Research Article

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Hyers-Ulam stability of first-order homogeneous linear dynamic equations on time scales

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Abstract: We establish the Hyers-Ulam stability (HUS) of certain first-order linear constant coefficient dynamic equations on time scales, which include the continuous (step size zero) and the discrete (step size constant and nonzero) dynamic equations as important special cases. In particular, for certain parameter values in relation to the graininess of the time scale, we find the minimum HUS constants. A few nontrivial examples are provided. Moreover, an application to a perturbed linear dynamic equation is also included.

Keywords: stability, first order, Hyers-Ulam, time scales

MSC: 34N05, 34A30, 34A05, 34D20

1 Introduction

In 1940, Ulam [1] posed the following problem concerning the stability of functional equations: give conditions for a linear mapping near an approximately linear mapping to exist. The problem for the case of approximately additive mappings was solved by Hyers [2] who proved that the Cauchy equation is stable in Banach spaces. This was later generalized by Rassias [3]. Since then there has been a significant amount of interest in Hyers-Ulam stability (HUS), especially in relation to ordinary differential equations; for example, see [4–15].

For \( a \in \mathbb{R} \), the equation

\[
x'(t) - ax(t) = 0, \quad t \in \mathbb{R}
\]

has HUS if and only if there exists a constant \( K > 0 \) with the following property:

For arbitrary \( \varepsilon > 0 \), if a function \( \phi : \mathbb{R} \to \mathbb{R} \) satisfies \( |\phi'(t) - a\phi(t)| \leq \varepsilon \) for all \( t \in \mathbb{R} \), then there exists a solution \( x : \mathbb{R} \to \mathbb{R} \) of (1.1) such that \( |\phi(t) - x(t)| \leq K\varepsilon \) for all \( t \in \mathbb{R} \).

Such a constant \( K \) is called an HUS constant for (1.1) on \( \mathbb{R} \). Recently, Onitsuka and Shoji [16] explored the minimum HUS constant for (1.1). Also, Onitsuka [17] investigated the influence of the constant step size \( h > 0 \) on HUS for the first-order homogeneous linear difference equation

\[
\Delta_h x(t) - ax(t) = 0
\]

on the uniformly discrete time scale \( h\mathbb{Z} \). In recent years, Hyers-Ulam stability of various difference equations, recurrences and dynamic equations on time scales have been studied; for example, see [18–22]. As a good overview explaining Hyers-Ulam stability, a summary written by Brilhouët-Belluot, Brzdęk and Ciepliński [23] would be appropriate. We propose to extend the results given in [16, 17] to all time scales, thus incorporating
the continuous and discrete (constant step size) cases as important corollaries. Shen [24] also investigated
HUS of first-order linear dynamic equations, but only on finite time scales, so the results that follow here
derf significantly.

2 Time scales

The following is a primer on time scales [25]. A time scale \( \mathbb{T} \) is any nonempty closed subset of \( \mathbb{R} \). It follows
that the jump operators \( \sigma, \rho : \mathbb{T} \to \mathbb{T} \)

\[
\sigma(t) = \inf \{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup \{s \in \mathbb{T} : s < t\}
\]

(supplemented by \( \inf \emptyset := \sup \mathbb{T} \) and \( \sup \emptyset := \inf \mathbb{T} \)) are well defined. The point \( t \in \mathbb{T} \) is left-dense, left-
scattered, right-dense, right-scattered if \( \rho(t) = t, \rho(t) < t, \sigma(t) = t, \sigma(t) > t \), respectively. If \( \mathbb{T} \) has a left-scattered
maximum \( M \), define \( \mathbb{T}^\kappa := \mathbb{T} - \{M\} \); otherwise, set \( \mathbb{T}^\kappa = \mathbb{T} \). The forward graininess is \( \mu(t) := \sigma(t) - t \).

For \( f : \mathbb{T} \to \mathbb{R} \) and \( t \in \mathbb{T}^\kappa \), the delta derivative of \( f \) at \( t \), denoted \( f^\Delta(t) \), is the number (provided it exists)
with the property that given any \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( t \) such that

\[
|f(\sigma(t)) - f(s) - f^\Delta(t)\sigma(t) - s| \leq \varepsilon \sigma(t) - s
\]

for all \( s \in U \). For \( \mathbb{T} = \mathbb{R} \), we have \( f^\Delta = f' \), the usual derivative, and for \( \mathbb{T} = h\mathbb{Z} \) we have the forward difference
operator, \( f^\Delta(t) = [f(t + h) - f(t)]/h \).

A function \( f : \mathbb{T} \to \mathbb{R} \) is right-dense continuous (rd-continuous) provided it is continuous at all right
dense points of \( \mathbb{T} \) and its left sided limit exists (finite) at left dense points of \( \mathbb{T} \). The set of all right dense
continuous functions on \( \mathbb{T} \) is denoted by \( C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}; \mathbb{R}) \); similarly, \( C^1_{rd}(\mathbb{T}) \) is the set of all delta-
differentiable right dense continuous functions on \( \mathbb{T} \). Bohner and Peterson [25] show that if \( f \) is rd-continuous,
then there is a function \( F(t) \) such that \( F^\Delta(t) = f(t) \). In this case, we define

\[
\int_a^b f(t)\Delta t = F(b) - F(a).
\]

**Definition 2.1.** A function \( p : \mathbb{T} \to \mathbb{R} \) is regressive provided \( 1 + \mu(t)p(t) \neq 0 \) for all \( t \in \mathbb{T}^\kappa \). Let

\[
\mathbb{R} := \{p \in C_{rd}(\mathbb{T}; \mathbb{R}) : 1 + \mu(t)p(t) \neq 0, t \in \mathbb{T}^\kappa\}.
\]

Also, \( p \in \mathbb{R}^+ \) if and only if \( 1 + \mu(t)p(t) > 0 \) for all \( t \in \mathbb{T}^\kappa \).

**Definition 2.2.** If \( p \in \mathbb{R}, t_0 \in \mathbb{T} \), define the generalized exponential function \( e_p(t, t_0) \) to be the unique
solution of the initial value problem

\[
x^\Delta = p(t)x, \quad x(t_0) = 1.
\]

Many of the properties of this generalized exponential function \( e_p(t, t_0) \) listed below in Theorem 2.3 are em-
ployed throughout this work.

