Approximation of functions and their conjugates in $L^p$ and uniform metric by Euler means

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Abstract: For $2\pi$-periodic functions from $L^p$ (where $1 < p < \infty$) we prove an estimate of approximation by Euler means in $L^p$ metric generalizing a result of L. Rempuska and K. Tomczak. Furthermore, we show that this estimate is sharp in a certain sense. We study the uniform approximation of functions by Euler means in terms of their best approximations in $p$-variational metric and prove the sharpness of this estimate under some conditions. Similar problems are treated for conjugate functions.

Keywords: Euler means, functions of bounded $p$-variation, best approximation, conjugate function, degree of approximation, $L^p$, sharpness

MSC: Primary 42A24; Secondary 40G05, 41A25, 42A50

1 Introduction

Let $1 < p < \infty$, $f$ be a $2\pi$-periodic real measurable bounded function, $\xi = \{x_0 < x_1 < \ldots < x_n = x_0 + 2\pi\}$ be a partition of a period and $\ell_\xi^p(f) := \left(\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p\right)^{1/p}$.

For $1 < p < \infty$ we define $\omega_{1-1/p}(f, \delta)$ to be

$$\omega_{1-1/p}(f, \delta) = \sup_{\xi} \ell_\xi^p(f) : \lambda(\xi) := \max(x_i - x_{i-1}) \leq \delta.$$ 

It is known that $\omega_{1-1/p}(f, n\delta) \leq n^{1/p} \omega_{1-1/p}(f, \delta)$ for $n \in \mathbb{N} = \{1, 2, \ldots\}$ and $\delta \in [0, 2\pi]$ (see [1, Lemma 1]) and this property explains the notation $\omega_{1-1/p}$. Further we will use also notations $V_p(f)$ for $\omega_{1-1/p}(f, 2\pi)$ and $V_p(f, [a, b])$ for $\omega_{1-1/p}(f, b - a)$ and $f$ defined on $[a, b]$.

For $1 < p < \infty$ let us introduce the space $V_p$ of all $2\pi$-periodic bounded functions with the property

$$\|f\|_{V_p} := \max(\|f\|_{\infty}, \omega_{1-1/p}(f, 2\pi)) < \infty$$

and $C_p = \{f \in V_p : \lim_{\delta \to 0} \omega_{1-1/p}(f, \delta) = 0\}$. Here $\|f\|_{\infty} = \sup_{x \in [0, 2\pi]} |f(x)|$. The space $V_p$ of functions of bounded $p$-variation was introduced in the case $p = 2$ by Wiener [2] while the space $C_p$ of $p$-absolutely continuous functions in another but equivalent form was considered by Young [3] (see also the paper of Love [4]). Both $V_p$ and $C_p$ are Banach spaces with respect to $\| \cdot \|_{V_p}$.

For $1 \leq p < \infty$ let $L^p$ be the space of $2\pi$-periodic measurable functions with finite norm

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p \, dx\right)^{1/p},$$

and for $k \in \mathbb{N}$, $\delta \in [0, 2\pi]$,

$$\omega_k^p(f, \delta) := \sup\{\|\Delta_k^p f(x)\| : |h| \leq \delta\},$$

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where $Δ_h^k f(x) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x + ih)$ is the $k$-th difference of $f$ with step $h$.

It is well known that for $f \in L^1$ the conjugate function

$$\tilde{f}(x) = \lim_{\varepsilon \to 0} (2\pi)^{-1} \int_{-\varepsilon}^{\varepsilon} (f(x - t) - f(x + t)) \cot(t/2) \, dt$$

exists a.e. By a theorem of M. Riesz [5, Ch. VIII, § 14] if $f \in L^p$, where $1 < p < \infty$, it follows that $\tilde{f} \in L^p$. For useful corollaries of Riesz’s theorem, see Lemma 8. It is known that for a function $f \in L^1$ with Fourier series $a_0/2 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$ and integrable $\tilde{f}$, the function $\tilde{f}$ has the Fourier series $\sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)$ (see [5, Ch. VIII, § 1]).

