Radosława Kranz* and Aleksandra Rzepka

On the approximation of integrable functions by some special matrix means of Fourier series

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Some special cases are also formulated as corollaries.

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1 Introduction

Let $L^p$ ($1 \leq p < \infty$) be the class of all $2\pi$–periodic real–valued functions, integrable in the Lebesgue sense, with $p$–th power over $Q = [-\pi, \pi]$ with the norm

$$\|f\|_{L^p} := \left( \int_Q |f(t)|^p \, dt \right)^{1/p} \quad \text{when} \quad 1 \leq p < \infty.$$

Consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

and denote by

$$S_kf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{k} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

the $k$-th partial sums of $Sf$.

Let $A := (a_{n,k})$ and $B := (b_{n,k})$ be infinite lower-triangular matrices with real entries, such that

$$a_{n,-1} = 0, \quad a_{n,k} \geq 0 \quad \text{and} \quad b_{n,k} \geq 0 \quad \text{when} \quad k = 0, 1, 2, \ldots, n,$$

$$a_{n,k} = 0 \quad \text{and} \quad b_{n,k} = 0 \quad \text{when} \quad k > n,$$

$$\sum_{k=0}^{n} a_{n,k} = 1 \quad \text{and} \quad \sum_{k=0}^{n} b_{n,k} = 1, \quad \text{where} \quad n = 0, 1, 2, \ldots$$

*Corresponding Author: Radosława Kranz: University of Zielona Góra, Faculty of Mathematics, Computer Science and Econometrics, 65-516 Zielona Góra, ul. Szafrana 4a, Poland, E-mail: R.Kranz@wmie.uz.zgora.pl

Aleksandra Rzepka: University of Zielona Góra, Faculty of Mathematics, Computer Science and Econometrics, 65-516 Zielona Góra, ul. Szafrana 4a, Poland, E-mail: A.Rzepka@wmie.uz.zgora.pl

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and let, for \( m = 0, 1, 2, \ldots, n, \)
\[
A_{n,m} = \sum_{k=0}^{m} a_{n,k} \quad \text{and} \quad \overline{A}_{n,m} = \sum_{k=m}^{n} a_{n,k}.
\]

Let the \( AB\)-transformation of \( (S_k f) \) be given by
\[
T_{n,A,B} f(x) := \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} S_k f(x) \quad (n = 0, 1, 2, \ldots).
\]

Following L. Leindler [1] a sequence \( c := (c_r) \) of nonnegative numbers tending to zero is called the \emph{Mean Rest Variation Sequence}, or briefly \( c \in \text{MRBVS} \), if it has the property
\[
\sum_{r=m}^{\infty} |c_r - c_{r+1}| \leq K(c) \frac{1}{m+1} \sum_{m \geq m/2} c_r,
\]  
for every positive integer \( m \).

Similarly, following W. Lenski and B. Szal [2], a sequence \( c := (c_r) \) of nonnegative numbers will be called the \emph{Mean Head Bounded Variation Sequence}, or briefly \( c \in \text{MHBVS} \), if it has the property
\[
\sum_{r=0}^{m-1} |c_r - c_{r+1}| \leq K(c) \frac{1}{m+1} \sum_{r=n-m}^{n} c_r,
\]  
for all positive integers \( m < n \), where the sequence \( c \) has finitely many nonzero terms and the last nonzero term is \( c_n \).

Moreover, we assume that the sequence \( (K(a_n))_{n=0}^{\infty} \) is bounded, that is, that there exists a constant \( K \), such that
\[
0 \leq K(a_n) \leq K
\]
holds for every \( n \), where \( K(a_n) \) denote the constants appearing in the inequalities (3) or (4) for the sequences \( a_n = (a_{n,r})_{r=0}^{n} \), \( n = 0, 1, 2, \ldots \).