**Theorem 2.3.** [25, Theorems 2.36 and 2.44] If \( p, q \in \mathbb{R} \) and \( s, t \in \mathbb{T} \), then

(i) \( e_0(t, s) \equiv 1 \) and \( e_p(t, t) \equiv 1 \);

(ii) \( e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s) \);

(iii) \( e_{p\circ p}(t, s) = e_p(t, s) \), where \( \circ p := \frac{\mathbb{R} \circ p}{\mathbb{R}p} \);

(iv) \( e_p(t, s) = e_{\frac{1}{p}}(s, t) = e_{\frac{1}{p}}(s, t) \);

(v) \( e_p(t, s)e_p(s, t) = e_p(t, r) \);

(vi) If \( p \in \mathbb{R}^+ \), then \( e_p(t, s) > 0 \) for all \( t \in \mathbb{T} \).
3 Hyers-Ulam stability on time scales

In this paper, we consider the Hyers-Ulam stability of first-order linear homogenous dynamic equations with constant coefficient

\[ x^A(t) - ax(t) = 0, \quad a \in \mathbb{R}. \] (3.1)

These exist on general time scales \( T \), which include \( T = \mathbb{R} \) and \( T = h\mathbb{Z} \) as important special cases. If \( T = \mathbb{R} \), then (3.1) is (1.1), and if \( T = h\mathbb{Z} \), then (3.1) is (1.2).

**Definition 3.1.** We say that (3.1) has Hyers-Ulam stability on \( T \) if and only if there exists a constant \( K > 0 \) with the following property. For arbitrary \( \varepsilon > 0 \), if a function \( \phi : T \to \mathbb{R} \) satisfies \( |\phi^A(t) - a\phi(t)| \leq \varepsilon \) for all \( t \in T^\kappa \), then there exists a solution \( x : T \to \mathbb{R} \) of (3.1) such that \( |\phi(t) - x(t)| \leq Ke \) for all \( t \in T \). Such a constant \( K \) is called an HUS constant for (3.1) on \( T \).

**Remark 3.2.** Note here that if a function \( x(t) \) exists on \( T \) and is delta differentiable, then \( x^A(t) \) exists on \( T^\kappa \), but if \( t \in T \) is right-scattered and \( a = \frac{1}{\mu(t)} \) holds, then we no longer have a first-order dynamic equation. Thus, throughout this paper, we assume that \( T \) and \( T^\kappa \) are nonempty sets of \( \mathbb{R} \), and \( a \neq \frac{1}{\mu(t)} \) at any right-scattered \( t \in T^\kappa \); that is to say, \( a \in \mathbb{R} \).

**Remark 3.3.** Let \( a = 0 \), let \( \varepsilon > 0 \) be given, and suppose \( T \) is unbounded above. Note that the function \( \phi(t) = \varepsilon t \) satisfies \( |\phi^A(t)| = \varepsilon \) for all \( t \in T \). As \( x(t) \equiv c \) is the general solution to \( x^A(t) = 0 \), we see that \( |\phi(t) - x(t)| \to \infty \) as \( t \to \infty \), so that (3.1) does not have Hyers-Ulam stability on \( T \) when \( a = 0 \).

**Lemma 3.4.** Suppose that \( a \neq 0 \) and \( a \in \mathbb{R}^+ \); that is to say, \( a > \frac{1}{\mu(t)} \) for any possible right-scattered \( t \in T^\kappa \). Let \( \varepsilon > 0 \) be a fixed arbitrary constant, \( t_0 \in T \), and let \( \phi \) be a real-valued function on \( T \). Then the inequality

\[ |\phi^A(t) - a\phi(t)| \leq \varepsilon \]

holds for all \( t \in T^\kappa \) if and only if the inequality

\[ 0 \leq \left[ \left( \phi(t) - \frac{\varepsilon}{a} \right) e_{\ominus a}(t, t_0) \right]^A \leq 2\varepsilon e_{\ominus a}(\sigma(t), t_0) \]

holds for all \( t \in T^\kappa \), where \( e_{\ominus a}(t, t_0) \) is defined in Theorem 2.3 (iii).

**Proof.** Using the \( A \)-derivative product rule [25, Theorem 1.20] and Theorem 2.3 above, we have that

\[
\left[ \left( \phi(t) - \frac{\varepsilon}{a} \right) e_{\ominus a}(t, t_0) \right]^A = \left( \phi(t) - \frac{\varepsilon}{a} \right) \left( \ominus a \right) e_{\ominus a}(t, t_0) + \phi^A(t) e_{\ominus a}(\sigma(t), t_0) \\
= \left( \phi(t) - \frac{\varepsilon}{a} \right) \left( \ominus a \right) e_{\ominus a}(t, t_0) + \phi^A(t) e_{\ominus a}(\sigma(t), t_0) \\
= \left( \phi(t) - \frac{\varepsilon}{a} \right) \left( -ae_{\ominus a}(\sigma(t), t_0) \right) + \phi^A(t) e_{\ominus a}(\sigma(t), t_0) \\
= \left( \phi(t) - \frac{\varepsilon}{a} \right) (-ae_{\ominus a}(\sigma(t), t_0) + \phi^A(t) e_{\ominus a}(\sigma(t), t_0)) \\
= \left[ \phi^A(t) - a\phi(t) + \varepsilon \right] e_{\ominus a}(\sigma(t), t_0)
\]

holds for all \( t \in T^\kappa \).

