Approximation solvability for a system of implicit nonlinear variational inclusions with $H$-monotone operators

Abstract: In this paper, we introduce and study a new system of variational inclusions which is called a system of nonlinear implicit variational inclusion problems with $A$-monotone and $H$-monotone operators in semi-inner product spaces. We define the resolvent operator associated with $A$-monotone and $H$-monotone operators and prove its Lipschitz continuity. Using resolvent operator technique, we prove the existence and uniqueness of solution for this new system of variational inclusions. Moreover, we suggest an iterative algorithm for approximating the solution of this system and discuss the convergence analysis of the sequences generated by the iterative algorithm under some suitable conditions.

Keywords: system of nonlinear implicit variational inclusion problem, $A$- and $H$-monotone operators, semi-inner product space, resolvent operator, iterative algorithm and convergence analysis

MSC: 49J40, 49J53, 47H06

1 Introduction

Variational inequality problems are among the most interesting and intensively studied classes of mathematical problems which have a wide range of applications in the fields of optimization and control, economics and transportation equilibrium and engineering sciences. Variational inequality problems have been generalized and extended in different directions using novel and innovative techniques. An important and useful generalization of variational inequality is a variational inclusion. Several numerical methods including projection methods, Wiener-Hopf equations, and descent and decomposition have been developed for solving variational inequalities.

In 1994, Hassouni and Moudafi [1] introduced and studied a class of variational inclusions and developed iterative algorithms for this class. Later on, Adly [2], Huang [3], Ding [4], Kazmi [5] and Kazmi and Bhat [6, 7] obtained some important extensions of the result in [1].

The projection method and its various generalizations have been widely used to solve variational inequalities, variational inclusions and their generalizations (see, for examples [1–34]). It is known that the monotonicity of the underlying operator plays a prominent role in solving different classes of variational inequality problems. In 2001, Huang and Fang [15] introduced the notion of generalized $m$-accretive mappings in Banach spaces. Later on in 2003, Fang and Huang [12] introduced and studied a new class of variational inclusions involving $H$-monotone operators in a Hilbert space. Using resolvent operator, they proposed an algorithm for solving the associated class of variational inclusions. Later on Kalia and Verma
also obtained some results for a system of variational inclusions in the setting of Banach space. A considerable research in approximation solvability using $A$-monotone operators, $H$-$\eta$-accretive operators and $H$-$\eta$-accretive operators has been carried out by Akram et al. [35], Bhat and Zahoor [8, 9], Ceng et al. [10], He et al. [14], Huang and Fang [15], Lan et al. [21], Li and Huang [23], Luo and Huang [24], Mohsen et al. [25], Shan et al. [28], Verma [29–32], and Zou and Huang [34].

Motivated and inspired by the above works, in this paper, we define the resolvent operators associated with $A$- and $H$-monotone operators given their Lipschitz continuity. As an application, we consider a class of system of nonlinear implicit variational inclusions involving $A$-monotone and $H$-monotone operators in semi-inner product spaces. Furthermore, we prove the existence and uniqueness of solution of the system of nonlinear implicit variational inclusions. Moreover, using resolvent operator technique, we suggest an iterative algorithm for approximating the solution of this system and discuss the convergence analysis of the sequences generated by the iterative algorithm. The results presented in this paper generalize and improve many well-known results in the literature (see, for example [1, 6, 13, 18–20, 26–32]) and the related references cited therein.

2 Resolvent operator and formulation of problem

We need the following definitions and well-known results for the main theorems.

**Definition 2.1.** [39] Let $X$ be a vector space over the field $F$ of real or complex numbers. A functional $[\cdot, \cdot] : X \times X \rightarrow F$ is called a semi-inner product if it satisfies the following:

(i) $[x + y, z] = [x, z] + [y, z], \quad \forall x, y, z \in X$;

(ii) $[\lambda x, y] = \lambda [x, y], \quad \forall \lambda \in F$ and $x, y \in X$;

(iii) $[x, x] > 0, \quad \text{for } x \neq 0$;

(iv) $||x||^2 \leq [x, x][y, y]$.

The pair $(X, [\cdot, \cdot])$ is said to be a semi-inner product space.

We observe that $||x|| = [x, x]^{\frac{1}{2}}$ is a norm on $X$. Hence every semi-inner product space is a normed linear space. On the other hand, in a normed linear space, one can generate semi-inner product in infinitely many different ways. Giles [38] had proved that if the underlying space $X$ is a uniformly convex smooth Banach space, then it is possible to find a semi-inner product, uniquely. Also the unique semi-inner product has the following nice properties:

(i) $[x, y] = 0$ if and only if $y$ is orthogonal to $x$, that is if and only if $||y|| \leq ||y + \lambda x||$, for all scalars $\lambda$.

(ii) Generalized Riesz representation theorem: If $f$ is a continuous linear functional on $X$ then there is a unique vector $y \in X$ such that $f(x) = [x, y]$, for all $x \in X$.

(iii) The semi-inner product is continuous, that is for each $x, y \in X$, we have $\text{Re}[y, x + \lambda y] \rightarrow \text{Re}[y, x]$ as $\lambda \rightarrow 0$.

Since the sequence space $l^p, \quad p > 1$ and the function space $L^p, \quad p > 1$ are uniformly convex smooth Banach spaces, we can define a semi-inner product on these spaces, uniquely.

**Example 2.2.** The real sequence space $l^p$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[x, y] = \frac{1}{||y||^{p-2}} \sum_{i} x_i y_i |y_i|^{p-2}, \quad x, y \in l^p.$$  

**Example 2.3** [38] The real Banach space $L^p(X, \mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[f, g] = \frac{1}{||g||^{p-2}} \int x f(x) g(x)|g(x)|^{p-2} \text{sgn}(g(x))d\mu, \quad f, g \in L^p.$$
Definition 2.4. [42] Let $X$ be a real Banach space. Then

(i) The modulus of smoothness of $X$ is defined as

$$\rho_X(t) = \sup \left\{ \frac{||x + y|| + ||x - y||}{2} - 1 : ||x|| = 1, ||y|| = t, \ t > 0 \right\}.$$  

(ii) $X$ is said to be uniformly smooth if $\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0$.

(iii) $X$ is said to be $p$-uniformly smooth if there exists a positive real constant $k$ such that $\rho_X(t) \leq kt^p$, $p > 1$.

Clearly, $X$ is 2-uniformly smooth if there exists a positive real constant $k$ such that $\rho_X(t) \leq kt^2$.

Lemma 2.5 [42] Let $p > 1$ be a real number and $X$ be a smooth Banach space. Then the following statements are equivalent:

(i) $X$ is 2-uniformly smooth.

(ii) There is a constant $k > 0$ such that for every $x, y \in X$, the following inequality holds

$$||x + y||^2 \leq ||x||^2 + 2(y, f_x) + k||y||^2, \quad (2.1)$$

where $f_x \in J(x)$ and $J(x) = \{ x^* \in X^* : \langle x, x^* \rangle = ||x||^2 \}$ and $||x^*|| = ||x||$ is the normalized duality mapping.