Next, we assume that for every \( n \) and \( 0 \leq m < n \)
\[
\sum_{r=m}^{n-1} |a_{n,r} - a_{n,r+1}| \leq K \frac{1}{m+1} \sum_{m \geq m/2} a_{n,r}
\]
or
\[
\sum_{r=0}^{m-1} |a_{n,r} - a_{n,r+1}| \leq K \frac{1}{m+1} \sum_{r=n-m}^{n} a_{n,r}
\]
hold if \( (a_{n,r})_{r=0}^{n} \) belongs to \( \text{MRBVS} \) or \( \text{MHBVS} \), for \( n = 1, 2, \ldots \), respectively.

Let
\[
|A|_{n,m} = \begin{cases} 
A_{n,m}, & \text{when} \ (a_{n,r})_{r=0}^{n} \in \text{MRBVS}, \\
\overline{A}_{n,n-m}, & \text{when} \ (a_{n,r})_{r=0}^{n} \in \text{MHBVS}
\end{cases}
\]
and
\[
\Phi_\pi(t) = \int_{0}^{t} |\varphi_\pi(u)| \, du,
\]
where
\[
\varphi_\pi(t) = f(x + t) + f(x - t) - 2f(x).
\]

As a measure of pointwise approximation of a function \( f \) by \( T_{n,A,B} f \) we will use the generalized pointwise modulus of continuity of \( f \) defined by the formula
\[
\omega_\pi [f, \beta] (t) := \frac{1}{t} \beta \left( \frac{\pi}{t} \right) \Phi_\pi(t) = \frac{1}{t} \beta \left( \frac{\pi}{t} \right) \int_{0}^{t} |\varphi_\pi(u)| \, du,
\]  
(5)
where β is a positive and non-decreasing function of t.

The deviation $T_{n,A,B}f(t) = f(t)$ with lower-triangular infinite matrices A and B was investigated among others by W. Łenski and B. Szal [3], Xh. Z. Krasniqi [4], R. Kranz and A. Rzepka [5]. In this paper the results corresponding to the following theorems of Xh. Z. Krasniqi [4] and W. Łenski and B. Szal [2] are shown.

The series $Sf(x)$ with partial sums $S_j f(x)$ is said to be summable to s by $A(E,r)$-means when the transformation

$$T_{n,A,(E,r)} f(x) := \sum_{n=0}^{\infty} a_{n,v} \left( \sum_{j=0}^{v} \left( \frac{v}{j} \right) \right)^r S_j f(x), \quad (r > 0, \ n = 0, 1, 2, \ldots),$$

tends to s, as $n \to \infty$, where s is a finite number.

**Theorem A** [4] Let $A := (a_{n,k})$ be a positive lower-triangular matrix, such that $\sum_{k=0}^{\infty} a_{n,k} = 1$ and $(a_{n,k})_{k=0}^{n}$ is a non-increasing sequence, $(n = 0, 1, 2, \ldots)$. Let $\beta(t)$ be a positive and non-decreasing function of $t$. If

$$\Phi_x(t) = o_x \left( \frac{t}{\beta \left( \frac{1}{t} \right)} \right), \quad as \ t \to 0^+,$$

and $\beta(t) \to \infty$, as $t \to \infty$, then a sufficient condition for the Fourier series $Sf(x)$ to be summable to $f(x)$ by $A(E,r)$ means is

$$\int_1^{\infty} \frac{A_n[u]}{u^\beta(u)} \ du = O(1), \quad as \ n \to \infty. \quad (6)$$

A lower-triangular matrix $C = (c_{n,k})$ is called a maximal hump matrix if, for each n, there exists an integer $k_0 = k_0(n)$, such that $(c_{n,k})_{k=0}^{k_0}$ is non-decreasing for $0 \leq k < k_0$, and $(c_{n,k})_{k=k_0}^{n}$ is non-increasing for $k_0 \leq k \leq n$. Denote by

$$w_{p}^d f(\delta) \beta = \left\{ \begin{array}{ll} \left( \frac{1}{\pi} \int_{0}^{\pi} \varphi_x(u) \sin^\beta \left( \frac{u}{2} \right) \ du \right)^{1/p} & \text{when } 1 \leq p < \infty, \\
\operatorname{ess sup}_{0 < u < \delta} | \varphi_x(u) \sin^\beta \left( \frac{u}{2} \right) | & \text{when } p = \infty. \end{array} \right.$$