If $T_n$ is the space of trigonometric polynomials of order at most $n$, then $n$-th best approximation in $V_p$ is defined by $E_n(f)_{V_p} := \inf_{t_n \in T_n} \|f - t_n\|_{V_p}$, $n \in \mathbb{Z}_+ = \{0, 1, \ldots\}$. The best approximation $E_n(f)_p$ in the space $L^p$ is defined similarly to $E_n(f)_{V_p}$, while $ω_n(f, \delta)_{V_p}$ is defined as $ω_k(f, \delta)_{V_p}$. The quantities $ω_k(f, \delta)_{X}$ and $E_n(f)_{X}$, where $X = V_p$ or $X = p$ (where $1 < p < \infty$), are connected by direct and inverse approximation theorems in corresponding spaces: for $k \in \mathbb{N}$ and $f \in C_p$ or $f \in L^p$ we have

$$E_n(f)_X \leq c(k)ω_k(f, 1/(n + 1))_X, \quad n \in \mathbb{Z}_+,$$

(1.1)

$$ω_k(f, 1/n)_X \leq c(k)n^{-k} \sum_{i=0}^{n} (i + 1)^{k-1} E_i(f)_X, \quad n \in \mathbb{N}.$$  

(1.2)

Inequalities (1.1) and (1.2) may be found in [6, Ch. 5, 6]. The problems of approximation in $C_p$ and $L^p$, where $1 < p < \infty$, are closely connected (see [7] and [8]).

Let $\{ε_n\}_{n=0}^{\infty}$ be a sequence decreasing to zero. We will write that $\{ε_n\}_{n=0}^{\infty}$ satisfies the Bary condition (B) if

$$\sum_{k=n}^{\infty} ε_k/(k + 1) = O(ε_n), \quad n \in \mathbb{Z}_+,$$

or satisfies the Bary-Stechkin condition $(B_m)$, where $m > 0$, if

$$\sum_{k=1}^{n} k^{m-1} ε_{k-1} = O(n^m ε_n), \quad n \in \mathbb{N}.$$  

These definitions are similar to the ones for moduli of continuity (see [9]). By definition, $f \in E_{C_p}(ε)$, if $E_n(f)_{V_p} \leq ε_n, n \in \mathbb{Z}_+$ (in the case $\lim_{n \to \infty} ε_n = 0, f \in E_{C_p}(ε)$ implies that $f \in C_p$). The class $E_p(ε) = \{f \in L^p : E_n(f)_{p} \leq ε_n, n \in \mathbb{Z}_+\}$ is defined in a similar way.

For $n \in \mathbb{Z}_+$ let $S_n(f)(x) = a_0/2 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$ be the partial sums of the Fourier series of a function $f \in L^1$. We will consider general Euler means (see [10, Ch. VIII])

$$e^q_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} q^{-k}(1 + q)^{-n} S_k(f)(x), \quad q \geq 0, \quad n \in \mathbb{Z}_+.$$  

Many mathematicians studied Lebesgue constants and approximation by Euler means in uniform and $L^p$ metric. We note here two papers. First, Chui and Holland proved in [11]

**Theorem A.** If $f \in Lip(\alpha), 0 < \alpha < 1$, is $2\pi$-periodic and

$$\frac{1}{2\pi/(n + 1)} \int_{t}^{\pi} \|\varphi_x(t) - \varphi_x(t + 2\pi/(n + 1))\|_\infty \, dt = O(n^{-\alpha}), \quad n \in \mathbb{N},$$

where $\varphi_x(t) = f(x + t) + f(x - t) - 2f(x)$, then $\|e^q_n(f) - f\|_\infty \leq cn^{-\alpha}, n \in \mathbb{N}$.

This theorem was generalized by Tyuleneva [12] to the case of general modulus of continuity $ω$ instead of $ω(\delta) = \delta^\alpha, 0 < \alpha < 1$.  

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Second, Rempulska and Tomczak [13] established the following result.

**Theorem B.** Let \( k \in \mathbb{N} \), and let \( p \) vary as \( 1 \leq p \leq \infty \). If \( p = \infty \), we let \( f \in C \), the space of continuous functions with uniform norm, otherwise let \( f \in L^p \). If \( p = 1, \infty \), then let \( A_{n,p} = \ln(n+2) \), and for \( 1 < p < \infty \) let \( A_{n,p} = 1 \). Then

\[
\left| e_n^\alpha(f) - f \right|_p \leq c A_{n,p} \omega_p(f, 1/n), \quad n \in \mathbb{N}.
\]

In our paper we extend Theorem B to the case \( e_n^\alpha(f) \), where \( q > 0 \), and prove the sharpness of a similar estimate in the \( L^p \) case \( (1 < p < \infty) \). Also we obtain the degree of approximation of bounded \( p \)-variation functions by Euler means in uniform norm and show its sharpness under some additional conditions. The case of conjugate function is also treated in both directions.

As usual, for \( n \in \mathbb{N} \), we write \( A_n \preceq B_n, n \in \mathbb{N} \), if \( A_n = O(B_n) \) and \( B_n = O(A_n) \).