Remark 2.6. Every normed linear space $X$ is a semi-inner product space (see [39]). In fact, by Hahn-Banach theorem, for each $x \in X$, there exists at least one functional $f_x \in X^*$ such that $\langle x, f_x \rangle = ||x||^2$. Given any such mapping $f$ from $X$ into $X^*$, we can verify that $\langle y, x \rangle = (y, f_x)$ defines a semi-inner product. Hence we can write the inequality (2.1) as

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x \rangle + k||y||^2, \quad \forall x, y \in X. \quad (2.2)$$

The constant $k$ is chosen with best possible minimum value. We call $k$, as the constant of smoothness of $X$.

The following example may be followed from Remark 2.6 above, Bynum [37] and Corollary 2(3.5)’ in [42], where we see that $J$, the duality mapping, becomes an identity mapping and consequently the duality pairing can be identified with the semi-inner product [..].

Example 2.7 [42] The function space $L^p$ is 2-uniformly smooth for $p \geq 2$ and it is $p$-uniformly smooth for $1 < p < 2$. If $2 \leq p < \infty$, then we have for all $x, y \in L^p$,

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x \rangle + (p - 1)||y||^2.$$  

Here the constant of smoothness is $p - 1$.

Definition 2.8. Let $X$ be a real 2-uniformly smooth Banach space. A mapping $T : X \to X$ is said to be

(i) $r$-strongly monotone if there exists a positive constant $r$ such that

$$[Tx - Ty, x - y] \geq r||x - y||^2, \quad \forall x, y \in X.$$  

(ii) $m$-relaxed monotone if there exists a positive constant $m$ such that

$$[Tx - Ty, x - y] \geq (-m)||x - y||^2, \quad \forall x, y \in X.$$  

Let $M : X \to 2^X$ be a set-valued map. We denote both the mapping and its graph by $M$, that is $M = \{(x, y) : y \in M(x)\}$. The domain of $M$ is defined by

$$D(M) = \{x \in X : \exists y \in X : (x, y) \in M\}.$$  

The range of $M$ is defined by

$$R(M) = \{y \in X : \exists x \in X : (x, y) \in M\}.$$  

The inverse $M^{-1}$ of $M$ is $\{(y, x) : (x, y) \in M\}$.  


For any two set-valued mappings $N$ and $M$, and any real number $\rho$, we define
\[
N + M = \{(x, y + z) : (x, y) \in N, (x, z) \in M\},
\]
\[
\rho M = \{(x, \rho y) : (x, y) \in M\}.
\]

For a map $A : X \to X$ and a set-valued map $M : X \to 2^X$, we define
\[
A + M = \{(x, y + z) : Ax = y \text{ and } (x, z) \in M\}.
\]

**Definition 2.9.** [27] Let $X$ be a real 2-uniformly smooth Banach space. The mapping $M : X \to 2^X$ is said to be
(i) monotone if
\[
[u^1 - v^1, u - v] \geq 0, \quad \forall (u, u^1), (v, v^1) \in M;
\]
(ii) $r$-strongly monotone if there exists a positive constant $r > 0$ such that
\[
[u^1 - v^1, u - v] \geq r \|u - v\|^2, \quad \forall (u, u^1), (v, v^1) \in M;
\]
(iii) $m$-relaxed monotone if there exists a positive constant $m$ such that
\[
[u^1 - v^1, u - v] \geq (-m) \|u - v\|^2, \quad \forall (u, u^1), (v, v^1) \in M.
\]

**Definition 2.10.** [27] Let $X$ be a real 2-uniformly smooth Banach space. Let $A : X \to X$ be a single-valued mapping and $M : X \to 2^X$ be a set-valued mapping on $X$. The map $M$ is said to be $A$-monotone if
(i) $M$ is $m$-relaxed monotone;
(ii) $(A + \rho M)(X) = X$, where $\rho > 0$ is a positive real number.

**Definition 2.11.** The resolvent operator $J_{\rho, A}^M : X \to X$ is defined by
\[
J_{\rho, A}^M(u) = (A + \rho M)^{-1}(u), \quad \forall u \in X.
\]

**Definition 2.12.** Let $H : X \to X$ be an $r$-strongly monotone operator. The map $M : X \to 2^X$ is said to be $H$-monotone if
(i) $M$ is monotone;
(ii) $(H + \rho M)(X) = X$, where $\rho$ is a positive real number.

**Definition 2.13.** The resolvent operator $J_{\rho, H}^M : X \to X$ is defined by
\[
J_{\rho, H}^M(u) = (H + \rho M)^{-1}(u), \quad \forall u \in X.
\]

Graph convergence plays a crucial role in variational problems, optimization problems and approximation theory. For details on graph convergence one may refer to Aubin and Frankowska [36], Rockafellar and Wets [41] and Verma [30].

**Definition 2.14.** [32] Let $A : X \to X$ be an $r$-strongly monotone and $s$-Lipschitz continuous operator. Let $\{M^n\}$ be a sequence of the $A$-monotone set-valued mappings $M^n : X \to 2^X$ for $n = 0, 1, 2, \cdots$. The sequence $\{M^n\}$ is graph convergent to $M$, denoted by $M^n \stackrel{AG}{\to} M$, if for every $(x, y) \in \text{graph}(M)$, there exists a sequence $\{(x_n, y_n)\} \in \text{graph}(M^n)$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

**Lemma 2.15** [32] Let $A : X \to X$ be an $s$-Lipschitz continuous and $r$-strongly monotone operator. Let $\{M^n\}$ be a sequence of the $A$-monotone set-valued mappings $M^n : X \to 2^X$ for $n = 0, 1, 2, \cdots$. Then the sequence $M^n \stackrel{AG}{\to} M$ if and only if $J_{\rho, A}^{M^n}(u) \to J_{\rho, A}^M(u)$ for all $u \in X$ and $\rho > 0$, where $J_{\rho, A}^M = (A + \rho M)^{-1}$.

**Definition 2.16.** Let $H : X \to X$ be an $r$-strongly monotone and $s$-Lipschitz continuous operator. Let $\{M^n\}$ be a sequence of the $H$-monotone set-valued mappings $M^n : X \to 2^X$ for $n = 0, 1, 2, \cdots$. The sequence
{M^n} is graph convergent to M, denoted by \( M^n \xrightarrow{HG} M \), if for every \((x, y) \in \text{graph}(M)\), there exists a sequence \(\{(x_n, y_n)\} \subseteq \text{graph}(M^n)\) such that \(x^n \to x\) and \(y^n \to y\) as \(n \to \infty\).

