Research Article

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Approximation of additive functional equations in NA Lie C*-algebras

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Abstract: In this paper, by using fixed point method, we approximate a stable map of higher *-derivation in NA C*-algebras and of Lie higher *-derivations in NA Lie C*-algebras associated with the following additive functional equation

\[ \sum_{i=1}^{m} T\left( u_i + \frac{1}{m} \sum_{j=1, j\neq i}^{m} u_j \right) + T\left( \frac{1}{m} \sum_{i=1}^{m} u_i \right) = 2 T\left( \sum_{i=1}^{m} u_i \right), \]

where \( m \geq 2 \).

Keywords: Fixed point method, approximation, higher *-derivations, Lie higher *-derivations, non-Archimedean Lie C*-algebras

MSC: 39B82, 39B52, 16W25, 46L05, 47H10

1 Introduction and Preliminaries

Najati and Eskandani [1] introduced the following additive functional equation

\[ \sum_{i=1}^{m} T\left( u_i + \frac{1}{m} \sum_{j=1, j\neq i}^{m} u_j \right) + T\left( \frac{1}{m} \sum_{i=1}^{m} u_i \right) = 2 T\left( \sum_{i=1}^{m} u_i \right), \]

where \( m \geq 2 \).

In this paper, using some ideas from [2–4], we first introduce the notions of higher *-derivations in non-Archimedean (shortly NA) C*-algebras and Lie higher *-derivations in NA Lie C*-algebras, respectively. Furthermore, we apply the fixed point method to investigate the stability results of higher *-derivations in NA C*-algebras and of Lie higher *-derivations in NA Lie C*-algebras associated with the additive functional equation (1.1).

Following [5–8], we recall some concepts and preliminary results concerning non-Archimedean (NA) normed spaces (NA Banach algebras), which will be used in this paper.

An NA field is a field \( \mathbb{K} \) equipped with a function (valuation) \( | \cdot | \) from \( \mathbb{K} \) into \( [0, \infty) \) such that \( |r| = 0 \) if and only if \( r = 0 \), \( |rs| = |r||s| \), and \( |r + s| \leq \max\{ |r|, |s| \} \) for all \( r, s \in \mathbb{K} \) (see [5, 7, 8]). Clearly, \( |1| = |-1| = 1 \) and \( |n| \leq 1 \) for all \( n \in \mathbb{N} \). By the trivial valuation we mean the function \( | \cdot | \) taking everything except for 0 to 1 and \( |0| = 0 \).

Definition 1.1. (cf. [5, 7, 8]). Let \( X \) be a vector space over a scalar field \( \mathbb{K} \) with an NA non-trivial valuation \( | \cdot | \). A function \( \| \cdot \| : X \to [0, \infty) \) is an NA norm (valuation) if it satisfies the following conditions:

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(I) \( \|x\| = 0 \) if and only if \( x = 0 \);

(II) \( \|rx\| = |r|\|x\| \), for any \( r \in \mathbb{K} \) and \( x \in X \);

(III) \( \|x + y\| \leq \max(\|x\|, \|y\|) \) (the strong triangle inequality), for all \( x, y \in X \).

Then \( (X, \|\cdot\|) \) is called an NA normed space.

Thanks to the below inequality

\[
\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1 \}
\]

holds for all \( x_n, x_m \in X \), where \( m, n \in \mathbb{N} \) with \( n > m \). Therefore, a sequence \( \{x_n\} \) is Cauchy if and only if \( \{x_{n+1} - x_n\} \) converges to zero in an NA normed space. By a complete NA normed space, we mean one in which every Cauchy sequence is convergent.

An NA Banach algebra is a complete NA algebra \( \mathcal{A} \) which satisfies \( \|ab\| \leq \|a\|\|b\| \), for all \( a, b \in \mathcal{A} \). For more detailed definitions of NA Banach algebras (see [9, 10]).