**Theorem B** [2] Let $f \in L^p$ with $1 \leq p \leq \infty$. If matrix A is a maximal hump matrix with $k_0 = 1$ ($n^{-1}$), such that $(a_{n,r})_{r=0}^{n} \in \text{MRBV}$ and

$$\left| \sum_{r=\mu}^{\nu} \sum_{k=0}^{\tau} b_{r,k} \sin \left( \frac{2k+1}{2} \right) \right| \lesssim \tau,$$

for $0 \leq \mu \leq \nu$ and $\tau = \left\lfloor \frac{\tau}{\pi} \right\rfloor$, with $t \in \left[ \frac{\pi}{n+\tau}, \tau \right]$ and $n = 1, 2, \ldots$, then

$$|T_{n,A,B}f(x) - f(x)| = O_{x} \left( (n+1)^\beta \left[ w_{p}^d f \left( \frac{\pi}{n+1} \right)^{\beta} + \frac{1}{n+1} \sum_{k=0}^{n} w_{p}^d f \left( \frac{\pi}{k+1} \right)^{\beta} \right] \right)$$

for almost all considered $x$ and $0 \leq \beta < 1 - \frac{1}{p}$, when $p > 1$, and $\beta = 0$, when $p = 1$.

We shall write $J_1 \ll J_2$, if there exists a positive constant $C$, sometimes depending on some parameters, such that $J_1 \leq CJ_2$.

## 2 Statement of the results

Let $w_x$ be a positive function of modulus continuity type, i.e., $w_x(0) = 0$, $w_x(t_1) \leq w_x(t_2) \leq w_x(t_1 + t_2) \leq w_x(t_1) + w_x(t_2)$, for any $0 \leq t_1 \leq t_2 \leq t_1 + t_2 \leq 2\pi$.

Now we can formulate our main results:

**Theorem 1.** Let $f \in L^p$ ($1 \leq p < \infty$) and let the entries of the matrices $A = (a_{n,r})$ and $B = (b_{r,k})$ satisfy the conditions (1) and (2). Additionally, let $(b_{r,k})$ be such that

$$\left| \sum_{r=\mu}^{\nu} \sum_{k=0}^{\tau} b_{r,k} \sin \left( \frac{2k+1}{2} \right) \right| \lesssim \tau \quad (7)$$
holds, for $0 \leq \mu \leq v \leq n$, and $\tau = \left[ \frac{n}{\Delta} \right]$, with $t \in \left[ \frac{n}{\Delta + 1}, \pi \right]$, when $n = 1, 2, \ldots$

If a sequence $(a_{n,r})_{r=0}^{n} \in MRBVS \cup MHBVS$,

$$|A|_{n,r} = O \left( \frac{t}{n + 1} \right),$$

and for some $x$

$$\frac{1}{t^{\beta}} \left( \frac{\pi}{t} \right) \int_{0}^{t} |\varphi_{x}(u)| \, du = O(w_{x}(t)),$$

where $w_{x}$ has a continuous derivative, $\beta(t)$ and $t^{\beta(t)}$ are positive and non-decreasing functions of $t$, then

$$|T_{n,A,B}f(x) - f(x)| = O_{x}(1) \left\{ w_{x} \left( \frac{\pi}{n + 1} \right) + \frac{1}{\beta(n + 1)} \int_{0}^{t} w_{x}(t) + tw'_{x}(t) \, dt \right\}$$

holds, for all considered $x$.