**Lemma 2.17 [26]** Let \( H : X \to X \) be \(s\)-Lipschitz continuous and \(r\)-strongly monotone. Let \(\{M^n\}\) be a sequence of the \(H\)-monotone set-valued mappings \( M^n : X \to 2^X \) for \( n = 0, 1, 2, \cdots \). Then the sequence \( M^n \xrightarrow{HG} M \) if and only if \( I^n_{p,H}(u) \to I^M_{p,H}(u) \) for all \( u \in X \) and \( p > 0 \), where \( I^n_{p,H} = (H + pM)^{-1} \).

**Lemma 2.18 [16]** Let \(\{a_n\}\), \(\{b_n\}\) and \(\{c_n\}\) be sequences of non-negative real numbers that satisfy: there exists a positive integer \(n_0\) such that for \(n \geq n_0\),

\[
a_{n+1} \leq (1 - t_n)a_n + b_nt_n + c_n,
\]

where \(t_n \in [0, 1]\), \(\sum_{n=0}^{\infty} t_n = \infty\), \(\lim_{n \to \infty} b_n = 0\) and \(\sum_{n=0}^{\infty} c_n < \infty\). Then \(a_n \to 0\) as \(n \to \infty\).

**Definition 2.19.** The Hausdorff metric \(\mathcal{H}(\cdot, \cdot)\) on \(CB(X)\), is defined by

\[
\mathcal{H}(A, B) = \max \left\{ \sup_{u \in A} \inf_{v \in B} d(u, v), \sup_{v \in B} \inf_{u \in A} d(u, v) \right\}, \quad A, B \in CB(X),
\]

where \(d(\cdot, \cdot)\) is the induced metric on \(X\) and \(CB(X)\) denotes the family of all nonempty closed and bounded subsets of \(X\).

**Definition 2.20.** [11] A set-valued mapping \(T : X \to CB(X)\) is said to be \(\gamma\)-\(\mathcal{H}\)-Lipschitz continuous, if there exists a constant \(\gamma > 0\) such that

\[
\mathcal{H}(T(x), T(y)) \leq \gamma \|x - y\|, \quad \forall x, y \in X.
\]

**Lemma 2.21 [40]** Let \((X, d)\) be a complete metric space and \(T : X \to CB(X)\) be a set-valued mapping on \(X\). Then

(i) For any given \(\xi > 0\) and for any given \(u, v \in X\) and \(x \in T(u)\), there exists \(y \in T(v)\) such that

\[
d(x, y) \leq (1 + \xi)\mathcal{H}(T(u), T(v));
\]

(ii) If \(T : X \to C(X)\), then (i) holds for \(\xi = 0\), (where \(C(X)\) denotes the family of all nonempty compact subsets of \(X\)).

The following lemmas play crucial roles for the proof of main results.

**Lemma 2.22.** If \(H : X \to X\) is \(r\)-strongly monotone and \(M : X \to 2^X\) is \(H\)-monotone, then the resolvent operator \(J^M_{p,H} = \frac{1}{r} I^M_{p,H}\) is \(\frac{1}{r}\)-Lipschitz continuous.

**Proof.** For any \(x, y \in X\), we have

\[
\left\{ \begin{array}{l}
I^n_{p,H}(x) = (H + pM)^{-1}(x), \\
I^n_{p,H}(y) = (H + pM)^{-1}(y).
\end{array} \right.
\]

This implies that

\[
\left\{ \begin{array}{l}
\frac{1}{r} (x - H(I^n_{p,H}(x))) \in M(I^n_{p,H}(x)), \\
\frac{1}{r} (y - H(I^n_{p,H}(y))) \in M(I^n_{p,H}(y)).
\end{array} \right.
\]
Since $H$ is $r$-strongly monotone and $M$ is $H$-monotone, we have
\[
\|x - y\| \|J_{p,H}^M(x) - J_{p,H}^M(y)\| \geq [x - y, J_{p,H}^M(x) - J_{p,H}^M(y)] \\
= [x - y, (Hf_{p,H}^M(x) - Hf_{p,H}^M(y)), J_{p,H}^M(x) - J_{p,H}^M(y)] \\
\geq 0 + [Hf_{p,H}^M(x) - Hf_{p,H}^M(y), J_{p,H}^M(x) - J_{p,H}^M(y)] \\
\geq r|J_{p,H}^M(x) - J_{p,H}^M(y)|^2.
\]
This implies that
\[
\|J_{p,H}^M(x) - J_{p,H}^M(y)\| \leq \frac{1}{r} \|x - y\|.
\]

**Lemma 2.23.** If $A : X \rightarrow X$ be $r$-strongly monotone and $M : X \rightarrow 2^X$ be $A$-monotone. Then the resolvent operator $J_{p,A}^M : X \rightarrow X$ is $\frac{1}{r - \rho m}$-Lipschitz continuous for $0 < \rho < \frac{r}{m}$, where $r$, $\rho$ and $m$ are positive constants.

**Proof.** For any $x, y \in X$, we have
\[
\begin{align*}
\{ J_{p,A}^M(x) &= (A + \rho M)^{-1}(x), \\
J_{p,A}^M(y) &= (A + \rho M)^{-1}(y).
\end{align*}
\]
This implies that
\[
\begin{align*}
\frac{1}{\rho} (x - A(J_{p,A}^M(x))) &\in M(J_{p,A}^M(x)), \\
\frac{1}{\rho} (y - A(J_{p,A}^M(y))) &\in M(J_{p,A}^M(y)).
\end{align*}
\]
Since $M$ is $A$-monotone, it is $m$-relaxed monotone. Hence we have
\[
\frac{1}{\rho} [(x - A(J_{p,A}^M(x))) - (y - A(J_{p,A}^M(y)))], J_{p,A}^M(x) - J_{p,A}^M(y)] = \frac{1}{\rho} [x - y - (A(J_{p,A}^M(x) - A(J_{p,A}^M(y))), J_{p,A}^M(x) - J_{p,A}^M(y)] \\
\geq -m|J_{p,A}^M(x) - J_{p,A}^M(y)|^2.
\]
Now we have
\[
\|x - y\| \|J_{p,A}^M(x) - J_{p,A}^M(y)\| \geq [x - y, J_{p,A}^M(x) - J_{p,A}^M(y)] \\
= [x - y - (A(J_{p,A}^M(x) - A(J_{p,A}^M(y))), J_{p,A}^M(x) - J_{p,A}^M(y)] \\
\geq -m|J_{p,A}^M(x) - J_{p,A}^M(y)|^2 + r|J_{p,A}^M(x) - J_{p,A}^M(y)|^2 \\
= (r - m)\|J_{p,A}^M(x) - J_{p,A}^M(y)\|^2.
\]
This implies that
\[
\|x - y\| \geq (r - \rho m)|J_{p,A}^M(x) - J_{p,A}^M(y)|.
\]
Hence we have
\[
\|J_{p,A}^M(x) - J_{p,A}^M(y)\| \leq \frac{1}{r - \rho m} \|x - y\|, \quad 0 < \rho < \frac{r}{m}.
\]

Taking $A = I$, the identity operator, we immediately have the following corollary:

**Corollary 2.24.** Let $M : X \rightarrow 2^X$ be $m$-relaxed monotone. Then the resolvent operator $J_{p,1}^M = (I + \rho M)^{-1} : X \rightarrow X$ is $\frac{1}{1 - \rho m}$-Lipschitz continuous for $0 < \rho < \frac{1}{m}$, where $\rho$ and $m$ are positive constants.
For each \(i = 1, 2, 3\), let \(X\) be a real 2-uniformly smooth Banach space and \(M_i : X \times X \to 2^X\) be a set-valued mapping. Let \(N_i : X \times X \to X\) be any single-valued mapping. Let \(g_i : X \to X\) be any mapping such that for each fixed \(z_i\), \(R(g_i) \cap D(M_i(., z_i)) \neq \emptyset\). Let \(B_i, C_i, G_i : X \to C(X)\) be multi-valued mappings for all \(i = 1, 2, 3\).