## 2 Auxiliary propositions

In this section we say that \( a(x) \) is a step function \((a \in G)\), if there exists a partition \( \{x_i\}_{i=0}^m \) of a period or segment such that \( a(x) \) is constant on all \([x_{i-1}, x_i]\), \( 1 \leq i \leq m - 1 \), and on \([x_{m-1}, x_m] \). Let us consider the space \( B_p \) of bounded \( 2\pi \)-periodic functions \( g \) such that

\[
\|g\|_{B_p} = \sup \left\{ \left| V_p^{-1}(\alpha) \sum_{i=1}^m a_i(g(x_i) - g(x_{i-1})) \right| : x \in \mathbb{R}, \alpha \in G \right\}
\]

is finite. Here we assume that \( a(x) \) is not a constant. For \( K_n(t) = \sum_{k=1}^n \sin kt/k \) Terekhin [14] obtained the asymptotics of \( \|K_n\|_{B_p} \). In particular he proved the following

**Lemma 1.** The sequence \( \{\|K_n\|_{B_p}\}_{n=1}^\infty \) is bounded for \( 1 < p < \infty \).

**Lemma 2.** (Terekhin, [14, Lemma 1]) Let \( 1 < p < \infty \) and \( 1/p + 1/p' = 1 \). If \( g \in B_p \), then \( g \in V_{p'} \) and \( \|g\|_{V_{p'}} \leq 2\|g\|_{B_p} \).

From Lemmas 1 and 2 we deduce the following

**Corollary 1.** Let \( 1 < p < \infty \). Then the sequence \( \{\|K_n\|_{V_{p'}}\}_{n=1}^\infty \) is bounded.

**Corollary 2.** Let \( 1 < p < \infty \), \( f \in C_p \). Then \( \|f - S_n(f)\|_\infty \leq c E_n(f)_{V_{p'}} \) and \( \|S_n(f)\|_\infty \leq c \|f\|_{V_{p'}} \), where \( c \) does not depend on \( n \in \mathbb{Z}_+ \) and \( f \).

The result of Corollary 2 is proved in [7]. It is easy to see that the subspace \( B_p^* \) containing all continuous from the right functions from \( B_p \) is dual to \( C_p \) in the following sense: every linear continuous functional on \( C_p \) has the form \( \int_0^{2\pi} f(x) \, dg(x) \), where \( g \in B_p^* \). Then the inequality \( \|S_n(f)\|_\infty \leq c \|f\|_{V_{p'}} \) follows from Lemma 1 while the first inequality of Corollary 2 follows from the second one in a standard way.

**Lemma 3.** Let \( 1 < p < \infty \), \( \{a_n\}_{n=0}^\infty \) be a sequence decreasing to zero and \( g(x) = \sum_{n=1}^\infty a_n \sin nx/n \). Then \( g \in C_p \) and \( E_n(g)_{V_{p'}} \leq ca_{n+1} \).

**Proof.** Using summation by parts we obtain

\[
g(x) - S_n(g)(x) = \sum_{j=n+1}^\infty a_j \sin jx = -a_{n+1} \sum_{j=1}^n \frac{\sin jx}{j} + \sum_{j=n+1}^\infty (a_j - a_{j+1}) \sum_{i=1}^j \frac{\sin ix}{i}.
\]

By Corollary 1 we have \( \|K_j\|_{V_{p'}} \leq c_1 \) and

\[
\|g - S_n(g)\|_{V_{p'}} \leq c_1 \left( a_{n+1} + \sum_{j=n+1}^\infty (a_j - a_{j+1}) \right) = 2c_1 a_{n+1}.
\]
Since trigonometric polynomials belong to \( C_p \), where \( 1 < p < \infty \), and the space \( C_p \) is complete we conclude that \( g \in C_p \) and \( E_n(g) \mathcal{V}_p \leq 2c_1a_{n+1} \). The lemma is proved.

\[ \square \]

**Remark 1.** For the space \( C \) of 2\( \pi \)-periodic continuous functions the result of Lemma 3 was established by Bary [15, Theorem 3]. Our proof follows arguments of Bary.

**Lemma 4.** Let \( 1 < p < \infty, f \in C_p \) and suppose the series \( \sum_{n=1}^{\infty} n^{-1} E_n(f) \mathcal{V}_p \) converges. Then \( \tilde{f} \in C_p \) and

\[
E_n(\tilde{f}) \mathcal{V}_p \leq c \left( E_n(f) \mathcal{V}_p + \sum_{k=n+1}^{\infty} \frac{E_k(f) \mathcal{V}_p}{k} \right).
\]

This lemma was proved by Golubov [16] together with other analogues of Bary results from [15].

**Corollary 3.** Let \( 1 < p < \infty, \{a_n\}_{n=0}^{\infty} \) be a sequence decreasing to zero and \( f(x) = \sum_{n=1}^{\infty} a_n \cos nx/n \). If \( \{a_n\}_{n=3}^{\infty} \) satisfies the Bary condition \( \sum_{k=n}^{\infty} a_k/k = O(a_n) \), then \( f \in C_p \) and \( E_n(f) \mathcal{V}_p \leq c a_{n+1} \).