More generally:

**Theorem 2.** Let $f \in L^{p}$ ($1 \leq p < \infty$) and let the entries of the matrices $A = (a_{n,r})$ and $B = (b_{r,k})$ satisfy the conditions (1) and (2). Additionally, let $(b_{r,k})$ satisfy (7), for $0 \leq \mu \leq v \leq n$, and $\tau = \left[ \frac{n}{\Delta} \right]$, with $t \in \left[ \frac{n}{\Delta + 1}, \pi \right]$, when $n = 1, 2, \ldots$

If a sequence $(a_{n,r})_{r=0}^{n} \in MRBVS \cup MHBVS$ satisfies (8), $t^{\beta(t)}$, $\beta(t)$ are positive and non-decreasing functions of $t$, then

$$|T_{n,A,B}f(x) - f(x)| = O_{x} \left( \frac{1}{\beta(n + 1)} \omega_{x} [f, \beta] \left( \frac{\pi}{n + 1} \right) + \frac{1}{n + 1} \sum_{k=1}^{n} \frac{1}{\beta(k + 1)} \omega_{x} [f, \beta] \left( \frac{\pi}{k} \right) + \frac{1}{n} \sum_{k=1}^{n} \frac{a_{n,k}}{\beta(k + 1)} \omega_{x} [f, \beta] \left( \frac{\pi}{k} \right) \right)$$

holds, for all considered $x$, where

$$a_{n,k} = \left\{ \begin{array}{ll} a_{n,k+1}, & \text{when } (a_{n,r})_{r=0}^{n} \in MRBVS, \\ a_{n,n-k+1}, & \text{when } (a_{n,r})_{r=0}^{n} \in MHBVS. \end{array} \right.$$

## 3 Corollaries

Finally, we give some corollaries as an application of our results.

**Corollary 1.** Theorem A, from Xh. Z. Krasniqi’s paper [4, Theorem 3.1], and Theorem 1 are comparable. Condition (6) is more general than (8), but condition (7) is satisfied by a more general class of summability methods.

**Corollary 2.** Taking $\beta(t) = t^{\alpha}$, $(0 < \alpha \leq 1)$ and assuming

$$\frac{1}{t^{\alpha}} \int_{0}^{t} |\varphi_{x}(u)| \, du = O_{x} \left( t^{\alpha} \right), \quad (0 < \alpha \leq 1)$$

we get

$$\omega_{x} [f, \beta] (t) = O_{x} \left( t^{\alpha-\eta} \right), \quad \text{for } \alpha > \eta.$$  

Hence, under the assumptions for the matrices $A = (a_{n,r})$ and $B = (b_{r,k})$ from Theorem 2, we have

$$|T_{n,A,B}f(x) - f(x)| = o_{x}(1)$$

for almost all considered $x$. 
Example 3. Let \( a_{n,r} = \frac{1}{n+1} \), when \( r = 0, 1, 2, \ldots, n, a_{n,r} = 0 \), when \( r > n \), \( b_{r,k} = \binom{r}{k} \frac{\gamma^k}{(1 + \gamma)^k} \), when \( k = 0, 1, 2, \ldots, r \), and \( b_{n,r} = 0 \), when \( k > r \) with \( \gamma > 0 \). Clearly, the sequence \( (a_{n,r})^n_{r=0} \in MRBVS \cup MHBVS \) and \( |A_{n,r}| = 0 \left( \frac{r}{n+1} \right) \), for \( \tau = \frac{\pi}{n+1} \), when \( n = 1, 2, \ldots \). W. Łenski and B. Szal proved in [2, proof of Corollary 2.4.1] that the sequence \( (b_{r,k})_{k=0} \) satisfies the condition

\[
\left| \sum_{r=0}^{v} \sum_{k=0}^{r} b_{r,k} \sin \left( \frac{(2k+1)t}{2} \right) \right| \ll \tau,
\]

