), (\( \theta_1 \)), \( a(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}) \geq 1 \), (3.13), (3.1) and the above inequality we obtain that

\[
\theta \left( s^2 \frac{E}{S} \right) \leq a(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}) \theta \left( s^2 \lim_{i \to \infty} G_b(\alpha m_i + 2, a_{n_i}, a_{n_i}) \right)
\]

\[
= a(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}) \theta \left( s^2 \lim_{i \to \infty} G_b(ga m_{i + 1}, ga m_{n - 1}, ga m_{n - 1}) \right)
\]

\[
\leq \left[ \theta \left( \lim_{i \to \infty} M(\alpha m_i + 1, a_{n_i - 1}, a_{n_i - 1}) \right) \right]^r \leq [\theta (E)]^r,
\]
a contradiction. Therefore, in all cases \( \{a_n\} \) is a \( G_b \)-Cauchy sequence, thus by \( G_b \)-completeness of \( X \) yields that \( \{a_n\} \) is \( G_b \)-convergent to a point \( a^* \in X \). By an argument similar to that in (3.10), we get either

\[
\frac{1}{3s^2} G_b(a_n, ga_n, ga_n) \leq G_b \left( a_n, x^*, a^* \right)
\]

or

\[
\frac{1}{3s^2} G_b(a_{n+1}, ga_{n+1}, ga_{n+1}) \leq G_b \left( a_{n+1}, a^*, a^* \right)
\]

holds for all \( n \in \mathbb{N} \). First, suppose that

\[
\frac{1}{3s^2} G_b(a_n, ga_n, ga_n) \leq G_b \left( a_n, a^*, a^* \right).
\]

Now,

\[
M \left( a_n, a^*, a^* \right) = \max \left\{ \frac{G_b(a_n, a^*, a^*)}{G_b(a_n, ga_n, ga_n)}, \frac{G_b(a_{n+1}, a^*, a^*)}{G_b(a_{n+1}, ga_{n+1}, ga_{n+1})} \right\}
\]

So, \( \lim_{n \to \infty} M \left( a_n, a^*, a^* \right) = 0 \). Hence from (3.1) and assertion (ii) of the theorem, we have

\[
1 \leq \theta \left( G_b \left( ga_n, ga^*, ga^* \right) \right) \leq \frac{1}{s^2} \theta \left( G_b \left( ga_n, ga^*, ga^* \right) \right) \leq a(a_n, a^*, a^*) \theta \left( s^2 G_b \left( ga_n, ga^*, ga^* \right) \right) \leq \left[ \theta \left( M \left( a_n, a^*, a^* \right) \right) \right]^r
\]

for all \( n \in \mathbb{N} \). Taking the limit as \( n \to \infty \), in the above inequality we get that

\[
\lim_{n \to \infty} \theta \left( G_b \left( ga_n, ga^*, ga^* \right) \right) = 1.
\]

This implies by \((\Theta_1)\) that

\[
\lim_{n \to \infty} G_b \left( ga_n, ga^*, ga^* \right) = 0.
\]

Hence, \( ga^* = \lim_{n \to \infty} ga_n = \lim_{n \to \infty} a_{n+1} = a^* \). Thus, we deduce that \( ga^* = a^* \).

Now if

\[
\frac{1}{3s^2} G_b(a_{n+1}, ga_{n+1}, ga_{n+1}) \leq G_b \left( a_{n+1}, a^*, a^* \right),
\]

holds, then by repeating the same process as above we can get \( ga^* = a^* \). Therefore, we proved that \( a^* \) is a fixed point of \( g \).

Now to prove uniqueness, suppose there exist \( u, v \in Fix(g) \) with \( u \neq v \), that is \( u = gu \) and \( v = gv \). Therefore by (iii), \( a(u, v, v) \geq 1 \) and so, by (3.1) and \((G_{b2})\) we have

\[
0 = \frac{1}{3s^2} G(u, gu, gu) \leq G(u, v, v)
\]

and

\[
\theta(G_b(u, v, v)) \leq a(u, v, v) \theta(s^2 G_b(gu, gv, gv)) \leq \left[ \theta(M(u, v, v)) \right]^r = \left[ \theta(G_b(u, v, v)) \right]^r < \theta(G_b(u, v, v)).
\]