If \( \mathcal{J} \) is an NA Banach algebra, then an involution on \( \mathcal{J} \) is a mapping \( t \to t^* \) from \( \mathcal{J} \) into \( \mathcal{J} \) which satisfies (see [5, 7]);

(I) \( t^{**} = t \) for \( t \in \mathcal{J} \);

(II) \( (as + \beta t)^* = \bar{\alpha} s^* + \bar{\beta} t^* \), for \( a, \beta \in \mathbb{C} \);

(III) \( (st)^* = t^* s^* \) for \( s, t \in \mathcal{J} \).

If, in addition, \( \|t^* t\| = \|t\|^2 \) for \( t \in \mathcal{J} \), then \( \mathcal{J} \) is an NA \( \mathcal{C}^* \)-algebra.

**Lemma 1.1.** ([11]). Let \( (E, d) \) be a complete generalized metric space and \( J : E \to E \) be a strictly contractive mapping with Lipschitz constant \( L < 1 \). Then for each given \( x \in E \), either

\[
d(f^n x, f^{n+1} x) = \infty,
\]

for all non-negative integers \( n \), or there exists a positive integer \( n_0 \), such that;

(i) \( d(f^n x, f^{n+1} x) < \infty \), for all \( n \geq n_0 \);

(ii) the sequence \( \{f^n x\} \) converges to a fixed point \( y^* \) of \( J \);

(iii) \( y^* \) is the unique fixed point of \( J \) in the set \( E^* := \{y \in E : d(f^{n_0} x, y) < \infty \} \);

(iv) \( d(y, y^*) = \frac{1}{1-L} d(y, Jy) \), for all \( y \in E^* \).

## 2 Higher \(*\)-derivations in NA \( \mathcal{C}^* \)-algebras

In this section, assume that \( \mathcal{A} \) is an NA \( \mathcal{C}^* \)-algebra with norm \( \|\cdot\|_A \) and that \( \mathcal{B} \) is an NA \( \mathcal{C}^* \)-algebra with norm \( \|\cdot\|_B \). For each given mapping \( f_n : \mathcal{A} \to \mathcal{B} \) and each \( n = 0, 1, \ldots \), we define

\[
\mathcal{D}_n f_n(x_1, \ldots, x_m) := \sum_{i=1}^{m} f_n \left( \mu_{x_i} + \frac{1}{m} \sum_{j=1}^{m} \mu_{x_j} \right) + f_n \left( \frac{1}{m} \sum_{i=1}^{m} \mu_{x_i} \right) - 2f_n \left( \frac{1}{m} \sum_{i=1}^{m} x_i \right),
\]

for all \( \mu \in \mathbb{T}^m := \{\mu \in \mathbb{C} : |\mu| = 1\} \) and all \( x_1, \ldots, x_m \in \mathcal{A} \). For more results and applications of functional equations we refer to [12–35].

We need the following definition and lemmas to prove the main results.

**Definition 2.1.** Let \( \mathbb{N} \) be the set of natural numbers. For each \( s \in \mathbb{N} \), a sequence \( F = \{f_0, f_1, \ldots, f_s\} \) (resp. \( F = \{f_0, f_1, \ldots, f_n, \ldots\} \)) of mappings from \( \mathcal{A} \) into \( \mathcal{B} \) is called a higher \(*\)-derivation of rank \( s \) (resp. infinite rank) from \( \mathcal{A} \) into \( \mathcal{B} \) if;

(i) \( f_n(x^s) = f_n(x)^s \), for all \( x \in \mathcal{A} \) and \( n = 0, 1, \ldots, s \) (resp. \( n = 0, 1, \ldots \));

(ii) \( f_n(xy) = \sum_{i=0}^{n} f_i(x)f_{n-i}(y) \) holds for \( n = 0, 1, \ldots, s \) (resp. \( n = 0, 1, \ldots \)) and all \( x, y \in \mathcal{A} \).

**Lemma 2.1.** (cf. [1]). Let \( X \) and \( Y \) be real vector spaces. A mapping \( f : X \to Y \) satisfies the functional equation (1.1) if and only if it is additive.
Lemma 2.2. (cf. [12]). Let \( f : \mathcal{A} \to \mathcal{A} \) be an additive mapping, such that \( f(\mu x) = \mu f(x) \), for all \( \mu \in \mathbb{T}^1 \) and all \( x \in \mathcal{A} \). Then the mapping \( f \) is \( C \)-linear.