**Lemma 3.5.** Suppose that \( a < 0 \) and \( a \in \mathbb{R}^+ \). Let \( t_0 \in T \). Then the inequality

\[ e_{a}(t, t_0) \geq 1 + a(t - t_0) \]

holds for all \( t \in (-\infty, t_0]_T \).
Proof. Let $\phi(t) = -1/a$ for all $t \in \mathbb{T}$. Then $|\phi^A(t) - a\phi(t)| = 1$ holds on $\mathbb{T}^\kappa$. Using Lemma 3.4, we obtain the inequality

$$0 \leq \left[ \left( \phi(t) - \frac{1}{a} \right) e_{\ominus a}(t, t_0) \right]_A^A = -\frac{2}{a} (e_{\ominus a}(t, t_0))^A$$

for all $t \in \mathbb{T}^\kappa$. Since $a < 0$ we have $(e_{\ominus a}(t, t_0))^A \geq 0$ on $\mathbb{T}^\kappa$. This and Theorem 2.3 imply that

$$0 < \frac{1}{e_a(t, t_0)} = e_{\ominus a}(t, t_0) \leq e_{\ominus a}(t_0, t_0) = \frac{1}{e_a(t_0, t_0)} = 1$$

for all $t \in (-\infty, t_0]_\mathbb{T}$. We now consider the function

$$y(t) = e_a(t, t_0) - a(t - t_0) - 1$$

on $\mathbb{T}$. From the result above, we see that

$$y^A(t) = (e_a(t, t_0))^A - a = a(e_a(t, t_0) - 1) \leq 0$$

for all $t \in (-\infty, \rho(t_0)]_\mathbb{T}$. This means that $y(t)$ is a nonincreasing function on $(-\infty, t_0]_\mathbb{T}$. Consequently, we obtain

$$y(t) \geq y(t_0) = e_a(t_0, t_0) - 1 = 0$$

for all $t \in (-\infty, t_0]_\mathbb{T}$. This completes the proof. $\square$

Proposition 3.6. Let $t_0 \in \mathbb{T}$. Fix the constant $a \in \mathbb{R} \setminus \{0\}$ with $a \in \mathbb{R}^+$, and let $\varepsilon > 0$ be a given arbitrary constant. Suppose that a function $\phi : \mathbb{T} \rightarrow \mathbb{R}$ satisfies $|\phi^A(t) - a\phi(t)| \leq \varepsilon$ for all $t \in \mathbb{T}^\kappa$. Then there exist a nondecreasing function $u : \mathbb{T} \rightarrow \mathbb{R}$ and a nonincreasing function $v : \mathbb{T} \rightarrow \mathbb{R}$ such that

$$\phi(t) = u(t)e_a(t, t_0) + \frac{\varepsilon}{a} = v(t)e_a(t, t_0) - \frac{\varepsilon}{a},$$

(3.2)

and one of the following hold.

(i) If $a > 0$ and max $\mathbb{T}^\kappa =: r^*$ exists, then the inequality

$$u(t) \leq u(r^*) < v(r^*) \leq v(t)$$

(3.3)

holds for all $t \in \mathbb{T}$.

(ii) If $a > 0$ and max $\mathbb{T}$ does not exist, then $\lim_{t \rightarrow \infty} u(t)$ and $\lim_{t \rightarrow \infty} v(t)$ exist, and

$$u(t) \leq \lim_{t \rightarrow \infty} u(t) = \lim_{t \rightarrow \infty} v(t) \leq v(t)$$

(3.4A)

holds for all $t \in \mathbb{T}$.

(iii) Suppose for all right-scattered $t \in \mathbb{T}$ that $\frac{-1}{\mu(t)} < a < 0$. If min $\mathbb{T}^\kappa =: r^*$ exists, then the inequality

$$v(t) \leq v(r^*) < u(r^*) \leq u(t)$$

(3.5)

holds for all $t \in \mathbb{T}$.

(iv) Suppose for all right-scattered $t \in \mathbb{T}$ that $\frac{-1}{\mu(t)} < a < 0$, and min $\mathbb{T}$ does not exist. Then $\lim_{t \rightarrow -\infty} u(t)$ and $\lim_{t \rightarrow -\infty} v(t)$ exist, and

$$v(t) \leq \lim_{t \rightarrow -\infty} v(t) = \lim_{t \rightarrow -\infty} u(t) \leq u(t)$$

(3.6)

holds for all $t \in \mathbb{T}$.

Proof. Fix the constant $a \in \mathbb{R} \setminus \{0\}$ with $a \in \mathbb{R}^+$; then $1 + a\mu(t) > 0$ for all $t \in \mathbb{T}$. Define the functions $u$ and $\nu$ on $\mathbb{T}$ via

$$u(t) := \left( \phi(t) - \frac{\varepsilon}{a} \right) e_{\ominus a}(t, t_0) \quad \text{and} \quad v(t) := \left( \phi(t) + \frac{\varepsilon}{a} \right) e_{\ominus a}(t, t_0)$$

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for $t \in T$, where $e_{\sigma}(t, t_0)$ is defined in Theorem 2.3 (iii). Then (3.2) holds, and thus for $t \in T$ we obtain

$$u(t) = v(t) - \frac{2\varepsilon}{a} e_{\sigma}(t, t_0), \quad u(t) \begin{cases} < v(t) : & a > 0, \\ > v(t) : & \mu(t) > 0, \quad \frac{1}{\mu(t)} < a < 0. \end{cases} \quad (3.7)$$

Using Lemma 3.4, we note that the inequality

$$0 \leq \left( \left( \phi(t) - \frac{\varepsilon}{a} \right) e_{\sigma}(t, t_0) \right)^\Delta \leq 2\varepsilon e_{\sigma}(\sigma(t), t_0)$$

holds for all $t \in T^\times$. Now, from Theorem 2.3 we have

$$\left( e_{\sigma}(t, t_0) \right)^\Delta = e_{\sigma}(t, t_0) = \frac{-a}{(1 + a\mu(t)) e_{\sigma}(t_0)} = -ae_{\sigma}(\sigma(t), t_0),$$

so that this together with (3.7) implies that

$$-2\varepsilon e_{\sigma}(\sigma(t), t_0) \leq v^\Delta(t) \leq 0, \quad t \in T.$$

Consequently, $u$ is nondecreasing and $v$ is nonincreasing.

Now we consider (i). From the results above and the assumptions in (i), we see that $u(\tau^*)$ is the maximum of $u$ on $T$, and $v(\tau^*)$ is the minimum of $v$ on $T$. Then (3.3) follows from (3.7), and (i) holds.