We consider the following system of nonlinear implicit variational inclusion problem (in short, SNVIP): For given \(\theta_i \in X\), find \((x_1, x_2, x_3, u_1, u_2, u_3, v_1, v_2, v_3, z_1, z_2, z_3)\) where \(x_i \in X\), \(u_i \in B_i(x_i)\), \(v_i \in C_i(x_i)\), \(z_i \in G_i(x_i)\) such that

\[
\begin{align*}
\theta_1 &\in N_1(u_1, v_1) + M_1(g_1(x_1), z_1), \\
\theta_2 &\in N_2(u_2, v_2) + M_2(g_2(x_2), z_2), \\
\theta_3 &\in N_3(u_3, v_3) + M_3(g_3(x_3), z_3).
\end{align*}
\] (2.3)

Some special cases:

**I.** For \(\theta_3 = 0\), \(N_3(u_3, v_3) \equiv 0\), \(M_3(g_3(x_3), z_3) \equiv 0\), for all \(x_3 \in X\). Then above problem (2.3) reduces to the following problem:

\[
\begin{align*}
\theta_1 &\in N_1(u_1, v_1) + M_1(g_1(x_1), z_1), \\
\theta_2 &\in N_2(u_2, v_2) + M_2(g_2(x_2), z_2).
\end{align*}
\] (2.4)

Problem (2.4) is the set-valued generalization of the variational inclusion problem considered and studied by Fang et al. [13].

**II.** For \(\theta_1 = \theta_2 = \theta_3 = \theta\), for all \(\theta \in X\), \(N_1(u_1, v_1) = N_2(u_2, v_2) = N_3(u_3, v_3) = S(u) - T(u)\), for all \(u \in X\) and \(M_1(g_1(x_1), z_1) = M_2(g_2(x_2), z_2) = M_3(g_3(x_3), z_3) = M(g(u), u)\), for all \(u \in X\), where \(M : X \to 2^X\) is a set-valued mapping, \(S, T, G : X \to X\) are single-valued mappings and \(g : X \to X\) is any mapping such that \(R(g) \cap D(M(., u)) \neq \emptyset\). Then problem (2.3) reduces to the following problem: For a given element \(\theta \in X\), find an element \(u \in X\) such that

\[
\theta \in S(u) - T(u) + M(g(u), u).
\] (2.5)

Problem (2.5) is the generalization of the problem considered and studied by Sahu et al. [27].

We remark that for suitable choices of different mappings \(M_i, N_i, g_i, B_i, C_i\) and the underlying space \(X\), we can obtain different classes of known and new classes of variational inequalities (inclusions) from SNVIP (2.3), see for example [1, 6, 12] and the related references cited therein.

## 3 Existence of solution

We give the following theorem which guarantees the existence of solution of SNVIP (2.3).

**Lemma 3.1.** Let \(X\) be a real 2-uniformly smooth Banach space. Suppose for each \(i = 1, 2, 3\), \(A_i : X \to X\) be \(r_i\)-strongly monotone and \(M_i : X \to 2^X\) be \(A_i\)-monotone. Let \(N_i : X \times X \to X\) and \(g_i : X \to X\) be any mapping such that \(R(g_i) \cap D(M_i(., z_i)) \neq \emptyset\). Then \((x_1, x_2, x_3, u_1, u_2, u_3, v_1, v_2, v_3)\) is a solution of SNVIP (2.3) where \(x_i \in X\), \(u_i \in B_i(x_i)\), \(v_i \in C_i(x_i)\) if and only if it satisfies

\[
g_i(x_i) = f_{\rho_i, A_i}^{M_i(., z_i)}(A_i(g_i(x_i)) - \rho_i N_i(u_i, v_i) + \rho_i \theta_i),
\]

where \(\rho_i\) is a positive real constant.

**Proof.** Suppose that for each \(i = 1, 2, 3\), \((x_i, u_i, v_i)\) is a solution of SNVIP (2.3). Then we have

\[
\theta_i \in N_i(u_i, v_i) + M_i(g_i(x_i), z_i).
\]
This implies that
\[ \rho_1 \theta_1 - \rho_1 \beta_i = \rho_1 M_i (g_i (x_i), z_i). \]
Hence we have
\[ A_i (g_i (x_i)) + \rho_i \theta_i - \rho_i \beta_i = A_i (g_i (x_i)) + \rho_i M_i (g_i (x_i), z_i), \]
which implies that
\[ J^{M_i} (A_i (g_i (x_i)) + \rho_i \theta_i - \rho_i \beta_i) = J^{M_i} (A_i + \rho_i M_i (g_i (x_i), z_i)). \]
Therefore, we have
\[ g_i (x_i) = J^{M_i} (A_i (g_i (x_i)) + \rho_i \theta_i - \rho_i \beta_i) \]
Conversely, assume that
\[ g_i (x_i) = J^{M_i} (A_i (g_i (x_i)) + \rho_i \theta_i - \rho_i \beta_i) \]
It implies that
\[ A_i (g_i (x_i)) + \rho_i \theta_i - \rho_i \beta_i \in (A_i + \rho_i M_i (., z_i)) (g_i (x_i)). \]
Hence we have
\[ \rho_i \theta_i - \rho_i \beta_i \in \rho_i M_i (g_i (x_i), z_i), \]
it means that
\[ \theta_i \in N_i (u_i, v_i) + M_i (g_i (x_i), z_i). \]
This completes the proof. \( \square \)