Theorem 2.1. Let \( \varphi : \mathcal{A}^m \to [0, \infty) \), \( \psi : \mathcal{A}^2 \to [0, \infty) \) and \( \eta : \mathcal{A} \to [0, \infty) \) be functions. Suppose that \( F = \{ f_0, f_1, \ldots, f_n, \ldots \} \) is a sequence of mappings from \( \mathcal{A} \) into \( \mathcal{B} \), such that for each \( n = 0, 1, \ldots ; \)

\[
\| D \mu f_n(x_1, \ldots, x_m) \|_\mathcal{B} \leq \varphi(x_1, \ldots, x_m) \tag{2.1}
\]

\[
\| f_n(xy) - \sum_{i=0}^n f_i(x)f_{n-i}(y) \|_\mathcal{B} \leq \psi(x, y) \tag{2.2}
\]

\[
\| f_n(x') - f_n(x) \|_\mathcal{B} \leq \eta(x) \tag{2.3}
\]

for all \( \mu \in \mathbb{T}^1 \) and all \( x_1, \ldots, x_m \in \mathcal{A} \). Assume that \( |m| < 1 \) is far from zero and there exists \( 0 < L < 1 \), such that:

\[
\varphi(mx_1, \ldots, mx_m) \leq |m|L\varphi(x_1, \ldots, x_m) \tag{2.4}
\]

\[
\psi(mx, my) \leq |m|^2 L\psi(x, y) \tag{2.5}
\]

\[
\eta(mx) \leq |m|L\eta(x) \tag{2.6}
\]

for all \( x, y, x_1, \ldots, x_m \in \mathcal{A} \). Then there exists a unique higher \(*\)-derivation \( H = \{ h_0, h_1, \ldots, h_n, \ldots \} \) of any rank from \( \mathcal{A} \) into \( \mathcal{B} \), such that for each \( n = 0, 1, \ldots ; \)

\[
\| f_n(x) - h_n(x) \|_\mathcal{B} \leq \frac{L}{1 - L} \varphi(0, \ldots, 0, x, 0, \ldots, 0), \tag{2.7}
\]

for all \( x \in \mathcal{A} \).

Proof. Consider the set \( E := \{ g : \mathcal{A} \to \mathcal{B} \} \). Introduce a generalized metric \( d \) on \( E \) as follows:

\[
d(g, q) := \inf \left\{ \delta \in (0, \infty) \mid \| g(x) - q(x) \|_\mathcal{B} \leq \delta \varphi(0, \ldots, 0, x, 0, \ldots, 0), \quad \forall x \in \mathcal{A} \right\}.
\]

It is easy to show that \((E, d)\) is a complete generalized metric space (see [36]). Now we consider the mapping \( \jmath : E \to E \) defined by

\[
\jmath g(x) := \frac{1}{m} g(mx),
\]

for all \( g \in E \) and \( x \in \mathcal{A} \). Let \( g, q \in E \), such that \( d(g, q) \leq \delta \), where \( \delta \in (0, \infty) \) is an arbitrary constant. Then we have

\[
\| g(x) - q(x) \|_\mathcal{B} \leq \delta \varphi(0, \ldots, 0, x, 0, \ldots, 0),
\]

for all \( x \in \mathcal{A} \). Hence,

\[
\| \jmath g(x) - \jmath q(x) \|_\mathcal{B} = \| \frac{1}{m} g(mx) - \frac{1}{m} q(mx) \|_\mathcal{B} = \frac{1}{|m|} \| g(mx) - q(mx) \|_\mathcal{B}
\]

\[
\leq L\delta \varphi(0, \ldots, 0, x, 0, \ldots, 0), \tag{2.9}
\]

for all \( x \in \mathcal{A} \). So \( d(\jmath g, \jmath q) \leq Ld(g, q) \), for all \( g, q \in E \). Thus, \( \jmath \) is a strictly contractive self-mapping on \( E \) with Lipschitz constant \( L \).