Next consider (ii). As $t_0 \in T$ is fixed, then (3.7) implies that for $a > 0$, we have

$$u(t) < v(t_0), \quad t \in T,$$

and thus $u$ is bounded above on $T$. As $T$ is unbounded in this case, the $\lim_{t \to \infty} u(t)$ exists. Since $a > 0$ we have $a \in \mathbb{R}^+$. By Bernoulli’s Inequality [25, Theorem 6.2],

$$0 < e_{\sigma}(\sigma(t), t_0) \leq e_{\sigma}(t, t_0) \leq \frac{1}{1 + a(t - t_0)^\sigma}, \quad t \in [t_0, \infty)_T.$$

Hence,

$$\lim_{t \to \infty} e_{\sigma}(\sigma(t), t_0) = \lim_{t \to \infty} e_{\sigma}(t, t_0) = 0.$$

Using (3.7) and the fact that $a > 0$, we get $\lim_{t \to \infty} u(t) = \lim_{t \to \infty} v(t)$. As a result, (3.4) is true for $t \in T$, since $u$ is nondecreasing and $v$ is nonincreasing; thus (ii) holds.

The arguments for (iii) and (iv) are similar to those given above for (i) and (ii), and thus are omitted. In particular, note that in these two cases we still have $a < 0$ and $a \in \mathbb{R}^+$, so that $e_{\sigma}(t, t_0) > 0$ for all $t \in T$ and $\lim_{t \to \infty} e_{\sigma}(t, t_0) = \infty$ by Theorem 2.3 (vi) and Lemma 3.5, respectively. This completes the proof. \qed

**Theorem 3.7.** Let $\varepsilon > 0$ be given and fix $t_0 \in T$. Suppose that a delta-differentiable function $\phi : T \to \mathbb{R}$ satisfies

$$|\phi^\Delta(t) - a\phi(t)| \leq \varepsilon, \quad t \in T^\times,$$

where $a \in \mathbb{R}^+$ and $a \neq 0$. Then one of the following holds.

(i) If $a > 0$ and $\tau^* := \max T$ exists, then any solution $x$ of (3.1) with $|\phi(\tau^*) - x(\tau^*)| < \varepsilon/a$ satisfies $|\phi(t) - x(t)| < \varepsilon/a$ for all $t \in T$.

(ii) If $a > 0$ and $\max T$ does not exist, then $\lim_{t \to \infty} \phi(t)e_{\sigma}(t, t_0)$ exists, and there exists a unique solution

$$x(t) := \left( \lim_{t \to \infty} \phi(t)e_{\sigma}(t, t_0) \right) e_{\sigma}(t, t_0)$$

of (3.1) such that $|\phi(t) - x(t)| \leq \varepsilon/a$ for all $t \in T$.

(iii) If $a \in \mathbb{R}^+$ with $a < 0$ and $\tau_* := \min T$ exists, then any solution $x$ of (3.1) with $|\phi(\tau_*) - x(\tau_*)| < \varepsilon/|a|$ satisfies $|\phi(t) - x(t)| < \varepsilon/|a|$ for all $t \in T$. 

(iv) If \( a \in \mathbb{R}^+ \) with \( a < 0 \) and \( \min T \) does not exist, then \( \lim_{t \to \infty} \phi(t)e_{\ominus a}(t, t_0) \) exists, and there exists a unique solution

\[
x(t) := \left( \lim_{t \to \infty} \phi(t)e_{\ominus a}(t, t_0) \right) e_a(t, t_0)
\]

of (3.1) such that \(|\phi(t) - x(t)| \leq \epsilon/|a| \) for all \( t \in T \).

**Proof.** From Proposition 3.6, we can find a nondecreasing function \( u : T \to \mathbb{R} \) and a nonincreasing function \( v : T \to \mathbb{R} \) such that (3.2) holds for \( t \in T \). Note here that \( e_a(t, t_0) > 0 \) on \( T \) by Theorem 2.3 (vi); if \( a > 0 \), \( \lim_{t \to \infty} e_a(t, t_0) = \infty \) holds by Bernoulli’s Inequality [25, Theorem 6.2]; if \( a < 0 \), \( \lim_{t \to \infty} e_a(t, t_0) = \infty \) by Lemma 3.5.

Now we consider (i). From Proposition 3.6 (i), we observe (3.3) holds for all \( t \in T \). Let \( x(t) \) be any solution of (3.1) with \(|\phi(t^*) - x(t^*)| < \epsilon/|a| \). That is, it is expressed as

\[
x(t) := x(t^*)e_{\ominus a}(t^*, t_0)e_a(t, t_0)
\]

for all \( t \in T \). Using (3.2), (3.3) and \(|\phi(t^*) - x(t^*)| < \epsilon/|a| \), we obtain

\[
u(t^*) < x(t^*)e_{\ominus a}(t^*, t_0) < v(t^*)
\]

From this inequality and the results above, we have

\[
\phi(t) - x(t) \leq u(t^*)e_a(t, t_0) + \frac{\epsilon}{|a|} - x(t^*)e_{\ominus a}(t^*, t_0)e_a(t, t_0)
\]

\[
= (u(t^*) - x(t^*)e_{\ominus a}(t^*, t_0))e_a(t, t_0) + \frac{\epsilon}{|a|}
\]

\[
< \frac{\epsilon}{|a|}
\]

and\[
\phi(t) - x(t) \geq v(t^*)e_a(t, t_0) - \frac{\epsilon}{|a|} - x(t^*)e_{\ominus a}(t^*, t_0)e_a(t, t_0)
\]

\[
= (v(t^*) - x(t^*)e_{\ominus a}(t^*, t_0))e_a(t, t_0) - \frac{\epsilon}{|a|}
\]

\[
> -\frac{\epsilon}{|a|}
\]

for all \( t \in T \). Thus (i) holds.