Thus the contradiction implies that the fixed point is unique. \qed
Theorem 3.16. Let \((X, G_b)\) be a \(G_b\)-complete metric space with \(s > 1\). Let \(\alpha : X \times X \times X \to (0, \infty)\) and \(g\) be a rectangular \(\alpha\)-admissible mapping. Suppose that there exist \(\theta \in \Omega\) and \(r \in (0, 1)\) such that

\[
\frac{1}{3s^2} G_b(u, gu, gw) \leq G_b(u, v, w) \Rightarrow \alpha(u, v, w) \theta \left( s^2 G_b(gu, gv, gw) \right) \leq [\theta(M(u, v, w))]^r
\]

for all \(x, y, z \in X\) with at least two of \(gx, gy\) and \(gz\) being not equal, where

\[
M(u, v, w) = \max \left\{ G_b(u, v, w), \frac{G_b(u, gu, gw) G_b(v, gv, gw)}{1 + G_b(u, gu, gw)} \right\} = \max \left\{ G_b(u, v, w), \frac{G_b(u, gu, gw) G_b(v, gv, gw)}{1 + G_b(u, gu, gw)} \right\}.
\]

Also, suppose that the following assertions hold:
(i) There exists \(a_0 \in X\) such that \(\alpha(a_0, ga_0, ga_0) \geq 1\).
(ii) For any convergent sequence \(\{a_n\}\) to \(a\) with \(\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1\) for all \(n \in \mathbb{N} \cup \{0\}\), we have \(\alpha(a_n, a, a) \geq 1\) for all \(n \in \mathbb{N} \cup \{0\}\).

Then \(g\) has a fixed point.

(iii) Moreover, if for all \(u, v \in \text{Fix}(g)\) implies \(\alpha(u, v, v) \geq 1\), then the fixed point is unique where \(\text{Fix}(g) = \{u; gu = u\}\).

Proof. Let \(a_0 \in X\) be such that \(\alpha(a_0, ga_0, ga_0) \geq 1\). Define a sequence \(\{a_n\}\) by \(a_n = g^n a_0\) for all \(n \in \mathbb{N}\).

Since \(g\) is an \(\alpha\)-admissible mapping and \(\alpha(a_0, a_1, a_1) = \alpha(a_0, ga_0, ga_0) \geq 1\), we deduce that \(\alpha(a_1, a_2, a_2) = \alpha(ga_0, ga_1, ga_1) \geq 1\). Continuing this process, we get that \(\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1\) for all \(n \in \mathbb{N} \cup \{0\}\).

Without loss of generality, assume that \(a_n \neq a_{n+1}\) for all \(n \in \mathbb{N} \cup \{0\}\). We shall show that \(\lim_{n \to \infty} G_b(a_n, a_{n+1}, a_n) = 0\).

Now,

\[
M(a_{n-1}, a_n, a_n) = \max \left\{ G_b(a_{n-1}, a_n, a_n), \frac{G_b(a_{n-1}, a_n, a_n) G_b(a_n, a_n, a_n)}{1 + G_b(a_{n-1}, a_n, a_n)} \right\}
\]

\[
= \max \left\{ G_b(a_{n-1}, a_n, a_n), \frac{G_b(a_{n-1}, a_n, a_n) G_b(a_n, a_n, a_n)}{1 + G_b(a_{n-1}, a_n, a_n)} \right\}
\]

Since, \(\frac{G_b(a_{n-1}, a_n, a_n)}{1 + G_b(a_{n-1}, a_n, a_n)} < 1\) and \(\frac{G_b(a_{n-1}, a_n, a_n)}{1 + G_b(a_{n-1}, a_n, a_n)} < 1\). Therefore,

\[
M(a_{n-1}, a_n, a_n) = \max \{ G_b(a_{n-1}, a_n, a_n), G_b(a_n, a_{n+1}, a_n) \}.
\]

If \(\max\{G_b(a_{n-1}, a_n, a_n), G_b(a_n, a_{n+1}, a_n)\} = G_b(a_n, a_{n+1}, a_n)\), then since \(\alpha(a_{n-1}, a_n, a_n) \geq 1\) for each \(n \in \mathbb{N}\), \(\frac{1}{3s^2} G_b(a_{n-1}, ga_{n-1}, ga_{n-1}) \leq G_b(a_{n-1}, a_n, a_n)\) and so by (3.15) we have

\[
\theta(G_b(a_n, a_{n+1}, a_n)) = \theta(G_b(ga_{n-1}, ga_{n-1}, ga_n)), \leq \alpha(a_{n-1}, a_n, a_n) \theta \left( s^2 G_b(ga_{n-1}, ga_{n-1}, ga_n) \right), \leq [\theta(M(a_{n-1}, a_n, a_n))]^r, = [\theta(G_b(a_n, a_{n+1}, a_n))]^r, < \theta(G_b(a_n, a_{n+1}, a_n))
\]

which is a contradiction since \(r \in (0, 1)\). Thus, \(M(a_{n-1}, a_n, a_n) = G_b(a_{n-1}, a_n, a_n)\).