Letting \( \mu = 1 \), \( x_j = mx \) and \( x_i = 0 \), for all \( 1 \leq i \leq m \) with \( i \neq j \) in (2.1), we get

\[
\| f_n(mx) - mf_n(x) \|_\mathcal{B} \leq \varphi(0, \ldots, 0, mx, 0, \ldots, 0), \tag{2.10}
\]
for each $n = 0, 1, \ldots$ and all $x \in \mathcal{A}$. It follows from (2.4) and (2.10) that $d(f_n, \eta f_n) \leq L$. By Lemma 1.1, the sequence $\eta f_n$ converges to a fixed point $h_n$ of $\eta$, that is,

$$\lim_{k \to \infty} \frac{1}{m^k} f_n(m^k x) = h(x) \tag{2.11}$$

and

$$h_n(mx) = mh_n(x), \tag{2.12}$$

for each $n = 0, 1, \ldots$ and all $x \in \mathcal{A}$. Also the mapping $h_n$ is the unique fixed point of $\eta$ in the set $E = \{ g \in E : d(f_n, g) < \infty \}$. This implies that $h_n$ is a unique mapping satisfying (2.12), such that there exists a $\delta \in (0, \infty)$ with

$$||f_n(x) - h_n(x)||_\mathcal{B} \leq \delta \varphi(0, \ldots, 0, x, 0, \ldots, 0),$$

for each $n = 0, 1, \ldots$ and all $x \in \mathcal{A}$. Also,

$$d(f_n, h_n) \leq \frac{L}{1 - L} d(f_n, \eta f_n) \leq \frac{L}{1 - L}.$$

This implies that inequality (2.7) holds. Furthermore, it follows from (2.1), (2.4) and (2.11) that

$$\|D_\mu h_n(x_1, \ldots, x_m)\|_\mathcal{B} = \lim_{k \to \infty} \| \frac{1}{m^k} D_\mu f_n(m^k x_1, \ldots, m^k x_m) \|_\mathcal{B} \leq \lim_{k \to \infty} \frac{1}{m^k} \| \varphi(m^k x_1, \ldots, m^k x_m) \| = 0$$

holds for each $n = 0, 1, \ldots$ and all $x_1, \ldots, x_m \in \mathcal{A}$ and $\mu \in \mathbb{T}^1$. So $D_\mu h_n(x_1, \ldots, x_m) = 0$, for all $x_1, \ldots, x_m \in \mathcal{A}$ and $\mu \in \mathbb{T}^1$. If we put $\mu = 1$ in the last equality, then $h_n$ is additive by Lemma 2.1. So letting $x_j = mx$ and $x_i = 0$, for all $1 \leq i \leq m$ with $i \neq j$ in the last equality, we obtain $h_n(mx) = \mu h_n(x)$. Now by using Lemma 2.2, we conclude that the mapping $h_n : \mathcal{A} \to \mathcal{B}$ is $C$-linear for each $n = 0, 1, \ldots$.

It follows from (2.2), (2.5) and (2.11) that

$$\|h_n(xy) - \sum_{l=0}^{n} h_l(x) h_{n-l}(y)\|_\mathcal{B} \leq \lim_{k \to \infty} \frac{1}{|m|^{2k}} \| f_n(m^{2k} xy) - \sum_{l=0}^{n} f_l(m^k x) f_{n-l}(m^k y) \|_\mathcal{B} \leq \lim_{k \to \infty} \frac{1}{|m|^{2k}} \| \psi(m^k x, m^k y) \| = 0,$$

for each $n = 0, 1, \ldots$ and all $x, y \in \mathcal{A}$. That is, we obtain that

$$h_n(xy) = \sum_{l=0}^{n} h_l(x) h_{n-l}(y),$$

for each $n = 0, 1, \ldots$ and all $x, y \in \mathcal{A}$. Also, by (2.3), (2.6), (2.11) and by a similar method to above, we obtain $h_n(x^*) = h_n(x)^*$, for each $n = 0, 1, \ldots$ and all $x, y \in \mathcal{A}$. This completes the proof of the theorem.