for \( 0 \leq \mu \leq v \). So conditions (7) and (8) are satisfied. If \( f \in L^p \) and (9) holds, where \( \beta(t) \) and \( t\beta(\frac{\pi}{n+1}) \) are positive and non-decreasing functions of \( t \), then

\[
\left| \sum_{r=0}^{n} \sum_{k=0}^{r} \binom{r}{k} \gamma^k S_k f(x) - f(x) \right| = O_x(1) \left\{ w_x \left( \frac{\pi}{n+1} \right) + \frac{1}{\beta(n+1)} \int \frac{w_x(t)}{t} \frac{\pi}{n+1} dt \right\}.
\]

Remark 1. Analogously, for sequences \( (a_{n,r})^n_{r=0} \) and \( (b_{r,k})_{k=0}^r \) defined as above, we can derive the following estimation from Theorem 2

\[
\left| \sum_{r=0}^{n} \sum_{k=0}^{r} \binom{r}{k} \gamma^k S_k f(x) - f(x) \right| = O_x \left( \frac{1}{\beta(n+1)} \omega_x[f, \beta] \left( \frac{\pi}{n+1} \right) + \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{\beta(k+1)} \omega_x[f, \beta] \left( \frac{\pi}{k} \right) \right),
\]

where \( \beta(t) \) and \( t\beta(\frac{\pi}{n+1}) \) are positive and non-decreasing functions of \( t \).

4 Auxiliary results

We begin this section with some notation following A. Zygmund [6, Section 5 of Chapter II].

It is clear that

\[
S_k f(x) = \frac{1}{n} \int_{-n}^{n} f(x + t) D_k(t) dt
\]

and

\[
T_{n,A,B} f(x) = \frac{1}{n} \int_{-n}^{n} f(x + t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} D_k(t) dt,
\]

where

\[
D_k(t) = \frac{1}{2} \sum_{v=1}^{k} \cos vt = \frac{\sin \left( \frac{(2k+1)t}{2} \right)}{2 \sin \frac{t}{2}}.
\]

Hence,

\[
T_{n,A,B} f(x) - f(x) = \frac{1}{n} \int_{0}^{n} \varphi_x(t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} D_k(t) dt.
\]

Now, we formulate some estimates of the considered kernel.

Lemma 1. (see [6]) If \( t \in \mathbb{R} \), then

\[
|D_k(t)| \leq k + \frac{1}{2}.
\]
Lemma 2. (see [2]) Let \( B = (b_{r,k})_{k=0}^r \) be such that condition (7) holds for \( 0 \leq \mu \leq \nu \). If \( (a_{n,k})_{k=0}^n \in \text{MRBVS} \), then
\[
\left| \sum_{r=0}^{n} a_{n,r} \sum_{k=0}^{r} b_{r,k} D_k(t) \right| \ll \tau A_{n,r}
\]
and if \( (a_{n,k})_{k=0}^n \in \text{MHBVS} \), then
\[
\left| \sum_{r=0}^{n} a_{n,r} \sum_{k=0}^{r} b_{r,k} D_k(t) \right| \ll \tau A_{n,n-r},
\]
where \( \tau = [\pi/t] \) and \( t \in \left[ \frac{\pi}{n+1}, \pi \right] \), for \( n = 0, 1, 2, \ldots \)

5 Proofs of the results

5.1 Proof of Theorem 1

Let
\[
T_{n,A,Bf}(x) - f(x) = \frac{1}{\pi} \left( \int_{0}^{\frac{\pi}{n}} \frac{n}{t} + \int_{\frac{\pi}{n}}^{\pi} \right) G_{n}(t) \varphi_{x}(t) dt,
\]
where
\[
G_{n}(t) = \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} D_k(t)
\]
and
\[
\left| T_{n,A,Bf}(x) - f(x) \right| \leq \frac{1}{\pi} \int_{0}^{\frac{\pi}{n}} G_{n}(t) \varphi_{x}(t) dt + \frac{1}{\pi} \int_{\frac{\pi}{n}}^{\pi} G_{n}(t) \varphi_{x}(t) dt = |I_1| + |I_2|.
\]