Next consider (ii). From Proposition 3.6 (ii), we observe (3.4) holds for all \( t \in T \). Since \( \lim_{t \to \infty} u(t) \) exists and \( \lim_{t \to \infty} e_{\ominus a}(t, t_0) = 0 \), we see that

\[
\lim_{t \to \infty} \phi(t)e_{\ominus a}(t, t_0) = \lim_{t \to \infty} \left( u(t) + \frac{\epsilon}{|a|}e_{\ominus a}(t, t_0) \right) = \lim_{t \to \infty} u(t)
\]

exists. Now, we consider the function

\[
x(t) := \left( \lim_{t \to \infty} \phi(t)e_{\ominus a}(t, t_0) \right) e_a(t, t_0)
\]

for all \( t \in T \). Then \( x(t) \) is a solution of (3.1). Using (3.2) and (3.4), we obtain

\[
\phi(t) - x(t) = \left( u(t) - \lim_{t \to \infty} \phi(t)e_{\ominus a}(t, t_0) \right) e_a(t, t_0) + \frac{\epsilon}{|a|} \leq \frac{\epsilon}{|a|}
\]

and

\[
\phi(t) - x(t) = \left( v(t) - \lim_{t \to \infty} \phi(t)e_{\ominus a}(t, t_0) \right) e_a(t, t_0) - \frac{\epsilon}{|a|} \geq -\frac{\epsilon}{|a|}
\]

for all \( t \in T \). That is, \(|\phi(t) - x(t)| \leq \epsilon/|a| \) holds for all \( t \in T \). We next show that \( x(t) \) is a unique solution of (3.1) such that \(|\phi(t) - x(t)| \leq \epsilon/|a| \) for all \( t \in T \). Let \( c \neq \lim_{t \to \infty} \phi(t)e_{\ominus a}(t, t_0) \) and \( y(t) = ce_a(t, t_0) \) for all \( t \in T \). Then \( y(t) \) means any solution of (3.1) except \( x(t) \), by the uniqueness of solutions for the initial value problem. Since

\[
\lim_{t \to \infty} (u(t) - c) = \lim_{t \to \infty} \phi(t)e_{\ominus a}(t, t_0) - c \neq 0,
\]
we see that
\[ \lim_{t \to \infty} |\phi(t) - y(t)| = \lim_{t \to \infty} \left| (u(t) - c)e_a(t, t_0) + \frac{\varepsilon}{a} \right| = \infty. \]
Consequently, \( x(t) \) is a unique solution of (3.1) such that \( |\phi(t) - x(t)| \leq \varepsilon/a \) for all \( t \in \mathbb{T} \). Thus (ii) holds.

The arguments for (iii) and (iv) are similar to those given above for (i) and (ii), and thus are omitted. This completes the proof. \( \square \)

By Theorem 3.7, we obtain the following result immediately.

**Corollary 3.8.** If \( a \in \mathbb{R}^+ \) and \( a \neq 0 \), then (3.1) has Hyers-Ulam stability with an HUS constant \( 1/|a| \) on \( \mathbb{T} \).

**Remark 3.9.** Let \( a \in \mathbb{R}^+ \) and \( a \neq 0 \). Suppose \( \mathbb{T} \) is unbounded above and below, and \( t_0 \in \mathbb{T} \). Then the minimum HUS constant for (3.1) on \( \mathbb{T} \) is \( 1/|a| \). Now we will show this fact. The function
\[ \phi(t) := e_a(t, t_0) + \frac{\varepsilon}{a}, \quad t \in \mathbb{T} \]
satisfies \( |\phi^s(t) - a\phi(t)| = \varepsilon \) for all \( t \in \mathbb{T} \). Note that \( x(t) \equiv ce_a(t, t_0) \) is the general solution of (3.1); if \( a > 0 \), \( \lim_{t \to \infty} e_a(t, t_0) = \infty \) holds by Bernoulli’s Inequality [25, Theorem 6.2]; if \( a < 0 \), \( \lim_{t \to \infty} e_a(t, t_0) = \infty \) by Lemma 3.5. From the facts above, we see that \( |\phi(t) - x(t)| < \infty \) whenever \( c = 1 \). If \( c = 1 \) then \( |\phi(t) - x(t)| = \varepsilon/|a| \) on \( \mathbb{T} \). This means that the minimum HUS constant for (3.1) on \( \mathbb{T} \) is greater than or equal to \( 1/|a| \). So that this together with Corollary 3.8 implies that the minimum HUS constant for (3.1) on \( \mathbb{T} \) is \( 1/|a| \).

If \( a \in \mathbb{R} \) but \( a \notin \mathbb{R}^+ \), then Theorem 3.7 and Corollary 3.8 do not apply. We do, however, have the following result.

**Theorem 3.10.** Let \( a \in \mathbb{R} \) but \( a \notin \mathbb{R}^+ \), and assume \( \mathbb{T} \) is a time scale such that \( \tau_* := \min \mathbb{T} \) exists and \( \mathbb{T} \) is unbounded above.

(i) If
\[ K := \sup_{t \in \mathbb{T}} \int_{\tau_*}^t |e_a(t, s)| \Delta s < \infty, \] (3.8)
then (3.1) has Hyers-Ulam stability on \([\tau_*, \infty)_\mathbb{T}\) with HUS constant \( K \).

(ii) If there exist constants \( 0 < m < M \) such that
\[ 0 < m \leq |e_a(t, \tau_*)| \leq M, \quad \text{for all } t \in \mathbb{T}, \] (3.9)
then (3.1) does not have HUS.

**Proof.** We modify the proof of [26, Theorem 5.1] and extend it. (i) Given \( \varepsilon > 0 \), suppose there exists a function \( \phi \in C^A_{\tau_*}[\tau_*, \infty)_\mathbb{T} \) that satisfies
\[ |\phi^s(t) - a\phi(t)| \leq \varepsilon, \quad t \in [\tau_*, \infty)_\mathbb{T}. \]
Set
\[ q(t) := \phi^s(t) - a\phi(t), \quad t \in [\tau_*, \infty)_\mathbb{T}. \]
Using the variation of constants formula [25, Theorem 2.77], we note that \( \phi \) is given by
\[ \phi(t) = \phi(\tau_*) e_a(t, \tau_*) + \int_{\tau_*}^t e_a(t, s) q(s) \Delta s. \]
Let \( x \in C^A_{\tau_*}[\tau_*, \infty)_\mathbb{T} \) be the unique solution of the initial value problem
\[ x^A - ax = 0, \quad x(\tau_*) = \phi(\tau_*). \]
Then
\[ x(t) = \phi(t) e \alpha(t, t_\ast), \quad t \in \left[ t_\ast, \infty \right) \mathbb{T}, \]
and
\[ |\phi(t) - x(t)| = \int_t^{t_\ast} e \alpha(t, \sigma(s)) q(s) ds \leq \varepsilon \int_t^{t_\ast} |e \alpha(t, \sigma(s))| ds. \]

Therefore, if condition (3.8) is met, then (3.1) has HUS on \( \mathbb{T} \) with HUS constant \( K \).