**Proof.** By Lemmas 3 and 4 we have for \( f(x) \) and \( g(x) = \sum_{n=1}^{\infty} a_n \sin nx/n \) that \( f \in C_p \) and

\[
E_n(f) \mathcal{V}_p = E_n(g) \mathcal{V}_p \leq c_1 \left( E_n(g) \mathcal{V}_p + \sum_{k=n+1}^{\infty} \frac{E_k(g) \mathcal{V}_p}{k} \right) \leq c_2 \left( a_{n+1} + \sum_{k=n+1}^{\infty} \frac{a_k}{k} \right) \leq c_3 a_{n+1}.
\]

The corollary is proved.

\[ \square \]

**Lemma 5.** Let \( 1 < p < \infty, f \in C_p, g \in L^{1/p}_2 \). Then \( f \ast g \in C_p \) and

\[
\omega_{l-1/p}(f \ast g, \delta) \leq \omega_{l-1/p}(f, \delta) \gamma_{l/1}(g) = \omega_{l-1/p}(f, \delta) \|g\|_1, \quad \delta \in [0, 2\pi].
\]

In particular, \( \|f \ast g\| \mathcal{V}_p \leq \|f\| \mathcal{V}_p \|g\|_1 \).

Lemma 5 was proved by Golubov [16]. Its proof may be also found in [17].

**Corollary 4.** Let \( 1 < p < \infty, f \in C_p \). Then

\[
\|f - S_n(f)\| \mathcal{V}_p \leq c \ln(n + 2) E_n(f) \mathcal{V}_p, \quad n \in \mathbb{Z}_+.
\]

**Proof.** Let \( t_n \in T_n \) be such that \( \|f - t_n\| \mathcal{V}_p = E_n(f) \mathcal{V}_p \) and \( D_n(x) := 1/2 + \sum_{k=1}^{n} \cos kx \). Using the equality

\[
\lim_{n \to \infty} \frac{D_n}{n} = \frac{4}{\pi} \left( 1 + \sum_{m=1}^{\infty} a_n m^{1/p} \right)
\]

and Lemma 5, we obtain

\[
\|f - S_n(f)\| \mathcal{V}_p \leq \|f - t_n\| \mathcal{V}_p + \|t_n - S_n(t_n)\| \mathcal{V}_p + \|D_n \ast (t_n - f)\| \mathcal{V}_p \leq (1 + \|D_n\|_1) \|f - t_n\| \mathcal{V}_p \leq c_1 \ln(n + 2) E_n(f) \mathcal{V}_p.
\]

The corollary is proved.

\[ \square \]

**Lemma 6.** (Konjushkov [19]) Let \( 1 < p < \infty, \{a_n\}_{n=1}^{\infty} \) be a sequence decreasing to zero, \( k \in \mathbb{N} \), and suppose the sum \( f \) of the series \( \sum_{n=1}^{\infty} a_n \cos nx \) (or \( \sum_{n=1}^{\infty} a_n \sin nx \)) belongs to \( L^p \). Then

\[
E_n(f)_p \leq c \left( n^{1/p} a_n + \left( \sum_{m=n}^{\infty} a_m^p m^{p-2} \right)^{1/p} \right)
\]

and \( a_{2n} \leq c n^{1/p-1} E_n(f)_p, a_{2n-1} \leq c n^{1/p-1} E_n(f)_p, n \in \mathbb{N} \).

Lemma 7 is a generalization of the inequality (20) in [13] in which the case \( q = 1 \) is considered.

**Lemma 7.** Let \( l \in \mathbb{N}, q > 0 \). Then there exists a constant \( c \), depending on \( l \) and \( q \) (not on \( n \)), such that

\[
\sum_{k=0}^{n} \binom{n}{k} q^{-k} \frac{1}{(k+1)^l} \leq c (q+1)^n \left( \frac{1}{n+1} \right)^l.
\]
Proof. Since \((k + m)/(k + 1) = 1 + (m - 1)/(k + 1) \leq m\) for \(k \geq 0\) and \(m \in \mathbb{N}\), we have
\[
\sum_{k=0}^{n} \binom{n}{k} q^{-k} \frac{1}{(k + 1)^l} \leq l! \sum_{k=0}^{n} \frac{n}{k} q^{-k} \frac{1}{(k + 1)(k + 2) \ldots (k + l)}
\]
\[
= l! \sum_{k=0}^{n} \binom{n}{k} q^{-k} \int_{0}^{1} \int_{0}^{1} \ldots \int_{0}^{1} u^k \, du \, dt_1 \ldots dt_{l-1}
\]
\[
= l! \int_{0}^{1} \int_{0}^{1} \ldots \int_{0}^{1} (q + u)^n \, du \, dt_1 \ldots dt_{l-1} \leq \frac{l!(q + 1)^{n+l}}{(n + 1) \ldots (n + l)} \leq c_1(q + 1)^n
\]
where \(c_1 = l!(q + 1)^l\). The lemma is proved. \(\square\)

**Lemma 8.** Let \(1 < p < \infty, f \in L^p\). Then \(\|S_n(f)\|_p \leq c\|f\|_p\) and \(\|f - S_n(f)\|_p \leq cE_n(f)_p, n \in \mathbb{Z}_+\).