**Theorem 2.2.** Suppose that $F = \{ f_0, f_1, \ldots, f_n, \ldots \}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ for which there exist functions $\varphi : \mathcal{A}^m \to [0, \infty)$, $\psi : \mathcal{A}^2 \to [0, \infty)$ and $\eta : \mathcal{A} \to [0, \infty)$, such that (2.1), (2.2) and (2.3) hold for each $n = 0, 1, \ldots$ all $\mu \in \mathbb{T}^1$ and all $x_1, \ldots, x_m, x, y \in \mathcal{A}$. Assume that $|m|<1$ is far from zero and there exists $0 < L < 1$, such that;

$$\varphi(x_1, \ldots, x_m) \leq \frac{L}{|m|} \varphi(mx_1, \ldots, mx_m) \tag{2.13}$$

$$\psi(x, y) \leq \frac{L}{|m|^2} \psi(mx, my) \tag{2.14}$$

$$\eta(x) \leq \frac{L}{|m|} \eta(mx) \tag{2.15}$$
for all \( x, y, x_1, \ldots, x_m \in A \). Then there exists a unique higher \( ^* \)-derivation \( H = \{ h_0, h_1, \ldots, h_n, \ldots \} \) of any rank from \( A \) into \( \mathcal{B} \), such that for each \( n = 0, 1, \ldots, \)

\[
\|f_n(x) - h_n(x)\|_{\mathcal{B}} \leq \frac{1}{1-L} \varphi(0, \ldots, 0, x, \underbrace{0, \ldots, 0}_{\ell n}),
\]

(2.16)

for all \( x \in A \).

**Proof.** Let \( E \) and \( d \) be as in the proof of Theorem 2.1. Then \((E, d)\) becomes a complete generalized metric space. Consider the mapping \( \mathcal{B} : E \to E \) defined by

\[
\mathcal{B}g(x) := mg(\frac{x}{m}), \quad \text{for all } g \in \Omega \text{ and } x \in A.
\]

Then, it is easy to see that \( d(\mathcal{B}g, \mathcal{B}q) \leq Ld(g, q) \), for all \( g, q \in E \). By (2.10) and (2.13), we obtain

\[
\|f_n(x) - mf_n(\frac{x}{m})\|_{\mathcal{B}} \leq \varphi(0, \ldots, 0, x, \underbrace{0, \ldots, 0}_{\ell n}),
\]

for each \( n = 0, 1, \ldots \) and all \( x \in A \). So, we have \( d(f_n, \mathcal{B}f_n) \leq 1 \). By Lemma 1.1, there exists a unique mapping \( h_n : A \to \mathcal{B} \), such that \( h_n(x) = mh_n(\frac{x}{m}) \), for each \( n = 0, 1, \ldots \) and all \( x \in A \), i.e., \( h_n \) is a unique fixed point of \( \mathcal{B} \). Moreover,

\[
h_n(x) = \lim_{k \to \infty} |m|^k f_n(\frac{x}{m^k}),
\]

(2.17)

for each \( n = 0, 1, \ldots \) and all \( x \in A \). Also

\[
d(f_n, h_n) \leq \frac{1}{1-L} d(f_n, \mathcal{B}f_n) \leq \frac{1}{(1-L)},
\]

which implies that (2.16) holds for each \( n = 0, 1, \ldots \) and all \( x \in A \). The remaining assertion is similar to the corresponding part of Theorem 2.1. This completes the proof. \( \square \)

**Corollary 2.1.** Let \( \ell \in (-1, 1) \), \( r \neq 1 \) and \( \theta \) be non-negative real numbers and let \( F = \{ f_0, f_1, \ldots, f_n, \ldots \} \) be a sequence of mappings from \( A \) into \( \mathcal{B} \), such that for each \( n = 0, 1, \ldots, \)

\[
\|Df_n(x_1, \ldots, x_m)\|_{\mathcal{B}} \leq \theta(||x_1||_A + ||x_2||_A + \cdots + ||x_m||_A)
\]

\[
\|f_n(xy) - \sum_{i=0}^{n} f_i(x)f_{n-i}(y)\|_{\mathcal{B}} \leq \theta \cdot (||x||^r_A \cdot ||y||^r_A)
\]