For the first term, from Lemma 1 and (9) we obtain
\[
|I_1| \leq \frac{1}{\pi} \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} |D_k(t)| \int_{0}^{\frac{\pi}{n}} |\varphi_{x}(t)| dt
\]
\[
\ll (n+1) \int_{0}^{\frac{\pi}{n}} |\varphi_{x}(t)| dt \ll \frac{1}{\beta(n+1)} \omega_x \left( \frac{n}{n+1} \right).
\]

For the second one, from (7) and Lemma 2 we get
\[
|I_2| \leq \frac{1}{\pi} \int_{\frac{\pi}{n}}^{\pi} |G_{n}(t)| |\varphi_{x}(t)| dt \ll \int_{\frac{\pi}{n}}^{\pi} \frac{|A|_{n,r} d}{dt} (\Phi_x(t)) dt
\]
\[
= \left[ \frac{|A|_{n,r}}{t} \Phi_x(t) \right]_{t=\frac{\pi}{n}}^{t=\pi} - \int_{\frac{\pi}{n}}^{\pi} \Phi_x(t) \frac{d}{dt} \left( \frac{|A|_{n,r}}{t} \right) dt
\]
\[
\ll \left\{ |A|_{n,1} \Phi_x(\pi) - (n+1) |A|_{n,n+1} \Phi_x \left( \frac{n}{n+1} \right) \right. \right.
\]
\[+ \int_{\frac{\pi}{n}}^{\pi} \Phi_x(t) \left[ -\frac{d}{dt} \left( \frac{|A|_{n,r}}{t \beta(\frac{\pi}{t})} \right) \beta(\frac{\pi}{t}) - \frac{d}{dt} \left( \beta(\frac{\pi}{t}) \right) \frac{|A|_{n,r}}{t \beta(\frac{\pi}{t})} dt \right] \right\}.
\]
Further, from (8) and by the monotonicity of the function
\[ \phi(t) - \phi\left(\frac{n}{n+1}\right) \]
and from (8) and (9), we obtain
\[ \int_{\frac{n}{n+1}}^{n} \beta(t) \phi(t) \left(-\frac{d}{dt} \left(\frac{|A_{n,t}|}{t \beta(t)}\right)\right) dt \]
\[ + \int_{\frac{n}{n+1}}^{n} \frac{\beta(t) \phi(t)}{t} \left(\frac{|A_{n,t}|}{\beta(t)}\right)^2 \left(-\frac{d}{dt} \left(\frac{\beta(t)}{t}\right)\right) dt \]
\[ = \Sigma_1 + \Sigma_2 + \Sigma_3. \]

Now we estimate \( \Sigma_1, \Sigma_2 \) and \( \Sigma_3 \) separately. Namely, from (8), (9) and by \( \frac{w_x(t)}{t} \leq 2 \frac{w_x(u)}{u} \), for \( 0 < u < t \),
\[ \Sigma_1 \leq |A_{n,1}| \phi_x(\pi) \leq \frac{1}{(n+1)\beta(1)} w_x(\pi) \leq \frac{1}{\beta(1)} w_x\left(\frac{\pi}{n+1}\right). \]

For the second term, integrating by parts and from (8) and (9), we obtain
\[ \Sigma_2 \leq \int_{\frac{n}{n+1}}^{n} tw_x(t) \left(-\frac{d}{dt} \left(\frac{|A_{n,t}|}{t \beta(t)}\right)\right) dt \]
\[ = \int_{\frac{n}{n+1}}^{n} tw_x(t) \left(\frac{|A_{n,t}|}{t \beta(t)}\right) dt \]
\[ = \frac{1}{\beta(n+1)} w_x\left(\frac{\pi}{n+1}\right) + \int_{\frac{n}{n+1}}^{n} \frac{w_x(t) + tw'_x(t)}{t} dt \]
\[ \leq \frac{1}{\beta(n+1)} w_x\left(\frac{\pi}{n+1}\right) + \int_{\frac{n}{n+1}}^{n} \frac{w_x(t) + tw'_x(t)}{t} dt. \]