The rest of the proof is the same as the proof of Theorem 3.15.

\[\square\]

Analogously, we can prove the following theorem.

Theorem 3.17. Let \((X, G_b)\) be a complete \(G_b\)-metric space with \(s > 1\). Let \(\alpha : X \times X \times X \to (0, \infty)\) and \(g\) be a rectangular \(\alpha\)-admissible mapping. Suppose that there exist \(\theta \in \Omega\) and \(r \in (0, 1)\) such that

\[
\frac{1}{3s^2} G_b(u, gu, gw) \leq G_b(u, v, w) \Rightarrow \alpha(u, v, w) \theta \left( s^2 G_b(gu, gv, gw) \right) \leq [\theta(M(u, v, w))]^r
\]
for all $u, v, w \in X$ with at least two of $gu$, $gv$ and $gw$ are not equal, where

$$M(u, v, w) = \max \left\{ \frac{G_b(u, v, w)}{G_b(u, v, w) + G_b(v, v, gw) + G_b(u, u, gw)} \right\}.$$  

Also, suppose that the following assertions hold:

(i) There exists $a_0 \in X$ such that $a(a_0, ga_0, ga_0) \geq 1$;

(ii) For any convergent sequence $\{a_n\}$ to $a$ with $a(a_n, a_{n+1}, a_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, we have $a(a_n, a, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then $g$ has a fixed point.

(iii) Moreover, if for all $u, v \in \text{Fix}(g)$ implies $a(u, v, v) \geq 1$, then the fixed point is unique where $\text{Fix}(g) = \{u; gu = u\}$.

Now, we give an example to support Theorem 3.1

**Example 3.18.** Let $X = [0, \infty)$ and $G_b : X \times X \times X \to R$ be a $G_b$-metric space defined by $G_b(u, v, w) = ((u - v) + (v - w) + (u - w))^2$. Clearly $(X, G_b)$ is a complete $G_b$-metric space with $s = 2$. Also let $r = \frac{1}{2}$ and define $g : X \to X$, $\alpha : X \times X \times X \to R$ and $\theta : [0, \infty) \to [1, \infty)$ by

$$g(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1] \\ x^2, & \text{otherwise}, \end{cases}$$
$$\alpha(u, v, w) = \begin{cases} 1, & \text{if } u, v, w \in [0, 1] \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\theta(t) = e^t.$$  

Assume that $\frac{1}{12} G_b(u, gu, gu) \leq G_b(u, v, w)$. If one of $u, v, w \notin [0, 1]$, then $a(u, v, w) = 0$ and so, the conclusion of (3.1) is satisfied. If $u, v, w \in [0, 1]$, then $gu, gv, gw \in [0, 1]$ and $a(u, v, w) \geq 1$ with $gu \neq gv \neq gw$. Hence,

$$a(u, v, w) \theta(4G_b(gu, gv, gw)) = e^{a(\frac{1}{2}(u-v) + (v-w) + (u-w))^2} = e^{\frac{1}{2}((u-v) + (v-w) + (u-w))^2} \leq e^{3((u-v) + (v-w) + (u-w))^2} = \left(e^{((u-v) + (v-w) + (u-w))^2}\right)^{\frac{1}{2}} = \left(e^{a_g(u,v,w)}\right)^{\frac{1}{2}} = \left(\theta(G_b(u, v, w))\right)^{\frac{1}{2}}.$$  

Thus all conditions of Theorem 3.15 are satisfied and $x = 0$ is the unique fixed point of $g$.

**Corollary 3.19.** Let $(X, G_b)$ be a complete $G_b$-metric space with $s > 1$. Let $\alpha : X \times X \times X \to (0, \infty)$ and $g$ be a rectangular $\alpha$-admissible mapping. Suppose that there exist $\theta \in \Omega$ and $r, \delta, \beta, \gamma \in (0, 1)$ with $\delta + \beta + \gamma < 1$ such that

$$\frac{1}{3s^2} G_b(u, gu, gu) \leq G_b(u, v, w) \Rightarrow a(u, v, w) \theta \left( s^2 G_b(gu, gv, gw) \right) \leq \left[ \theta \left( \delta G_b(u, v, w) + \beta \frac{G_b(u, gu, gu) G_b(v, gv, gw)}{1 + G_b(u, u, w)} + G_b(u, gu, gu) G_b(v, gv, gw) \right) \right]^r$$  

for all $u, v, w \in X$ with at least two of $gu$, $gv$ and $gw$ being not equal. Also, suppose that the following assertions hold:
(i) There exists $a_0 \in X$ such that $a(a_0, g a_0, g a_0) \geq 1$;
(ii) For any convergent sequence $\{a_n\}$ to $a$ with $a(a_n, a_{n+1}, a_{n+2}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, we have $a(a_n, a, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then $g$ has a fixed point.