**Theorem 3.2.** Let \( X \) be a real 2-uniformly smooth Banach space. Suppose for each \( i = 1, 2, 3 \), \( A_i : X \to X \) be \( r_i \)-strongly monotone map and \( M_i : X \times X \to 2^X \) be \( A_i \)-monotone set-valued map. Let \( g_i : X \to X \) be a map such that \( \text{R}(g_i) \cap \text{D}(M_i (., z_i)) \neq \emptyset \) and \( g_i \) be \( \beta_i \)-Lipschitz continuous and \( q_i \)-strongly monotone. Suppose that \( N_i : X \times X \to X \) is \( \xi_i \)-Lipschitz continuous in the first argument, \( \gamma_i \)-Lipschitz continuous in the second argument and \( \delta_i \)-strongly monotone w.r.t \( A_i(g_i) \) in the first argument and that \( A_i(g_i) \) be \( \sigma_i \)-Lipschitz continuous. Let \( B_i : X \to C(X) \) be \( L_i^{-\gamma_i} \)-Lipschitz continuous, \( C_i : X \to C(X) \) be \( L_i^{-\xi_i} \)-Lipschitz continuous and \( G_i : X \to C(X) \) be \( L_i^{-\delta_i} \)-Lipschitz continuous. Suppose that the following condition holds:

**Condition A**
\[ ||J^{M_i} (x_i) - J^{M_i} (x_i')|| \leq b ||x_i - x_i'||, \quad \forall b_i > 0. \]
In addition if \( r_i - \rho_i m_i > 0, 1 - 2q_i + k \beta_i^2 > 0 \) and \( 0 < 1 - 2q_i + k \beta_i^2 + \frac{1}{\mu_1 (r_i - \rho_i m_i)} + L_{G_i} < 1 \), where \( \rho_i \) is a positive real constant and \( k \) is the constant of smoothness of the Banach space \( X \). Then SNVIP (2.3) has a solution.

**Proof.** Define the mapping \( F_i : X \to X \) by
\[ F_i (x_i) = x_i - g_i (x_i) + J^{M_i} (A_i (g_i (x_i)) - \rho_i N_i (u_i, v_i) + \rho_i \theta_i). \] (3.1)
Then for any \( x_i, x_i' \in X \), using Condition A, we have
\[ ||F_i (x_i) - F_i (x_i')|| = \ ||(x_i - g_i (x_i)) + J^{M_i} (A_i (g_i (x_i)) - \rho_i N_i (u_i, v_i) + \rho_i \theta_i)|| - \ ||(x_i' - g_i (x_i')) + J^{M_i} (A_i (g_i (x_i')) - \rho_i N_i (u_i', v_i') + \rho_i \theta_i)|| \]
\[ \leq ||(x_i - x_i') - (g_i (x_i) - g_i (x_i'))|| + ||J^{M_i} (A_i (g_i (x_i)) - \rho_i N_i (u_i, v_i) + \rho_i \theta_i)|| + ||J^{M_i} (A_i (g_i (x_i')) - \rho_i N_i (u_i', v_i') + \rho_i \theta_i)|| \]
\[ \leq ||(x_i - x_i') - (g_i (x_i) - g_i (x_i'))|| + ||J^{M_i} (A_i (g_i (x_i)) - \rho_i N_i (u_i, v_i) + \rho_i \theta_i)|| + ||J^{M_i} (A_i (g_i (x_i')) - \rho_i N_i (u_i', v_i') + \rho_i \theta_i)|| + b_i ||z_i - z_i'||. \] (3.2)
Since \( X \) is 2-uniformly smooth Banach space, we have
\[
\| (x_i - x'_i) - (g_i(x_i) - g_i(x'_i)) \|^2 \leq \| x_i - x'_i \|^2 - 2 \| g_i(x_i) - g_i(x'_i), x_i - x'_i \| + k \| g_i(x_i) - g_i(x'_i) \|^2 \\
\leq \| x_i - x'_i \|^2 - 2 q_i \| x_i - x'_i \|^2 + k \beta_i^2 \| x_i - x'_i \|^2 \\
= (1 - 2 q_i + k \beta_i^2) \| x_i - x'_i \|^2.
\]
Hence we have
\[
\| (x_i - x'_i) - (g_i(x_i) - g_i(x'_i)) \| \leq \sqrt{1 - 2 q_i + k \beta_i^2} \| x_i - x'_i \|. 
\tag{3.3}
\]
Since the resolvent operator \( J^M_{\rho_i A_i} \) is \( \frac{1}{(r_i - \rho_i m_i)} \)-Lipschitz continuous, we have
\[
\| J^M_{\rho_i A_i}(A_i(g_i(x_i))) - \rho_i N_i(u_i, v_i) + \rho_i \theta_i - J^M_{\rho_i A_i}(A_i(g_i(x'_i))) - \rho_i N_i(u'_i, v'_i) + \rho_i \theta_i \| \\
\leq \frac{1}{(r_i - \rho_i m_i)} \| (A_i(g_i(x_i))) - \rho_i N_i(u_i, v_i) + \rho_i \theta_i - (A_i(g_i(x'_i))) - \rho_i N_i(u'_i, v'_i) + \rho_i \theta_i \| \\
= \frac{1}{(r_i - \rho_i m_i)} \left\{ \| (A_i(g_i(x_i))) - A_i(g_i(x'_i)) \| - \rho_i \| N_i(u_i, v_i) - N_i(u'_i, v'_i) \| + \rho_i \| N_i(u'_i, v'_i) - N_i(u'_i, v'_i) \| \right\}. 
\tag{3.4}
\]
Again, since \( A_i(g_i) \) is \( \sigma_i \)-Lipschitz continuous, \( N_i(\cdot, \cdot) \) is \( \delta_i \)-strongly monotone w.r.t \( A_i(g_i) \) in the first argument and \( \xi_i \)-Lipschitz continuous in the first argument, we have
\[
\| (A_i(g_i(x_i))) - A_i(g_i(x'_i)) \| - \rho_i \| N_i(u_i, v_i) - N_i(u'_i, v'_i) \| \\
\leq \| A_i(g_i(x_i))) - A_i(g_i(x'_i)) \|^2 - 2 \rho_i \| N_i(u_i, v_i) - N_i(u'_i, v'_i), A_i(g_i(x_i))) - A_i(g_i(x'_i)) \| \\
+ k \| N_i(u_i, v_i) - N_i(u'_i, v'_i) \|^2 \\
\leq \sigma_i^2 \| x_i - x'_i \|^2 - 2 \rho_i \delta_i \| x_i - x'_i \|^2 + k \rho_i^2 \xi_i^2 \| u_i - u'_i \|^2 \\
\leq \sigma_i^2 \| x_i - x'_i \|^2 - 2 \rho_i \delta_i \| x_i - x'_i \|^2 + k \rho_i^2 \xi_i^2 (\gamma_i(B_i(x_i), B_i(x'_i))^2 \\
\leq \sigma_i^2 \| x_i - x'_i \|^2 - 2 \rho_i \delta_i \| x_i - x'_i \|^2 + k \rho_i^2 \xi_i^2 \| x_i - x'_i \|^2 \\
= (\sigma_i^2 - 2 \rho_i \delta_i + k \rho_i^2 \xi_i^2 \| L_{B_i} \| \| x_i - x'_i \|^2. 
\]
Hence we have
\[
\| (A_i(g_i(x_i))) - A_i(g_i(x'_i)) \| - \rho_i \| N_i(u_i, v_i) - N_i(u'_i, v'_i) \| \\
\| x_i - x'_i \| \leq \frac{\sqrt{\sigma_i^2 - 2 \rho_i \delta_i + k \rho_i^2 \xi_i^2 \| L_{B_i} \| \| x_i - x'_i \|}}{\| x_i - x'_i \|} \leq \frac{1}{\lambda_i} \| x_i - x'_i \|, 
\tag{3.5}
\]
where \( \lambda_i = \frac{1}{\sqrt{\sigma_i^2 - 2 \rho_i \delta_i + k \rho_i^2 \xi_i^2 \| L_{B_i} \|}}. \)