By the monotonicity of the function \( t \beta(t) \) we get
\[ \Sigma_2 \leq \frac{1}{\beta(n+1)} w_x\left(\frac{\pi}{n+1}\right) + \int_{\frac{n}{n+1}}^{n} \frac{w_x(t) + tw'_x(t)}{t} dt. \]

Further, from (8) and by the monotonicity of the function \( w_x \)
\[ \Sigma_3 \leq \int_{\frac{n}{n+1}}^{n} w_x(t) t \frac{|A_{n,t}|}{\beta(t)} \left(-\frac{d}{dt} \left(\frac{\beta(t)}{t}\right)\right) dt \leq \]
\[ \leq \int_{\frac{n}{n+1}}^{n} w_x(t) \left(\frac{1}{\beta(t)}\right)^2 \left(-\frac{d}{dt} \left(\frac{\beta(t)}{t}\right)\right) dt \leq w_x\left(\frac{\pi}{n+1}\right) \int_{\frac{n}{n+1}}^{n} \left(\frac{1}{\beta(t)}\right)^2 \left(-\frac{d}{dt} \left(\frac{\beta(t)}{t}\right)\right) dt \leq \frac{1}{\beta(1)} w_x\left(\frac{\pi}{n+1}\right). \]

Combining these estimates we obtain the desired result. \( \square \)

### 5.2 Proof of Theorem 2

As usual let
\[ |T_{n,A,B}f(x) - f(x)| \leq |I_1| + |I_2|. \]
From Lemma 1 and (5), we get
\[ |I_1| \leq \frac{1}{\beta(n + 1)} \omega_x [f, \beta] \left( \frac{\pi}{n + 1} \right). \]

For the second term, from Lemma 2 and by (10) in the previous proof we have
\[ |I_2| \leq \frac{1}{n} \int_{\pi/n}^{\pi} |G_n(t)| |\varphi_x(t)| \, dt \ll \frac{1}{n} \int_{\pi/n}^{\pi} \frac{|A|_{n,r}}{t} \left( \frac{\beta(\pi/t)}{\pi} \right) \, dt \]
\[ \leq \left| A_{n,1} \right| \Omega_x(\pi) - (n + 1) \Phi_x \left( \frac{\pi}{n + 1} \right) + \int_{\pi/n}^{\pi} \beta \left( \frac{\pi}{t} \right) \Omega_x(t) \left[ -\frac{d}{dt} \left( \frac{|A|_{n,r}}{t \beta(\pi/t)} \right) \right] \, dt \]
\[ + \int_{\pi/n}^{\pi} \frac{|A|_{n,r}}{\beta \left( \frac{\pi}{t} \right)} \Phi_x(t) \left[ -\frac{d}{dt} \left( \frac{\pi}{t} \right) \right] \, dt = \Lambda_1 + \Lambda_2 + \Lambda_3. \]

Now we estimate \( \Lambda_1, \Lambda_2 \) and \( \Lambda_3 \). For \( \Lambda_1 \) from (5) and (8) we obtain
\[ \Lambda_1 \ll \frac{1}{n + 1} \Phi_x(\pi) - (n + 1) \Phi_x \left( \frac{\pi}{n + 1} \right) \ll \frac{1}{n + 1} \frac{\pi}{\beta(1)} \omega_x [f, \beta](\pi) \]
\[ \ll \frac{1}{n + 1} \sum_{k=1}^{n} \frac{1}{\beta(k + 1)} \omega_x [f, \beta] \left( \frac{\pi}{k} \right). \]