(ii) Let \( 0 < m < M \) be as in (3.9), and let \( \varepsilon > 0 \) be given. Then
\[ \phi(t) := \frac{e}{M} e \alpha(t, t_\ast), \quad t \in \mathbb{T} \]
satisfies the inequality
\[ |\phi^a(t) - a \phi(t)| = \frac{e}{M} |e \alpha(t, t_\ast)| \leq \varepsilon \]
for all \( t \in \mathbb{T} \). Since \( x(t) = ce \alpha(t, t_\ast) \) is the general solution to (3.1), then
\[ |\phi(t) - x(t)| = \frac{e}{M} t - c \cdot |e \alpha(t, t_\ast)| \geq m \left| \frac{e}{M} t - c \right| \to \infty \]
as \( t \to \infty \) for \( t \in \mathbb{T} \) and for any \( c \in \mathbb{R} \). In this case, (3.1) lacks HUS.

\[ \square \]

4 Examples

In this section we explore some examples on time scales of interest, to illustrate some of the results in the preceding section.

**Example 4.1.** Consider a time scale \( \mathbb{T} \) with bounded graininess that exhibits both continuous and discrete properties [25, Example 1.38]. In particular, for \( \alpha, \beta > 0 \), let
\[ \mathbb{T} = \mathbb{P}_{\alpha, \beta} = \bigcup_{k \in \mathbb{Z}} \left[ k(\alpha + \beta), k(\alpha + \beta) + \alpha \right]. \]

Then for \( t \in \mathbb{T} \) we have
\[ \mu(t) = \begin{cases} 0 : & t \in \left[ k(\alpha + \beta), k(\alpha + \beta) + \alpha \right), \\ \beta : & t = k(\alpha + \beta) + \alpha, \end{cases} \]
and for \( a \in \mathbb{R} \setminus \{0, -1/\beta\} \) the exponential function \( e \alpha(t, 0) \) is given by
\[ e \alpha(t, 0) = \left( \frac{1 + a \beta}{e^{a \beta}} \right)^k e^{a t}, \quad t \in \left[ k(\alpha + \beta), k(\alpha + \beta) + \alpha \right], \quad k \in \mathbb{Z}. \quad (4.1) \]

If \( a \in (-1/\beta, 0) \cup (0, \infty) \), then \( a \in \mathbb{R}^+ \) and \( a \neq 0 \). By Corollary 3.8, (3.1) has Hyers-Ulam stability with an HUS constant of \( 1/|a| \) on \( \mathbb{P}_{\alpha, \beta} \) for any \( \alpha, \beta > 0 \).

Now, for \( t = k(\alpha + \beta) \),
\[ e \alpha(k(\alpha + \beta), 0) = \left( \frac{1 + a \beta}{e^{a \beta}} \right)^k e^{a(k(\alpha + \beta))} = \left[ (1 + a \beta) e^{a \alpha} \right]^k. \]

Set
\[ f(a) := (1 + a \beta) e^{a \alpha}. \]

Then \( f'(a) = e^{a \alpha}(a + \beta + a a \beta) \), and \( f \) has a global minimum at \( a_\ast = \frac{\alpha + \beta}{-a \beta} \), which is
\[ f(a_\ast) := \frac{\beta}{-a} e^{-1 - \frac{\beta}{\alpha}}. \]
Note, however, that if
\[
    a = a_\ast = \frac{a + \beta}{-a\beta} = \frac{1 + a/\beta}{-a},
\]
then \(a \not\in \mathbb{R}^+\), since
\[
    1 + a\mu(t) = -\frac{\beta}{a} < 0, \quad t = k(a + \beta) + a.
\]
Also note that
\[
    e_\alpha(k(a + \beta), 0) = \left(\frac{-\beta}{a} e^{-1 - k}\right)^k, \quad \text{for all } k \in \mathbb{Z}.
\]
Thus, if
\[
    \frac{a}{\beta} = \text{ProductLog}(1/e) = 0.278465428\cdots,
\]
where ProductLog is the Lambert \(W\) function, then
\[
    e_\alpha(k(a + \beta), 0) = (-1)^k, \quad \text{for all } k \in \mathbb{Z}.
\]
We claim that if \(a\) and \(\alpha/\beta\) are as stated in (4.2) and (4.3), respectively, then (3.1) does not have HUS on \(\mathbb{P}_{a,\beta}\).
To see this, let \(\varepsilon > 0\) be given. For fixed \(k \in \mathbb{Z}\), we have from (4.1) that the exponential function is
\[
    e_\alpha(t, 0) = \left(\frac{-\beta}{a} e^{-1e}\right)^k e^{-\frac{\alpha(e)k}{x(r)}} = (-1)^k e^{-\frac{\alpha(e)k}{x(r)}}, \quad t \in [k(a + \beta), k(a + \beta) + \alpha], \quad k \in \mathbb{Z},
\]
which is bounded above and below but alternates sign based on the parity of \(k\). Now, for each fixed \(k \in \mathbb{N}_0\), the function
\[
    \phi(t) := \varepsilon e_\alpha(t, 0), \quad t \in [k(a + \beta), k(a + \beta) + \alpha]
\]
satisfies
\[
    \phi^{\Delta}(t) - a\phi(t) = \begin{cases}
        \varepsilon (-1)^k : t = k(a + \beta), \\
        \varepsilon(-1)^k e^{-\frac{\alpha(e)k}{x(r)}} : t \in (k(a + \beta), k(a + \beta) + \alpha),
    \end{cases}
\]
so that
\[
    |\phi^{\Delta}(t) - a\phi(t)| = \begin{cases}
        \varepsilon : t = k(a + \beta), \\
        \varepsilon e^{-\frac{\alpha(e)\alpha}{x(r)}} : x \in (0, \alpha].
    \end{cases}
\]
This implies \(|\phi^{\Delta}(t) - a\phi(t)| \leq \varepsilon\) for all \(t \in [k(a + \beta), k(a + \beta) + \alpha]\). Since the general solution of \(x^{\Delta} - ax = 0\) is \(x(t) = ce_\alpha(t, 0)\), we have
\[
    |\phi(k(a + \beta)) - x(k(a + \beta))| = |\varepsilon k(a + \beta) - c| \to \infty
\]
as \(k \to \infty\), just as predicted by Theorem 3.10 (ii).