Now, since \( N_i(\cdot, \cdot) \) is \( \gamma_i \)-Lipschitz continuous w.r.t second argument and \( C_i \) is \( L_{G_i, \gamma_i} \)-Lipschitz continuous and \( G_i \) is \( L_{G_i, \gamma_i} \)-Lipschitz continuous, we have
\[
\| N_i(u'_i, v_i) - N_i(u'_i, v'_i) \| \leq \gamma_i \| v_i - v'_i \| \leq \gamma_i \| G_i(C_i(x_i), C_i(x'_i)) \| \leq \gamma_i \| L_{C_i} \| \| x_i - x'_i \|. 
\tag{3.6}
\]
and
\[
\| z_i - z'_i \| \leq \| G_i(G_i(x_i), G_i(x'_i)) \| \leq \| L_{G_i} \| \| x_i - x'_i \|. 
\tag{3.7}
\]
Combining (3.4), (3.5) and (3.6), we get
\[
\| J^M_{\rho_i A_i}(A_i(g_i(x_i))) - \rho_i N_i(u_i, v_i) + \rho_i \theta_i - J^M_{\rho_i A_i}(A_i(g_i(x'_i))) - \rho_i N_i(u'_i, v'_i) + \rho_i \theta_i \| \\
\leq \frac{1}{(r_i - \rho_i m_i)} \left( \frac{1}{\lambda_i} + \rho_i \gamma_i \| L_{C_i} \| \right) \| x_i - x'_i \| \\
\leq \frac{1}{\mu_i (r_i - \rho_i m_i)} \| x_i - x'_i \|. 
\tag{3.8}
\]
where \( \mu_i = \left( \frac{1}{L_i} + \rho_i \gamma_i L_{G_i} \right)^{-1} \).

Using (3.3), (3.7) and (3.8) in (3.2), we get

\[
|F_i(x_i) - F_i(x'_i)| \leq \sqrt{1 - 2q_i + k\beta_i^2} \|x_i - x'_i\| + \frac{1}{\mu_i(r_i - \rho_i m_i)} \|x_i - x'_i\|
\]

\[
\leq \left( \sqrt{1 - 2q_i + k\beta_i^2} + \frac{1}{\mu_i(r_i - \rho_i m_i)} + L_{G_i} \right) \|x_i - x'_i\|.
\]

Since \( r_i - \rho_i m_i > 0 \), \( 1 - 2q_i + k\beta_i^2 > 0 \) and \( 0 < \sqrt{1 - 2q_i + k\beta_i^2} + \frac{1}{\mu_i(r_i - \rho_i m_i)} + L_{G_i} < 1 \), the map \( F_i : X \to X \) (defined by (3.1)) is a contraction and thus it has a fixed point say \( x_i \in X \). Hence we have \( g_i(x_i) = f_{\rho_i, A_i}^{M, C}(A_i(g_i(x_i))) - \rho_i N_i(u_i, v_i) + \rho_i \theta_i) \). Hence, it follows from Lemma 3.1 that SNVIP (2.3) has a solution.

When \( X = L^p(R) \), \( 2 \leq p < \infty \), we have the following corollary:

**Corollary 3.3.** Let for each \( i = 1, 2, 3 \), \( A_i : L^p \to L^p \) be an \( r_i \)-strongly monotone map and \( M_i : L^p \to 2^{L^p} \) be \( A_i \)-monotone set-valued map. Suppose that \( g_i : L^p \to L^p \) is a map such that \( D(M_i) \cap R(g_i) \neq \emptyset \) and \( g_i \) is \( \beta_i \)-Lipschitz continuous and \( q_i \)-strongly monotone. Suppose that \( N_i : L^p \times L^p \to L^p \) is \( \xi_i \)-Lipschitz continuous in the first argument and \( \gamma_i \)-Lipschitz continuous in the second argument and \( \delta_i \)-strongly monotone w.r.t \( A_i(g_i) \) in the first argument and that \( A_i(g_i) \) be \( \alpha_i \)-Lipschitz continuous. Let \( B_i : L^p \to C(L^p) \) be \( L_{B_i} - \mathcal{G}(L_i) \)-Lipschitz continuous, \( C_i : L^p \to C(L^p) \) be \( L_{C_i} - \mathcal{G}(L_i) \)-Lipschitz continuous and \( G_i : L^p \to C(L^p) \) be \( L_{G_i} - \mathcal{G}(L_i) \)-Lipschitz continuous. In addition, suppose Condition A of Theorem 3.2 holds and if \( r_i - \rho_i m_i > 0 \), \( 1 - 2q_i + (p - 1)\beta_i^2 > 0 \) and \( 0 < \sqrt{1 - 2q_i + (p - 1)\beta_i^2} + \frac{1}{\mu_i(r_i - \rho_i m_i)} + L_{G_i} < 1 \), where \( \rho_i \) is a positive real constant and \( (p - 1) \) is the constant of smoothness of the function space \( L^p \), then SNVIP (2.3) has a solution.

When the set-valued map \( M_i : X \times X \to 2^X \) is \( H_i \)-monotone, then we have the following corollary:

**Corollary 3.4.** Let \( X \) be a real 2-uniformly smooth Banach space. Suppose for each \( i = 1, 2, 3 \), \( H_i : X \to X \) be \( r_i \)-strongly monotone map and \( M_i : X \to 2^X \) be \( H_i \)-monotone set-valued map. Let \( g_i : X \to X \) be a map such that \( R(g_i) \cap D(M_i, x_i) \neq \emptyset \) and \( g_i \) be \( \beta_i \)-Lipschitz continuous and \( q_i \)-strongly monotone. Suppose that \( N_i : X \times X \to X \) is \( \xi_i \)-Lipschitz continuous in the first argument, \( \gamma_i \)-Lipschitz continuous in the second argument and \( \delta_i \)-strongly monotone w.r.t \( H_i(g_i) \) in the first argument and that \( H_i(g_i) \) be \( \alpha_i \)-Lipschitz continuous. Let \( B_i : X \to C(X) \) be \( L_{B_i} - \mathcal{G}(L_i) \)-Lipschitz continuous, \( C_i : X \to C(X) \) be \( L_{C_i} - \mathcal{G}(L_i) \)-Lipschitz continuous and \( G_i : X \to C(X) \) be \( L_{G_i} - \mathcal{G}(L_i) \)-Lipschitz continuous. In addition, suppose Condition A of Theorem 3.2 holds and if \( 1 - 2q_i + (p - 1)\beta_i^2 > 0 \) and \( 0 < \sqrt{1 - 2q_i + (p - 1)\beta_i^2} + \frac{1}{\mu_i(r_i - \rho_i m_i)} + L_{G_i} < 1 \), where \( k \) is the constant of smoothness of the Banach space \( X \), then SNVIP (2.3) has a solution.