(iii) Moreover, if for all $u, v \in \text{Fix}(g)$ implies $a(u, v, v) \geq 1$, then the fixed point is unique where $\text{Fix}(g) = \{u; gu = u\}$.

**Corollary 3.20.** Let $(X, G_b)$ be a complete $G_b$-metric space with $s > 1$. Let $a : X \times X \times X \to (0, \infty)$ and $g$ be a rectangular $a$-admissible mapping. Suppose that there exist $\theta \in \Omega$ and $r, \delta, \beta, \gamma \in (0, 1)$ with $\delta + \beta + \gamma < 1$ such that

$$
\frac{1}{3^s} G_b(u, gu, gu) \leq G_b(u, v, w) \Rightarrow a(u, v, w) \theta \left( s^2 G_b(gu, gv, gw) \right)
$$

$$
\leq \left[ \theta \left( \delta G_b(u, v, w) + \beta \frac{G_b(a(u, v, w) G_b(v, v, v) G_b(v, v, v)}{1 + G_b(u, v, w) G_b(v, v, v) G_b(v, v, v)} \right) \right]^\gamma
$$

for all $u, v, w \in X$ with at least two of $gu, gv$ and $gw$ being not equal. Also, suppose that the following assertions hold:

(i) There exists $a_0 \in X$ such that $a(a_0, g a_0, g a_0) \geq 1$;

(ii) for any convergent sequence $\{a_n\}$ to $a$ with $a(a_n, a_{n+1}, a_{n+2}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, we have $a(a_n, a, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then $g$ has a fixed point.

(iii) Moreover, if for all $u, v \in \text{Fix}(g)$ implies $a(u, v, v) \geq 1$, then the fixed point is unique where $\text{Fix}(g) = \{u; gu = u\}$.

**Corollary 3.21.** Let $(X, G_b)$ be a complete $G_b$-metric space with $s > 1$. Let $a : X \times X \times X \to (0, \infty)$ and $g$ be a rectangular $a$-admissible mapping. Suppose that there exist $\theta \in \Omega$ and $r, \delta, \beta, \gamma \in (0, 1)$ with $\delta + \beta + \gamma < 1$ such that

$$
\frac{1}{3^s} G_b(u, gu, gu) \leq G_b(u, v, w) \Rightarrow a(u, v, w) \theta \left( s^2 G_b(gu, gv, gw) \right)
$$

$$
\leq \left[ \theta \left( \delta G_b(u, v, w) + \beta \frac{G_b(a(u, v, w) G_b(v, v, v) G_b(v, v, v)}{1 + G_b(u, v, w) G_b(v, v, v) G_b(v, v, v)} \right) \right]^\gamma
$$

for all $u, v, w \in X$ with at least two of $gu, gv$ and $gw$ being not equal. Also, suppose that the following assertions hold:

(i) There exists $a_0 \in X$ such that $a(a_0, g a_0, g a_0) \geq 1$;

(ii) for any convergent sequence $\{a_n\}$ to $a$ with $a(a_n, a_{n+1}, a_{n+2}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, we have $a(a_n, a, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then $g$ has a fixed point.

(iii) Moreover, if for all $u, v \in \text{Fix}(g)$ implies $a(u, v, v) \geq 1$, then the fixed point is unique where $\text{Fix}(g) = \{u; gu = u\}$.

Taking $\theta(t) = e^t$ for all $t > 0$, in the above corollaries we get the following new results.

**Corollary 3.22.** Let $(X, G_b)$ be a complete $G_b$-metric space with $s > 1$. Let $a : X \times X \times X \to (0, \infty)$ and $g$ be a rectangular $a$-admissible mapping. Suppose that there exist $\theta \in \Omega$ and $r, \delta, \beta, \gamma \in (0, 1)$ with $\delta + \beta + \gamma < 1$ such that

$$
\frac{1}{3^s} G_b(u, gu, gu) \leq G_b(u, v, w) \Rightarrow a(u, v, w) + s^2 G_b(gu, gv, gw)
$$

$$
\leq r \left[ \delta G_b(u, v, w) + \beta \frac{G_b(a(u, v, w) G_b(v, v, v) G_b(v, v, v)}{1 + G_b(u, v, w) G_b(v, v, v) G_b(v, v, v)} + \gamma \frac{G_b(a(u, v, w) G_b(v, v, v) G_b(v, v, v)}{1 + G_b(gu, gv, gw)} \right]
$$

for all $u, v, w \in X$ with at least two of $gu, gv$ and $gw$ being not equal. Also, suppose that the following assertions hold:
(i) There exists \( a_0 \in X \) such that \( a(a_0, g a_0, g a_0) \geq 1 \);
(ii) For any convergent sequence \( \{ a_n \} \) to \( a \) with \( a(a_n, a_{n+1}, a_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \), we have \( a(a_n, a, a) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \).