Two statements of Lemma 8 are also often called the M. Riesz theorem (see [5, Ch. VIII, § 20]).

### 3 Degree of approximation

**Theorem 1.** Let \(1 < p < \infty, k \in \mathbb{N}\). Then

(i) For \(f \in L^p\) we have
\[
\|f - e_n^q(f)\|_p \leq c\omega_k(f, 1/n)_p, \quad n \in \mathbb{N},
\]
\[
(3.1)
\]
\[
\|f - e_n^q(f)\|_p \leq c n^{k^{-1}} E_{k-1}(f)_p, \quad n \in \mathbb{N}.
\]
\[
(3.2)
\]

(ii) For \(f \in C_p\) we have
\[
\|f - e_n^q(f)\|_{V_p} \leq c \ln(n + 2) \omega_k(f, 1/n)_{V_p},
\]
\[
(3.3)
\]
\[
\|f - e_n^q(f)\|_{V_p} \leq c \ln(n + 2)^{-k} E_{k-1}(f)_{V_p},
\]
\[
(3.4)
\]

**Proof.** (i) By definition, Lemma 8 and (1.1) we obtain
\[
\|f - e_n^q(f)\|_p = \left\| (1 + q)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} (f - S_j(f)) \right\|_p
\]
\[
\leq c_1(1 + q)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} E_j(f)_p \leq \frac{c_2}{(1 + q)^n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \omega_k \left( f, \frac{1}{n+j} \right)_p.
\]
\[
(3.5)
\]
By the property of moduli of smoothness of order \(k\) (see [6, Ch. 3, §3.3]) we have \(\omega_k(f, \lambda \delta)_p \leq (\lambda + 1)^k \omega_k(f, \delta)_p\). Due to this property and Lemma 7 we find that
\[
\|f - e_n^q(f)\|_p \leq \frac{c_2}{(1 + q)^n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \left( \frac{n+1}{j+1} \right)^k \omega_k \left( f, \frac{1}{n+1} \right)_p
\]
\[
\leq \frac{c_3}{(1 + q)^n} \omega_k \left( f, \frac{1}{n+1} \right)_p (n + 1)^k \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \frac{1}{(j+1)^k} \leq c_4 \omega_k \left( f, \frac{1}{n} \right)_p,
\]
\[
(3.6)
\]
i.e. (3.1) is proved. Applying (1.2) we deduce (3.2) from (3.6).
(ii) In the case $f \in C_p$ we have $\|f - S_j(f)\|_{L_p} \leq c_3 \ln(j + 2)E_j(f)_{L_p}$ by Corollary 4. Using this inequality and Lemma 7 again, we see that

$$\|f - e_n^q(f)\|_{L_p} \leq \frac{c_4 \ln(n + 2)}{(1 + q)^n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \left( \frac{n + 1}{j + 1} + 1 \right) k \omega_k \left( f, \frac{1}{n+1} \right)_{L_p} \leq c_7 \ln(n + 2) \omega_k \left( f, \frac{1}{n} \right)_{L_p}.$$ 

Thus, (3.3) is established and (3.4) follows from (3.3) and (1.2). The theorem is proved.

\[\boxplus\]

**Theorem 2.** Let $1 < p < \infty$, $k \in \mathbb{N}$. Then

(i) For $f \in L^p$ we have

$$\|f - e_n^q(f)\|_p \leq c \omega_k(f, 1/n)_p, \quad n \in \mathbb{N},$$

and

$$\|f - e_n^q(f)\|_p \leq c n^{-k} \sum_{i=1}^{n} i^{k-1} E_{i-1}(f)_p, \quad n \in \mathbb{N}.$$ (3.8)

(ii) If $\{\varepsilon_i\}_{i=0}^\infty$ is a sequence decreasing to zero and satisfies Bary condition (B) and $f \in E_{C_p}(\varepsilon)$, then

$$\|f - e_n^q(f)\|_{L_p} \leq c \ln(n + 2)n^{-k} \sum_{i=1}^{n} i^{k-1} \varepsilon_{i-1}, \quad n \in \mathbb{N}.$$ (3.9)

\[\boxplus\]

**Proof.** (i) By the Riesz’s theorem $\|f\|_p \leq c_1 \|f\|_p$ and so the inequality $\omega_k(f, \delta)_p \leq c_1 \omega_k(f, \delta)_p$ holds. Substituting this estimate into (3.1) yields (3.7), while (3.8) follows from (3.7) and (1.2).