For the second term, by the monotonicity of the function \( \beta \left( \frac{\pi}{t} \right) \) and \( \Phi_x(t) \) we get
\[ \Lambda_2 \leq \sum_{k=1}^{n} \int_{\pi/k}^{\pi} \frac{1}{\beta(\pi/t)} \Phi_x(t) \left[ -\frac{d}{dt} \left( \frac{|A|_{n,r}}{t \beta(\pi/t)} \right) \right] \, dt \]
\[ \ll \sum_{k=1}^{n} \frac{k + 1}{n} \beta(k + 1) \Phi_x \left( \frac{\pi}{k} \right) \int_{\pi/k}^{\pi} \left[ -\frac{d}{dt} \left( \frac{|A|_{n,r}}{t \beta(\pi/t)} \right) \right] \, dt. \]

Further, integrating by parts and using (5) and (8) we obtain
\[ \Lambda_2 \leq \sum_{k=1}^{n} \omega_x [f, \beta] \left( \frac{\pi}{k} \right) \left[ \frac{|A|_{n,k+1}}{\beta(k + 1)} - \frac{|A|_{n,k}}{\beta(k)} + \frac{\pi}{n + 1} \int_{\pi/k}^{\pi} \frac{1}{t^2 \beta(\pi/t)} \, dt \right]. \]

Since
\[ \frac{|A|_{n,k+1}}{\beta(k + 1)} - \frac{|A|_{n,k}}{\beta(k)} \leq \frac{|A|_{n,k+1} - |A|_{n,k}}{\beta(k + 1)} \]
\[ = \begin{cases} \frac{a_n}{\beta(k + 1)}, & \text{when } (a_n)_{n=0} \in MRBVS \\ \frac{a_{n,k+1}}{\beta(k+1)}, & \text{when } (a_n)_{n=0} \in MHBVS \end{cases} \]
\[ := \begin{cases} a_n / \beta(k + 1), & \text{when } (a_n)_{n=0} \in MRBVS \\ a_{n,k+1} / \beta(k+1), & \text{when } (a_n)_{n=0} \in MHBVS \end{cases}, \]

therefore
\[ \Lambda_2 \ll \sum_{k=1}^{n} \omega_x [f, \beta] \left( \frac{\pi}{k} \right) \left[ \frac{a_n}{\beta(k + 1)} + \frac{1}{n + 1} \frac{1}{\beta(k + 1)} \right]. \]

For the third term by (5) and (8) we obtain
\[ \Lambda_3 \leq \sum_{k=1}^{n} \int_{\pi/k}^{\pi} \Phi_x(t) \left[ -\frac{d}{dt} \left( \frac{\pi}{t \beta(\pi/t)} \right) \right] \, dt \]
\[ \ll \frac{\pi}{n + 1} \sum_{k=1}^{n} \int_{\pi/k}^{\pi} \Phi_x(t) \frac{1}{t \beta(\pi/t)} \left[ -\frac{d}{dt} \left( \frac{\pi}{t \beta(\pi/t)} \right) \right] \, dt. \]
On the approximation of integrable functions by some special matrix means of Fourier series

\[
\sum_{k=1}^{n} k + 1 \frac{1}{\pi} \int_{\pi}^{\pi} \left[ -\frac{d}{dt} \left( \beta \left( \frac{\pi}{k} \right) \right) \right] dt
\]

\[
\beta (k + 1) - \beta (k) = \frac{1}{k} \left( (k + 1) \beta (k + 1) - k \beta (k) \right) 
\]

\[
= \frac{1}{k} \beta (k).
\]

So

\[
\Lambda_3 \leq \frac{1}{n + 1} \sum_{k=1}^{n} \frac{1}{\beta (k + 1)} \omega_x \int_{\pi}^{\pi} \left[ -\frac{d}{dt} \left( \beta \left( \frac{\pi}{k} \right) \right) \right] dt
\]

The desired result follows by combining these estimates.

References