**Proof.** The proof is similar to the proof of Theorem 3.2 but here we have to use Lemma 2.23 instead of Lemma 2.24.

**4 Algorithm and convergence analysis**

Based on Lemmas 3.1 and 2.21, we give an iterative method for finding an approximate solution of SNVIP (2.3).

**Algorithm 4.1.** For each \( i = 1, 2, 3 \), given \( (x^0_i, u^0_i, v^0_i, z^0_i) \) where \( x^0_i \in X, u^0_i \in B_i(x^0_i), v^0_i \in C_i(x^0_i) \) and \( z^0_i \in G_i(x^0_i) \) such that \( B_i, C_i, G_i : X \to C(X) \), compute the sequences \( \{x^s_i\}, \{u^s_i\}, \{v^s_i\}, \{z^s_i\} \) defined by

\[
\begin{align*}
  x^{s+1}_i &= (1 - a^n)x^n_i + a^n(x^n_i - g_i(x^n_i)) + M^{\mathcal{G}(\cdot,x^n_i)}_{\rho_i}(A_i(g_i(x^n_i)) - \rho_i N_i(u^n_i, v^n_i) + \rho_i \theta_i)), \\
  u^{s+1}_i &\in B_i(x^{s+1}_i) : \|u^{s+1}_i - u^n_i\| \leq \mathcal{H}(B_i(x^{s+1}_i), B_i(x^n_i)), \\
  v^{s+1}_i &\in C_i(x^{s+1}_i) : \|v^{s+1}_i - v^n_i\| \leq \mathcal{H}(C_i(x^{s+1}_i), C_i(x^n_i)), \\
  z^{s+1}_i &\in G_i(x^{s+1}_i) : \|z^{s+1}_i - z^n_i\| \leq \mathcal{H}(G_i(x^{s+1}_i), G_i(x^n_i)),
\end{align*}
\]
where \( M^n_i : X \times X \rightarrow 2^X \) are \( A_i \)-monotone set-valued mappings for \( n = 0, 1, 2, \ldots \), \( f_{\rho_i,A_i}^{M^n_i(z_i)} = (A_i + \rho_i M^n_i(\cdot, z_i))^{-1} \) and \( \{a^n\} \) be a sequence of real numbers such that \( a^n \in [0, 1] \) and \( \sum_{n=0}^{\infty} a^n = \infty \).

Now, we give the convergence analysis of the sequences generated by the iterative Algorithm 4.1.

**Theorem 4.2.** Let \( X \) be a real 2-uniformly smooth Banach space. Suppose for each \( i = 1, 2, 3 \), \( A_i : X \rightarrow X \) be an \( r_i \)-strongly monotone map and \( \delta_i \)-Lipschitz continuous. Let \( M^n_i : X \times X \rightarrow 2^X \) be a sequence of \( A_i \)-monotone set-valued maps such that \( M^n_i(\cdot, z_i) \xrightarrow{\Delta} M_i(\cdot, z_i) \) as \( n \rightarrow \infty \). Suppose that \( g_i : X \rightarrow X \) is \( q_i \)-strongly monotone and \( \beta_i \)-Lipschitz continuous and \( N_i : X \times X \rightarrow X \) is \( \xi_i \)-Lipschitz continuous in the first argument, \( \gamma_i \)-Lipschitz continuous in the second argument and \( \delta_i \)-strongly monotone w.r.t \( A_i(g_i) \) in the first argument and that \( A_i(g_i) \) be \( \sigma_i \)-Lipschitz continuous. Let \( B_i : X \rightarrow C(X) \) be \( L_{B_i}, \delta_i \)-Lipschitz continuous, \( C_i : X \rightarrow C(X) \) is \( L_{C_i}, \gamma_i \)-Lipschitz continuous and \( G_i : X \rightarrow C(X) \) be \( L_{G_i}, \delta_i \)-Lipschitz continuous in addition, if \( r_i - \rho_i m_i > 0, 1 - 2q_i + k\beta_i > 0 \) and \( 0 < \sqrt{1 - 2q_i + k\beta_i^2} + \frac{1}{\mu_i(r_i - \rho_i m_i)} < 1 \), where \( \rho_i \) is a positive real constant and \( k \) is the constant of smoothness of the Banach space \( X \). Then for each \( i = 1, 2, 3 \), the sequences \( \{x^n_i\}, \{u^n_i\}, \{v^n_i\}, \{z^n_i\} \) generated by Iterative Algorithm 4.1 converges strongly to \( x_i, u_i, v_i, z_i \) where \( (x_1, x_2, x_3, u_1, u_2, u_3, v_1, v_2, v_3, z_1, z_2, z_3) \) is a solution of SNVIP (2.3).

**Proof.** Let \( x_i \) be a solution of SNVIP (2.3). Then by Algorithm 4.1, we have

\[
\begin{align*}
||x_{i+1}^n - x_i|| & = ||(1 - a^n)x^n_i + a^n(x^n_i - g_i(x^n_i)) + f_{\rho_i,A_i}^{M^n_i(z^n_i)}(A_i(x^n_i)) - \rho_i N_i(u^n_i, v^n_i) + \rho_i \theta_i)|| \\
& = (1 - a^n)||x^n_i - g_i(x^n_i)|| + a^n||f_{\rho_i,A_i}^{M^n_i(z^n_i)}(A_i(x^n_i)) - \rho_i N_i(u^n_i, v^n_i) + \rho_i \theta_i)|| \\
& \leq (1 - a^n)||x^n_i - g_i(x^n_i)|| + a^n||f_{\rho_i,A_i}^{M^n_i(z^n_i)}(A_i(x^n_i)) - \rho_i N_i(u^n_i, v^n_i) + \rho_i \theta_i)|| \\
& \leq (1 - a^n)||x^n_i - g_i(x^n_i)|| + a^n||f_{\rho_i,A_i}^{M^n_i(z^n_i)}(A_i(x^n_i)) - \rho_i N_i(u^n_i, v^n_i) + \rho_i \theta_i)||. \\
\end{align*}
\]