(ii) Since $\{\varepsilon_i\}_{i=0}^\infty$ satisfies Bary condition (B) and $f \in E_{C_p}(\varepsilon)$, Lemma 4 yields

$$\tilde{f} \in E_{C_p}(\varepsilon).$$

By (3.4) and Lemma 4 we have

$$\|f - e_n^q(f)\|_{L_p} \leq \frac{c_2 \ln(n + 2)}{n^k} \sum_{i=1}^{n} i^{k-1} E_{i-1}(\tilde{f})_{L_p} \leq \frac{c_3 \ln(n + 2)}{n^k} \sum_{i=1}^{n} i^{k-1} \sum_{j=1}^{\infty} E_j(f)_{L_p} \leq \frac{c_3 \ln(n + 2)}{n^k} \sum_{i=1}^{n} i^{k-1} \varepsilon_{i-1} + \frac{c_4 \ln(n + 2)}{n^k} \sum_{i=1}^{n} i^{k-1} \varepsilon_{i-1} + \frac{c_5 \ln(n + 2)}{n^k} \sum_{i=1}^{n} i^{k-1} \varepsilon_{n},$$

since $\sum_{i=1}^{n} i^{k-1} \geq c_7 n^k$. Hence (3.9) is established. The theorem is proved.

\[\boxplus\]

**Theorem 3.** Let $1 < p < \infty$, $f \in C_p$, $k \in \mathbb{N}$. Then the following inequalities hold

$$\|f - e_n^q(f)\|_\infty \leq c_1 (1 + q)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} E_j(f)_{L_p}, \quad n \in \mathbb{N},$$ (3.10)

$$\|f - e_n^q(f)\|_\infty \leq c_2 \omega_k(f, 1/n)_p, \quad n \in \mathbb{N}.$$ (3.11)

Let $\{\varepsilon_n\}_{n=0}^\infty$ be a sequence decreasing to zero which satisfies the Bary condition (B), and let $f \in E_{C_p}(\varepsilon)$.

Then

$$\|f - e_n^q(f)\|_\infty \leq c_3 2^{-n} \sum_{j=0}^{n} \binom{n}{j} \varepsilon_j.$$ (3.12)
Proof. By Corollary 2 we obtain

$$||f - e_n^q(f)||_\infty \leq (1 + q)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j}||f - S_j(f)||_\infty \leq c_4(1 + q)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j}E_j(f)_{V_p},$$

hence we have proven (3.10). Applying (3.10), (1.1) and Lemma 7 as in the proof of Theorem 1 we find that

$$||f - e_n^q(f)||_\infty \leq c_5(1 + q)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \omega_k \left(f, \frac{1}{j+1} \right)_{V_p} \leq$$

$$\leq c_6(1 + q)^{-n}(n + 1)^k \omega_k \left(f, \frac{1}{n + 1} \right)_{V_p} \leq c_7 \omega_k \left(f, \frac{1}{n + 1} \right)_{V_p}.$$

Thus, (3.11) is also proved.

For $q = 1$, using the decreasing nature of $E_j(f)_{V_p}$ and Lemma 4, we have

$$||\tilde{f} - e_1^1(f)||_\infty \leq c_42^{-n} \sum_{j=0}^{n} \binom{n}{j} E_j(f)_{V_p} \leq 2c_42^{-n} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j} E_j(f)_{V_p}$$

$$\leq c_82^{-n} \left( \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j} E_j + \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j} \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{E_{i-1}}{i} \right)$$

$$\leq c_82^{-n} \left( \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j} E_j + \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{j} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{E_{i-1}}{i} \right) c_9 E_{\lfloor n/2 \rfloor}$$

$$\leq c_82^{-n} \left( 1 + c_9 \sum_{j=0}^{n} \binom{n}{j} E_j + \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{j} E_i \right) \leq c_{10}2^{-n} \sum_{j=0}^{n} \binom{n}{j} E_j.$$  

Here we use the fact that $\binom{n}{j}$ increases with respect to $j \in [0, \lfloor n/2 \rfloor]$. The theorem is proved.  

\square

4 Sharpness of estimates

Let us prove that (3.2) and (3.8) are sharp under some additional conditions.