\[
\|f_n(x^*) - f_n(x)^*\|_{\mathcal{B}} \leq \theta \cdot ||x||^r_A,
\]

for all \( \mu \in T^1 \), and \( x_1, \ldots, x_m, x, y \in A \). Then there exists a unique higher \( ^* \)-derivation \( H = \{ h_0, h_1, \ldots, h_n, \ldots \} \) of any rank from \( A \) into \( \mathcal{B} \) such that for each \( n = 0, 1, \ldots \), \( \ell r > \ell \),

\[
\|f_n(x) - h_n(x)\|_{\mathcal{B}} \leq \frac{|m|^r}{\ell(|m| - |m^r|)} \theta \cdot ||x||^r_A,
\]

(2.18)

for all \( x \in A \).

**Proof.** The proof follows from Theorem 2.1 and Theorem 2.2 by taking

\[
\varphi(x_1, \ldots, x_m) = \theta(||x_1||^r_A + ||x_2||^r_A + \cdots + ||x_m||^r_A)
\]

and

\[
\psi(x, y) = \theta \cdot (||x||^r_A \cdot ||y||^r_A), \quad \eta(x) = \theta \cdot ||x||^r_A,
\]

for all \( x_1, \ldots, x_m, x, y \in A \). Choosing \( L = |m|^{(\ell-1)} \), we obtain the desired result. \( \square \)
3 Lie higher *-derivations in NA Lie $C^*$-algebras

An NA $C^*$-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] = \frac{xy - yx}{2}$ on $\mathcal{C}$, is called an NA Lie $C^*$-algebra. In this section, assume that $\mathcal{A}$ is an NA Lie $C^*$-algebra with norm $\| \cdot \|_\mathcal{A}$ and $\mathcal{B}$ is an NA Lie $C^*$-algebra with norm $\| \cdot \|_\mathcal{B}$. Before proceeding to the proofs of the main results, we first introduce the following definition:

**Definition 3.1.** Let $\mathbb{N}$ be the set of natural numbers, for each $s \in \mathbb{N}$, a sequence $F = \{f_0, f_1, \ldots, f_s\}$ (resp. $F = \{f_0, f_1, \ldots, f_n\}$) of mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a Lie higher *-derivation of rank $s$ (resp. infinite rank) from $\mathcal{A}$ into $\mathcal{B}$ if:

(i) $f_n(x^s) = f_n(x)^s$ for all $x \in \mathcal{A}$ and for each $n = 0, 1, \ldots, s$ (resp. $n = 0, 1, \ldots$);

(ii) $f_n((x, y)) = \sum_{i=0}^{n} [f_i(x), f_{n-i}(y)]$ holds for each $n = 0, 1, \ldots, s$ (resp. $n = 0, 1, \ldots$) and all $x, y \in \mathcal{A}$.

**Theorem 3.1.** Let $\varphi : \mathcal{A}^m \rightarrow [0, \infty)$, $\psi : \mathcal{A}^2 \rightarrow [0, \infty)$ and $\eta : \mathcal{A} \rightarrow [0, \infty)$ be functions. Suppose that $F = \{f_0, f_1, \ldots, f_n\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$, such that for each $n = 0, 1, \ldots$;

$$\|\partial^m f_n(x_1, \ldots, x_m)\|_\mathcal{B} \leq \varphi(x_1, \ldots, x_m)$$

and

$$\|f_n((x, y)) - \sum_{i=0}^{n} [f_i(x), f_{n-i}(y)]\|_\mathcal{B} \leq \psi(x, y)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \ldots, x_m, y \in \mathcal{A}$. Assume that $|m| < 1$ is far from zero and that there exists $0 < L < 1$ such that (2.4), (2.5) and (2.6) hold for all $x, y, x_1, \ldots, x_m \in \mathcal{A}$. Then there exists a unique Lie higher *-derivation $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$, such that for each $n = 0, 1, \ldots, (2.7)$ holds for all $x \in \mathcal{A}$.