Then \( g \) has a fixed point.

(iii) Moreover, if for all \( u, v \in \text{Fix}(g) \) implies \( a(u, v, v) \geq 1 \), then the fixed point is unique where \( \text{Fix}(g) = \{ u; gu = u \} \).

**Corollary 3.23.** Let \((X, G_b)\) be a complete \( G_b \)-metric space (with parameter \( s > 1 \)). Let \( a : X \times X \times X \to (0, \infty) \) and \( g \) be a rectangular \( a \)-admissible mapping. Suppose that there exist \( \theta \in \Omega \), and \( r, \delta, \beta, \gamma \in (0, 1) \) with \( \delta + \beta + \gamma < 1 \) such that

\[
\frac{1}{3s^2} G_b(u, gu, gu) \leq G_b(u, gu, gu) \Rightarrow \ln a(u, v, w) + s^2 G_b(u, gu, gu) \\
\leq r \left[ \delta G_b(u, v, w) + \beta G_b(v, gu, gu) + \gamma G_b(v, v, gu, gu) \right]
\]

for all \( u, v, w \in X \) with at least two of \( gu, gv \) and \( gw \) being not equal. Also, suppose that the following assertions hold:

(i) There exists \( a_0 \in X \) such that \( a(a_0, g a_0, g a_0) \geq 1 \);
(ii) For any convergent sequence \( \{ a_n \} \) to \( a \) with \( a(a_n, a_{n+1}, a_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \), we have \( a(a_n, a, a) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \).

Then \( g \) has a fixed point.

(iii) Moreover, if for all \( u, v \in \text{Fix}(g) \) implies \( a(u, v, v) \geq 1 \), then the fixed point is unique where \( \text{Fix}(g) = \{ u; gu = u \} \).

**Corollary 3.24.** Let \((X, G_b)\) be a complete \( G_b \)-complete metric space with \( s > 1 \). Let \( a : X \times X \times X \to (0, \infty) \) and \( g \) be a rectangular \( a \)-admissible mapping. Suppose that there exist \( \theta \in \Omega \), and \( r, \delta, \beta, \gamma \in (0, 1) \) with \( \delta + \beta + \gamma < 1 \) such that

\[
\frac{1}{3s^2} G_b(u, gu, gu) \leq G_b(u, gu, gu) \Rightarrow \ln a(u, v, w) + s^2 G_b(u, gu, gu) \\
\leq r \left[ \delta G_b(u, v, w) + \beta G_b(v, gu, gu) + \gamma G_b(v, v, gu, gu) \right]
\]

for all \( u, v, w \in X \) with at least two of \( gu, gv \) and \( gw \) being not equal. Also, suppose that the following assertions hold:

(i) There exists \( a_0 \in X \) such that \( a(a_0, g a_0, g a_0) \geq 1 \);
(ii) For any convergent sequence \( \{ a_n \} \) to \( a \) with \( a(a_n, a_{n+1}, a_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \), such that \( a_n \to x \) as \( n \to \infty \), we have \( a(a_n, a, a) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \).

Then \( g \) has a fixed point.

(iii) Moreover, if for all \( u, v \in \text{Fix}(g) \) implies \( a(u, v, v) \geq 1 \), then the fixed point is unique where \( \text{Fix}(g) = \{ u; gu = u \} \).

**Competing interests:** The authors declare that they have no competing interests.

**Authors’ contributions:** All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

**Acknowledgement:** The authors sincerely thank the referees for valuable suggestions which improved the presentation of the paper.
References

[26] Hussain N., Parvaneh V., Vetro C., Some fixed point theorems for generalized contractive mappings in complete metric spaces, Fixed Point Theory Appl., 2015, 2015:185
[27] Onsod W., Saleewong T., Ahmad J., Al-Mazrooei A., Poom Kumam, Fixed points of a $\Theta$-contraction on metric spaces with a graph, Commun. Nonlinear Anal., 2016, 2, 139-149