**Theorem 4.** Let $1 < p < \infty$ and $(\varepsilon_k)_{k=0}^{\infty}$ be a sequence decreasing to zero which satisfies Bary condition (B) and Bary-Steckin condition (B_k) for some $k \in \mathbb{N}$. Then

$$\sup_{f \in E_p(\varepsilon_k)} ||f - e_0^q(f)||_p \asymp \sup_{f \in E_p(\varepsilon_k)} ||\tilde{f} - e_0^q(f)||_p \asymp \varepsilon_n \asymp n^{-k} \sum_{i=1}^{n} i^{k-1} \varepsilon_{i-1}.$$

**Proof.** The upper estimates for $\sup_{f \in E_p(\varepsilon_k)} ||f - e_0^q(f)||_p$ and $\sup_{f \in E_p(\varepsilon_k)} ||\tilde{f} - e_0^q(f)||_p$ follow from (3.2) in Theorem 1 and (3.8) in Theorem 2. For lower estimates we consider

$$f_0(x) = \sum_{n=1}^{\infty} \varepsilon_n n^{1/p-1} \cos(nx).$$

Since $(\varepsilon_n n^{1/p-1})_{n=1}^{\infty}$ is a sequence decreasing to zero we obtain by Lemma 6 and the condition (B) that

$$E_n(f_0)_p \leq c_1 \left(n^{1-1/p} \varepsilon_n n^{1/p-1} + \left( \sum_{i=n+1}^{\infty} (\varepsilon_i i^{1/p-1})^{1/p} \right)^{1/p} \right) \leq c_2 \varepsilon_n, \quad n \in \mathbb{N}. $$
Here we use the fact that if \( \{ \varepsilon_k \}_{k=0}^\infty \) satisfies the Bary condition \((B)\), then \( \{ \varepsilon_k \}_{k=0}^\infty \) also satisfies this condition (see conditions (L) or (P) in Lemma 2 from [9]). Thus, 
\( f_0/c_2 \in E_p(\varepsilon) \). On the other hand, by the obvious inequality \( \| f - e_n^q(f) \|_p \geq E_n(f)_p \) and the second statement of Lemma 6, we have \( \| f - e_n^q(f) \|_p \geq c_3 \varepsilon_{2n} \). However,

\[
\varepsilon_n \leq c_4 n^{-k} \sum_{i=1}^{2n} i^{-1} \varepsilon_{i-1} \leq 2^k c_4 (2n)^{-k} \sum_{i=1}^{2n} i^{-1} \varepsilon_{i-1} \leq c_5 \varepsilon_{2n}, \quad n \in \mathbb{N},
\]
due to the condition \((B_k)\). Thus, \( \sup_{f \in E_p(\varepsilon)} \| f - e_n^q(f) \|_p = \varepsilon_n, n \in \mathbb{N} \). By Riesz’s theorem (see Introduction) we have also \( E_n(f_0)_p \leq c_6 E_n(f_0)_p \leq c_7 \varepsilon_n, n \in \mathbb{Z}_+ \), and \( \| f_0 - e_n^q(f_0) \|_p \geq E_n(f_0)_p \geq c_8 \varepsilon_n, n \in \mathbb{N} \). The theorem is proved.

**Remark 2.** It is obvious that \( \{ \varepsilon_j \}_{j=0}^{\infty} = \{ (j+1)^{-\alpha} \}, k > \alpha > 0 \), satisfy both conditions \((B)\) and \((B_k)\). If a modulus of continuity type function \( \omega \) (i.e. \( \omega(0) = 0 \), \( \omega \) increases and is continuous on \([0, 2\pi]\) is such that \( \{ \omega((j+1)^{-\alpha}) \}_{j=0}^{\infty} \) satisfies the condition \((B_k)\), then the assertions \( \omega(f, (j+1)^{-\alpha}) = O(\omega((j+1)^{-\alpha})) \) and \( E(f)_p = O(\omega((j+1)^{-\alpha})) \), \( j \in \mathbb{Z}_+ \), are equivalent (see [6, § 7.13]). Therefore, if \( \{ \omega((j+1)^{-1}) \}_{j=0}^{\infty} \) satisfies the conditions \((B)\) and \((B_k)\), then by the method of the proof of Theorem 4 we can obtain the sharpness of estimate \((3.1)\) and a similar estimate for conjugate functions.

Now we prove the sharpness of \((3.10)\) and \((3.12)\).