**Proof.** By the same reasoning as in the proof of Theorem 2.1, there exists a mapping $h_n : \mathcal{A} \rightarrow \mathcal{B}$ which is *-preserving for each $n = 0, 1, \ldots$ and satisfies (2.7) for all $x \in \mathcal{A}$. The mapping $h_n : \mathcal{A} \rightarrow \mathcal{B}$ is given by

$$h_n(x) = \lim_{k \rightarrow \infty} \frac{1}{|m|^k} f_n(m^k x),$$

for each $n = 0, 1, \ldots$ and all $x \in \mathcal{A}$. By (2.5) and (3.2), we have

$$\|h_n((x, y)) - \sum_{i=0}^{n} [h_i(x), h_{n-i}(y)]\|_\mathcal{B} = \lim_{k \rightarrow \infty} \frac{1}{|m|^{2k}} \|f_n(m^{2k}x, m^{2k}y) - \sum_{i=0}^{n} [f_i(m^k x), f_{n-i}(m^k y)]\|_\mathcal{B} \leq \lim_{k \rightarrow \infty} \frac{1}{|m|^{2k}} \psi(m^k x, m^k y) = 0,$$

for each $n = 0, 1, \ldots$ and all $x, y \in \mathcal{A}$. So

$$h_n((x, y)) = \sum_{i=0}^{n} [h_i(x), h_{n-i}(y)],$$

for each $n = 0, 1, \ldots$ and all $x, y \in \mathcal{A}$. Thus, $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ is Lie higher *-derivation, as desired.

**Theorem 3.2.** Suppose that $F = \{f_0, f_1, \ldots, f_n\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ for which there exist functions $\varphi : \mathcal{A}^m \rightarrow [0, \infty)$, $\psi : \mathcal{A}^2 \rightarrow [0, \infty)$ and $\eta : \mathcal{A} \rightarrow [0, \infty)$, such that (3.1), (3.2) and (3.3) hold for each $n = 0, 1, \ldots$, all $\mu \in \mathbb{T}^1$ and all $x_1, \ldots, x_m, y \in \mathcal{A}$. Assume that $|m| < 1$ is far from zero and that there exists $0 < L < 1$ such that (2.13), (2.14) and (2.15) hold, for all $x, y, x_1, \ldots, x_m \in \mathcal{A}$. Then there exists a unique Lie higher *-derivation $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$, such that, for each $n = 0, 1, \ldots$, (2.16) holds, for all $x \in \mathcal{A}$.

**Proof.** The proof is similar to the proof of Theorem 3.1. The result follows from Theorem 2.2.
Corollary 3.1. Let \( \ell \in \{ -1, 1 \}, r \neq 1 \) and \( \theta \) be non-negative real numbers and let \( F = \{ f_0, f_1, \ldots, f_n, \ldots \} \) be a sequence of mappings from \( \mathcal{A} \) into \( \mathcal{B} \), such that, for each \( n = 0, 1, \ldots \),
\[
\| 2\mu f_n(x_1, \ldots, x_m) \|_B \leq \theta (\| x_1 \|_A^{\ell} + \| x_2 \|_A^\ell + \cdots + \| x_m \|_A^\ell)
\]
\[
\| f_n((x, y)) - \sum_{i=0}^n [f_i(x), f_{n-i}(y)] \|_B \leq \theta \cdot (\| x \|_A^{\ell} \cdot \| y \|_A^\ell)
\]
\[
\| f_n(x') - f_n(x)^\ell \|_B \leq \theta \cdot \| x \|_A^\ell,
\]
for all \( \mu \in \mathbb{T}^1 \), and \( x_1, \ldots, x_m, x, y \in \mathcal{A} \). Then there exists a unique Lie higher \( \ast \)-derivation \( H = \{ h_0, h_1, \ldots, h_n, \ldots \} \) of any rank from \( \mathcal{A} \) into \( \mathcal{B} \), such that, for each \( n = 0, 1, \ldots \), if \( \ell \geq 0 \), (2.18) holds, for all \( x \in \mathcal{A} \).

Proof. The proof is similar to the proof of Corollary 2.1. The result follows from Theorem 3.1 and Theorem 3.2.

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References