**Example 4.2.** In this example we consider a discrete time scale with bounded graininess that tends to zero near infinity. Let \(\mathbb{T} = \sqrt{\mathbb{N}_0} = \{0, 1, \sqrt{2}, \sqrt{3}, \cdots\}\), and fix \(n_0 \in \mathbb{N}_0\). Then for \(\sqrt{n} \in \mathbb{T}\) we have
\[
    \mu(\sqrt{n}) = \sqrt{n + 1} - \sqrt{n},
\]
and the exponential function \(e_\alpha(\sqrt{n}, \sqrt{m})\) is given by
\[
    e_\alpha(\sqrt{n}, \sqrt{m}) = \prod_{k=1}^{n-n_0} \left(1 + a\mu(\sqrt{k - 1})\right), \quad n, n_0 \in \mathbb{N}_0.
\]
Let \(a \in (-1, 0) \cup (0, \infty)\); then \(a \in \mathbb{R}^+\) and \(a \not\in 0\), so that Corollary 3.8 holds. In particular, given \(\varepsilon > 0\) and a fixed \(n_0 \in \mathbb{N}_0\), suppose that a delta-differentiable function \(\phi: \mathbb{T} \to \mathbb{R}\) satisfies
\[
    |\phi^{\Delta}(\sqrt{n}) - a\phi(\sqrt{n})| \leq \varepsilon, \quad \text{for all } n \in \mathbb{N}_0.
\]
Then by Theorem 3.7 (ii) we know that \( \lim_{n \to \infty} \phi (\sqrt{n}) e^{\Delta a \left( \sqrt{n}, \sqrt{n_0} \right)} \) exists, and there exists a unique solution \( x (\sqrt{n}) := \left( \lim_{n \to \infty} \phi (\sqrt{n}) e^{\Delta a \left( \sqrt{n}, \sqrt{n_0} \right)} \right) e^{a \left( \sqrt{n}, \sqrt{n_0} \right)} \) of (3.1) such that \( |\phi (\sqrt{n}) - x (\sqrt{n})| \leq \varepsilon / a \) for all \( n \in \mathbb{N}_0 \). In other words, (3.1) has HUS for \( a > 0 \) with HUS constant \( 1/a \) on \( T = \sqrt{\mathbb{N}_0} \).

Now let \( n_0 = 0 \), fix \( k_0 \in \mathbb{N} \), and let \( a \in \left( \frac{-1}{\mu (\sqrt{k_0})}, \frac{-1}{\mu (\sqrt{k_0} - 1)} \right) \). Then

\[
1 + a \mu (0) < 1 + a \mu (1) < \cdots < 1 + a \mu \left( \sqrt{k_0} - 1 \right) < 0 < 1 + a \mu \left( \sqrt{k_0} \right) < \cdots < 1,
\]

and we have

\[
e^{a \left( \sqrt{n}, 0 \right)} = \prod_{k=1}^{n} \left( 1 + a \mu \left( \sqrt{k - 1} \right) \right) \left\{ \begin{array}{ll}
< 0 : & k_0 \text{ odd}, \\
> 0 : & k_0 \text{ even},
\end{array} \right.
\]

so that Theorem 3.7 does not apply, as \( a \in \mathbb{R} \) but \( a \not\in \mathbb{R}^+ \). By Theorem 3.10 (i), if

\[
\sup_{t \in T} \int_{0}^{t} |e^{a \left( t, \sigma (s) \right)}| \Delta s = \sup_{n \in \mathbb{N}} \sum_{s=0}^{n-1} \left( \frac{1}{\sqrt{s + T} + \sqrt{s}} \right) \left| \prod_{k=1}^{n-s-1} \left( 1 + \frac{a}{\sqrt{k + \sqrt{k - 1}}} \right) \right| < \infty
\]

(4.4)

for all \( n \in \mathbb{N} \), then (3.1) has HUS for this time scale. Numerical evidence in FIGURE 1 with \( a = -1.25 \) suggests that (4.4) indeed holds, and thus (3.1) has HUS with HUS constant \( K = 1 \) for \( T = \sqrt{\mathbb{N}} \) for these \( a < 0 \) values.
5 Perturbed linear dynamic equation

As an application of our results, we next consider the first-order perturbed linear dynamic equation

$$\phi^A(t) - a\phi(t) = f(t, \phi(t)), \quad a \in \mathbb{R},$$  \hspace{1cm} (5.1)

where \(f(t, \phi)\) is a real-valued function on \(T \times \mathbb{R}\). If max \(T\) does not exist, while \(a\) and \(f(t, \phi)\) satisfy suitable conditions, then we claim that Theorem 3.7 (iii) implies the uniform-ultimate boundedness of solutions to (5.1). We say that the solutions of (5.1) are uniform-ultimately bounded for a bound \(L\) if there exists a \(B > 0\) and, for any \(a > 0\), there exists a \(T(a) > 0\) such that \(t_0 \in T\) and \(|\phi_{t_0}| < a\) imply that \(|\phi(t)| < B\) for all \(t \geq t_0 + T(a)\) and \(t \in T\), where \(\phi(t)\) is a solution of (5.1) satisfying \(\phi(t_0) = \phi_0\) and \((t_0, \phi_0) \in T \times \mathbb{R}\). See [27, 28] for the definition of the uniform-ultimate boundedness for differential equations and some dynamical systems.

**Corollary 5.1.** Let \(\delta > 0\) be an arbitrary constant. Suppose that max \(T\) does not exist, and there exists an \(L > 0\) such that \(|f(t, \phi)| \leq L\) for all \((t, \phi) \in T \times \mathbb{R}\). Suppose also that all solutions of (5.1) exist on \(T\). If \(a \in \mathbb{R}^+\) with \(a < 0\), then all solutions of (5.1) are uniform-ultimately bounded for a bound \(\frac{L}{|a|} + \delta\).

Before proving the corollary, we give a lemma as follows.

**Lemma 5.2.** Suppose that \(a < 0\) and \(a \in \mathbb{R}^+\). Let \(t_0 \in T\). Then the inequality

$$e_a(t, t_0) \leq \frac{1}{1 - a(t - t_0)}$$

holds for all \(t \in [t_0, \infty)_T\).