Using Lemma 2.23, \( \sigma_i \)-Lipschitz continuity of \( A_i(g_i) \), \( \delta_i \)-Strongly monotonicity of \( N_i(\cdot, \cdot) \) w.r.t \( A_i(g_i) \) in the first argument, \( \xi_i \)-Lipschitz continuity of \( N_i(\cdot, \cdot) \) in the first argument and \( \gamma_i \)-Lipschitz continuity of \( N_i(\cdot, \cdot) \) in the second argument, we have

\[
\begin{align*}
||f_{\rho_i,A_i}^{M^n_i(z^n_i)}(A_i(x^n_i)) - \rho_i N_i(u^n_i, v^n_i) + \rho_i \theta_i)|| & = \frac{1}{\mu_i(r_i - \rho_i m_i)}||(A_i(x^n_i)) - \rho_i N_i(u^n_i, v^n_i) + \rho_i \theta_i)|| \\
& \leq \frac{1}{\mu_i(r_i - \rho_i m_i)}||x^n_i - x_i||. \\
\end{align*}
\]

Combining (4.1) and (4.2), we get

\[
\begin{align*}
||x_{i+1}^n - x_i|| & \leq (1 - a^n)||x^n_i - g_i(x^n_i)|| + a^n||f_{\rho_i,A_i}^{M^n_i(z^n_i)}(A_i(x^n_i)) - \rho_i N_i(u^n_i, v^n_i) + \rho_i \theta_i)|| \\
& \quad + a^n||f_{\rho_i,A_i}^{M^n_i(z^n_i)}(A_i(x^n_i)) - \rho_i N_i(u^n_i, v^n_i) + \rho_i \theta_i)||. \\
\end{align*}
\]

where

\[
f_{i}^n = a^n||f_{\rho_i,A_i}^{M^n_i(z^n_i)}(A_i(x^n_i)) - \rho_i N_i(u^n_i, v^n_i) + \rho_i \theta_i)||. 
\]

with \( f_{i}^n \rightarrow 0 \) as \( n \rightarrow \infty \). Again, since \( g_i \) is \( q_i \)-strongly monotone and \( \beta_i \)-Lipschitz continuous, we have

\[
||x^n_i - x_i - (g_i(x^n_i) - g_i(x_i))|| \leq \sqrt{1 - 2q_i + k\beta_i^2} ||x^n_i - x_i||. 
\]
By using (4.3) and (4.4), we get
\[
||x_{i+1}^n - x_i|| \leq (1 - \alpha^n)||x_i^n - x_i|| + a^n \sqrt{1 - 2q_i + k\beta_i^2} ||x_i^n - x_i|| + \frac{\alpha^n}{\mu_i(r_i - \rho_i m_i)} ||x_i^n - x_i|| + a^n f_i^n
\]
\[
= \left(1 - \alpha^n \left(1 - \sqrt{1 - 2q_i + k\beta_i^2} - \frac{1}{\mu_i(r_i - \rho_i m_i)} \right)\right) ||x_i^n - x_i|| + a^n f_i^n
\]
(4.5)
where $h_i := \sqrt{1 - 2q_i + k\beta_i^2} + \frac{1}{\mu_i(r_i - \rho_i m_i)}$ with $h_i < 1$ by assumption. Hence
\[
||x_{i+1}^n - x_i|| \leq (1 - \alpha^n (1 - h_i))||x_i^n - x_i|| + a^n (1 - h_i) \frac{f_i^n}{(1 - h_i)}.
\]
(4.6)
If $m_i^n = ||x_i^n - x_i||$, $n_i^n = \frac{f_i^n}{(1 - h_i)}$ and $t_i^n = \alpha^n (1 - h_i)$, then we have
\[
m_i^{n+1} \leq (1 - t_i^n)m_i^n + t_i^n n_i^n.
\]
Using Lemma 2.18, we have $m_i^n \to 0$ as $n \to \infty$ and thus $x_i^n \to x_i$ as $n \to \infty$. Hence $\{x_i^n\}$ converges strongly to a solution of SNVIP (2.3).

Since $B_i$ is $L_{B_i}$-\(\beta\)-Lipschitz continuous, it follows from Algorithm 4.1 that
\[
||u_i^n - u_i|| \leq \beta((B_i(x_i^n), B_i(x_i)))
\]
\[
\leq L_{B_i}||x_i^n - x_i||.
\]
This implies that $u_i^n \to u_i$ as $n \to \infty$. Further, we claim that $u_i \in B_i(x_i)$
\[
d(u_i, B_i(x_i)) \leq ||u_i - u_i^n|| + d(u_i^n, B_i(x_i))
\]
\[
\leq ||u_i - u_i^n|| + \beta((B_i(x_i^n), B_i(x_i)))
\]
\[
\leq ||u_i - u_i^n|| + L_{B_i}||x_i^n - x_i||
\]
\[
\to 0 \text{ as } n \to \infty.
\]
Since, $B_i(x_i)$ is compact, we have $u_i \in B_i(x_i)$.

Similarly, we can prove that $v_i \in C_i(x_i)$ and $z_i \in G_i(x_i)$. Thus the approximate solution $(x_i^n, u_i^n, v_i^n, z_i^n)$ generated by iterative Algorithm 4.1 converges strongly to $(x_i, u_i, v_i, z_i)$ giving a solution of SNVIP (2.3). This completes the proof. 

Similar results can be obtained for $H_i$-monotone operators. We state the result for $H_i$-monotone operators in the form of a corollary.

**Corollary 4.3.** Let $X$ be a real 2-uniformly smooth Banach space. Suppose for each $i = 1, 2, 3$, $H_i : X \to X$ be $r_i$-strongly monotone map and $s_i$-Lipschitz continuous. Let $M_i^n : X \to 2^X$ be a sequence of $H_i$-monotone set-valued maps such that $M_i^n \rightharpoonup M_i$ as $n \to \infty$. Suppose that $g_i : X \to X$ is $q_i$-strongly monotone and $\beta_i$-Lipschitz continuous and $N_i : X \times X \to X$ is $\xi_i$-Lipschitz continuous in the first argument, $\gamma_i$-Lipschitz continuous in the second argument and $\delta_i$-strongly monotone w.r.t $H_i(g_i)$ in the first argument and that $H_i(g_i)$ be $\sigma_i$-Lipschitz continuous. Let $B_i : X \to C(X)$ be $L_{B_i}$-\(\beta\)-Lipschitz continuous and $C_i : X \to C(X)$ be $L_{C_i}$-\(\beta\)-Lipschitz continuous. In addition if $1 - 2q_i + k\beta_i^2 > 0$ and $0 < \sqrt{1 - 2q_i + k\beta_i^2} + \frac{1}{\mu_i r_i} < 1$, where $k$ is the constant of smoothness of the Banach space $X$. Then for each $i = 1, 2, 3$, the sequences $\{x_i^n\}$, $\{u_i^n\}$, $\{v_i^n\}$, $\{z_i^n\}$ generated by Iterative Algorithm 4.1 converge strongly to $x_i, u_i, v_i, z_i$ where $(x_1, x_2, x_3, u_1, u_2, u_3, v_1, v_2, v_3, z_1, z_2, z_3)$ is a solution of SNVIP (2.3).
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