**Theorem 5.** Let \( 1 < p < \infty \), \( \{ \varepsilon_k \}_{k=0}^{\infty} \) be a sequence decreasing to zero and let \( \varepsilon_k \leq C \varepsilon_{k+1} \) for \( k \in \mathbb{Z}_+ \). If \( \{ \varepsilon_k \}_{k=0}^{\infty} \) also satisfies the two-sided Bary condition

\[
\sum_{k=n}^{\infty} \varepsilon_k/(k+1) \asymp \varepsilon_n, \quad n \in \mathbb{Z}_+,
\]

then

\[
\sup_{f \in E_p(\varepsilon)} \| f - e_n^q(f) \|_\infty \asymp (1 + q)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \varepsilon_j, \quad n \in \mathbb{N},
\]

and

\[
\sup_{f \in E_p(\varepsilon)} \| f - e_n^q(\tilde{f}) \|_\infty \asymp 2^{-n} \sum_{j=0}^{n} \binom{n}{j} \varepsilon_j, \quad n \in \mathbb{N}.
\]

**Proof.** (i) The upper estimate for the left-hand side of \((4.1)\) follows from \((3.10)\). Let us consider the function \( f_0(x) = \sum_{k=1}^{\infty} \varepsilon_k \cos kx/k \). By Corollary 3 we have \( E_j(f_0)V_p \leq c_1 \varepsilon_j, j \in \mathbb{Z}_+ \), i.e. \( f_0/c_1 \in E_{C_q}(\varepsilon) \). On the other hand,

\[
\| f_0 - e_n^q(f_0) \|_\infty \leq (1 + q)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \| f_0 - S_j(f_0) \|_\infty
\]

\[
\leq (1 + q)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \sum_{k=j+1}^{\infty} \varepsilon_k/k =: (1 + q)^{-n} \sum_{j=0}^{\infty} \binom{n}{j} q^{n-j} A_{j+1}.
\]

We note that \( f_0 \) has non-negative Fourier coefficients, hence

\[
f_0(0) - e_n^q(f_0)(0) = (1 + q)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} (f_0(0) - S_j(f_0)(0))
\]

\[
= (1 + q)^{-n} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \sum_{k=j+1}^{\infty} \varepsilon_k/k = (1 + q)^{-n} \sum_{j=0}^{\infty} \binom{n}{j} q^{n-j} A_{j+1}.
\]
From (4.3) and the last equality we deduce that
\[
\|f_0 - e_n^q(f_0)\|_\infty = f_0(0) - e_n^q(f_0)(0) = (1 + q)^n \sum_{j=0}^{\infty} \binom{n}{j} q^{n-j} \sum_{k=n+1}^{\infty} \frac{E_k}{k} + (1 + q)^n \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \sum_{k=j+1}^{n} \frac{E_k}{k} = A_{n+1} + (1 + q)^n \sum_{k=1}^{n} \binom{n}{k} q^{n-k} \frac{E_k}{k}.
\]

(4.4)

We put \( B_k = \sum_{j=0}^{k-1} \binom{n}{j} q^{n-j}, 1 \leq k \leq n + 1, B_0 := 0 \). By the condition of Theorem 5 and (4.4) we have
\[
(1 + q)^n \|f_0 - e_n^q(f_0)\|_\infty = (1 + q)^n A_{n+1} + \sum_{k=1}^{n} B_k (A_k - A_{k+1})
\]
\[
= A_{n+1} B_{n+1} + \sum_{k=1}^{n+1} A_k (B_k - B_{k-1}) - B_n A_{n+1} = \sum_{k=1}^{n+1} A_k (B_k - B_{k-1}) = \sum_{k=1}^{n+1} \binom{n}{k-1} q^{n-k+1} A_k \geq c_2 \sum_{k=1}^{n+1} \binom{n}{k-1} q^{n-k+1} E_k = c_2 \sum_{j=0}^{n} \binom{n}{j} q^{n-j} E_{j+1} \geq c_3 \sum_{j=0}^{n} \binom{n}{j} q^{n-j} E_j.
\]

(4.5)

For \( f_0/c_1 \), we also have an inequality similar to (4.5) with the constant \( c_3/c_1 \). Therefore, (4.1) is proved.

(ii) The upper bound for the right-hand side of (4.2) follows from (3.12). Let us consider the function
\[
f_1(x) = \sum_{k=1}^{\infty} E_k \sin kx/k.
\]
By Lemma 3 we have \( E_j(f_1) \leq c_4 E_{j+1} \leq c_4 \epsilon_j, j \in \mathbb{Z}^+, \) i.e. \( f_1 \in E_{c_4}(\epsilon) \). Since \( f_1 \) has Fourier series \(- \sum_{k=1}^{\infty} \cos kx/k, \) from the proof of (i) (see (4.4) and (4.5)) we deduce
\[
\|\tilde{f} - e_n^q(\tilde{f})\|_\infty = 2^{-n} \sum_{j=0}^{n} \binom{n}{j} \sum_{k=j+1}^{\infty} \frac{E_k}{k} \geq c_5 2^{-n} \sum_{j=0}^{n} \binom{n}{j} E_j.
\]

Therefore, (4.2) is valid. The theorem is proved. \( \square \)

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