**Proof.** From Theorem 2.3 and \(a < 0\) we have

$$(e_{\ominus a}(t, t_0))^A = \ominus a e_{\ominus a}(t, t_0) = \frac{-a}{(1 + a\mu(t)) e_a(t, t_0)} = \frac{-a}{e_a(\sigma(t), t_0)} = -ae_{\ominus a}(\sigma(t), t_0) > 0$$

on \(T\), this implies that \(e_{\ominus a}(t, t_0)\) is an increasing function on \(T\). Thus, we have

$$1 = \frac{1}{e_a(t_0, t_0)} = e_{\ominus a}(t_0, t_0) \leq e_{\ominus a}(t, t_0) \leq e_{\ominus a}(\sigma(t), t_0)$$

for all \(t \in [t_0, \infty)_T\). We now consider the function

$$y(t) = e_{\ominus a}(t, t_0) + a(t - t_0) - 1$$

on \(T\). Using the results above, we can conclude that

$$y^A(t) = (e_{\ominus a}(t, t_0))^A + a = -ae_{\ominus a}(\sigma(t), t_0) + a = -a(e_{\ominus a}(\sigma(t), t_0) - 1) \geq 0$$

for all \(t \in [t_0, \infty)_T\). Consequently, we obtain

$$y(t) \geq y(t_0) = e_{\ominus a}(t_0, t_0) - 1 = 0$$

for all \(t \in [t_0, \infty)_T\). This completes the proof. \(\square\)

**Proof of Corollary 5.1.** Let \(B = L/|a| + \delta\), and let \(a > 0\) be any positive constant. We consider the solution \(\phi(t)\) of (5.1) with \(\phi(t_0) = \phi_0\) and \(|\phi_0| < a\), where \((t_0, \phi_0) \in T \times \mathbb{R}\). By the assumptions, \(\phi(t)\) exists on \(T\), and

$$|\phi^A(t) - a\phi(t)| = |f(t, \phi(t))| \leq L$$

for all \(t \in T\). Using Theorem 3.7 (iii) for \([t_0, \infty)_T\), there exists a solution \(x(t)\) of (3.1) with \(|\phi_0 - x(t_0)| < L/|a|\) such that \(|\phi(t) - x(t)| < L/|a|\) for \(t \in [t_0, \infty)_T\). Note that \(x(t)\) is written by \(x(t_0)e_a(t, t_0)\) on \(T\). Since

$$|x(t)| \leq (|x(t_0) - \phi_0| + |\phi_0|)e_a(t, t_0) < \left(\frac{L}{|a|} + |\phi_0|\right) e_a(t, t_0) < \left(\frac{L}{|a|} + a\right) e_a(t, t_0)$$
for \( t \in [t_0, \infty)_T \), we see that
\[
|\phi(t)| \leq |\phi(t) - x(t)| + |x(t)| < \frac{L}{|a|} + \left( \frac{L}{|a|} + \alpha \right) e_\alpha(t, t_0)
\]
for \( t \in [t_0, \infty)_T \). From Lemma 5.2 and \( a < 0 \), we obtain
\[
|\phi(t)| < \frac{L}{|a|} + \left( \frac{L}{|a|} + \alpha \right) \frac{1}{1 + |a|(t - t_0)}
\]
(5.2)
for \( t \in [t_0, \infty)_T \). If \( L/|a| + \alpha < \delta \), then \( |\phi(t)| < B \) holds for \( t \in [t_0, \infty)_T \). That is, we can choose a \( T(a) > 0 \) arbitrarily. Next, we consider the case \( L/|a| + \alpha > \delta \). Let
\[
T(a) = \frac{1}{|a|} \left( \frac{L}{|a|} + \alpha - \delta \right) > 0.
\]
Using this and (5.2), we obtain the inequality \( |\phi(t)| < B \) for \( t \geq t_0 + T(a) \) and \( t \in [t_0, \infty)_T \). This completes the proof of Corollary 5.1.

**Remark 5.3.** In Corollary 5.1, we cannot choose \( \delta = 0 \) as the following example shows. Let \( T = h\mathbb{Z} \) and \( t_0 \in T \). Consider the equation
\[
\phi^\Delta(t) - a\phi(t) = 1,
\]
where \( a \in \mathbb{R}^+ \) and \( a < 0 \). In this case, all assumptions in Corollary 5.1 are satisfied. Thus, we conclude that all solutions are uniform-ultimately bounded for a bound \( \frac{L}{|a|} + \delta = \frac{1}{|a|} + \delta \). Note that the function
\[
\phi(t) = (ah + 1)^\frac{1}{a} - \frac{1}{a}
\]
is a solution on \( T \). Since \( a \in \mathbb{R}^+ \) and \( a < 0 \), we have \( 0 < ah + 1 < 1 \), and thus \( \phi(t) > 1/|a| = L/|a| \) for all \( t \in T \). This means that we cannot choose \( \delta = 0 \).

**Example 5.4.** In this example we consider the first-order nonlinear dynamic equation
\[
\phi^\Delta(t) - a\phi(t) = (1 + a) \sin(\phi(t))
\]
(5.3)
on \( T = \sqrt{n_0} \), where \(-1 < a < 0 \). Since \( \mu(\sqrt{n}) = \sqrt{n + 1} - \sqrt{n} \leq 1 \) and \( \mu(\sqrt{n}) \) is decreasing for \( n \in \mathbb{N}_0 \), we have
\[
0 < 1 + a\mu(0) < 1 + a\mu(1) < \cdots < 1,
\]
and thus \( a \in \mathbb{R}^+ \). It is clear that \( \phi(t) \equiv 0 \) is a trivial solution of (5.3). If \( \phi \neq 0 \) then
\[
\phi + \mu(t)(a\phi + (1 + a) \sin \phi) \neq 0
\]
holds for all \( t \in \sqrt{n_0} \). Since \( \sqrt{n_0} \) is a discrete time scale and \( \phi + \mu(t)(a\phi + (1 + a) \sin \phi) \) is increasing for \( \phi \in \mathbb{R} \), all solutions of (5.3) exist uniquely on \( \sqrt{n_0} \). Using Corollary 5.1, we see that all solutions of (5.3) are uniform-ultimately bounded for a bound \( \frac{1 + a}{|a|} + \delta \), where \( \delta > 0 \) is an arbitrary constant.

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**References**