SMALLEST REGULAR GRAPHS OF GIVEN DEGREE AND DIAMETER

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Abstract

In this note we present a sharp lower bound on the number of vertices in a regular graph of given degree and diameter.

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1. Introduction

The degree/diameter problem consists in determination of the largest order \( N(d, k) \) of a graph with (maximum) degree \( d \) and diameter \( k \). An upper bound for \( N(d, k) \) is the Moore bound

\[
M(d, k) = 1 + d + d(d - 1) + \cdots + d(d - 1)^{k-1}
\]

and graphs achieving this bound are called Moore graphs. As shown in [1, 3, 5], Moore graphs exist only when \( d = 2 \) or \( k = 1 \) or when \( k = 2 \) and the degree is either 3 or 7 or possibly 57. For all other pairs \((d, k)\) we have \( N(d, k) \leq M(d, k) - 2 \), see [2, 4]. Recently, there are plenty of papers dealing with the degree/diameter problem, some of them constructing “large” graphs of given degree and diameter, which increases the lower bound for \( N(d, k) \) for special pairs \((d, k)\), other decreasing \( N(d, k) \) for special classes of graphs. For a nice survey see [7].

In this note we consider the inverse of degree/diameter problem. Since usually the degree/diameter problem is formulated for regular graphs (although some authors require only that \( d \) is the maximum degree), we ask what is the minimum

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order $n(d, k)$ of a regular graph of degree $d$ and diameter $k$. In this note we answer this question completely.

We start with some notation. Let $G$ be a graph, $G = (V(G), E(G))$. For two of its vertices, say $x$ and $y$, by $\text{dist}_G(x, y)$ we denote their distance in $G$. By $N_i(x)$ we denote the set of vertices that are at distance $i$ from $x$. As usual, $N_1(x)$ is often abbreviated to $N(x)$. The longest distance in $G$ is the diameter $\text{diam}(G)$.

The complete graph on $n$ vertices is denoted by $K_n$ and the discrete graph on $n$ vertices (the complement of $K_n$) is denoted by $D_n$. If $G$ is a graph, then by $G[−1]$ (and $G[−2]$) we denote a graph obtained from $G$ by removing all the edges of one 1-factor (one 2-factor).

If $G$ and $H$ are graphs, then $G + H$ denotes the join of $G$ and $H$, that is, a graph obtained from the disjoint union of $G$ and $H$ by adding all edges $xy$, where $x \in V(G)$ and $y \in V(H)$. The sequential join of graphs $G_1, G_2, \ldots, G_r$ is denoted by $G_1 + G_2 + \cdots + G_r$ and is defined by

$$G_1 + G_2 + \cdots + G_r = (G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{r-1} + G_r).$$

Thus, one can obtain $G_1 + G_2 + \cdots + G_r$ from the disjoint union $G_1 \cup G_2 \cup \cdots \cup G_r$ by adding all edges $xy$ where $x \in V(G_i)$ and $y \in V(G_{i+1})$ for $i = 1, 2, \ldots, r-1$.

To simplify the expressions, instead of

$$\underbrace{\cdots + G + G + \cdots + G + \cdots}_{\text{k times}}$$

we write

$$\cdots + (G)_k + \cdots.$$

Finally, denote by $G \div H$ a graph obtained from the disjoint union of $G$ and $H$ by adding all edges of one 1-factor, every edge of which joins a vertex of $G$ with a vertex of $H$. Obviously, $G \div H$ is defined only if $|V(G)| = |V(H)|$.

Analogously as in the case of join, by $G_1 \div G_2 \div \cdots \div G_r$ we denote the graph $(G_1 \div G_2) \cup (G_2 \div G_3) \cup \cdots \cup (G_{r-1} \div G_r)$. We can form also more complicated expressions using both $+$ and $\div$. In such a way, $K_1 + D_2 \div D_2 \div K_2$ is a cycle of length 7; see Figure 1.

![Figure 1. The graph $K_1 + D_2 \div D_2 \div K_2$.](image-url)
2. Results

For small diameters we have the following statement.

**Proposition 1.** Let \( d \geq 2 \). We have

(i) \( n(d,1) = d + 1 \);

(ii) if \( d \) is even, then \( n(d,2) = d + 2 \);

(iii) if \( d \) is odd, then \( n(d,2) = d + 3 \);

(iv) \( n(d,3) = 2d + 2 \).

**Proof.** The case \( k = 1 \) is obvious since \( K_{d+1} \) is the unique graph of diameter 1 and degree \( d \).

Let \( k = 2 \). Let \( G \) be a \( d \)-regular graph of diameter 2, and let \( x, y \in V(G) \) such that \( \text{dist}_G(x,y) = 2 \). Then \( \{x\} \cup N(x) = N_0(x) \cup N_1(x) \), which gives \( |N_0(x)| + |N_1(x)| = d + 1 \). Since \( y \in N_2(x) \), we have \( |V(G)| = |N_0(x)| + |N_1(x)| + |N_2(x)| \geq d + 2 \), which gives \( n(d,2) \geq d + 2 \). However, if \( d \) is odd then \( |V(G)| \) cannot be odd and so \( n(d,2) \geq d + 3 \) in this case. If \( d \) is even then \( K_{d+2}^{(-1)} \) is a \( d \)-regular graph of diameter 2 on \( d + 2 \) vertices, which shows \( n(d,2) \leq d + 2 \); while if \( d \) is odd then \( K_{d+3}^{(-2)} \) is a \( d \)-regular graph of diameter 2 on \( d + 3 \) vertices, which shows \( n(d,2) \leq d + 3 \).

Finally, let \( k = 3 \). Analogously as above, let \( G \) be a \( d \)-regular graph of diameter 3, and let \( x, y \in V(G) \) such that \( \text{dist}_G(x,y) = 3 \). Then \( \{x\} \cup N(x) = N_0(x) \cup N_1(x) \), which gives \( |N_0(x)| + |N_1(x)| = d + 1 \), and \( \{y\} \cup N(y) \subseteq N_2(x) \cup N_3(x) \), which gives \( |N_2(x)| + |N_3(x)| \geq d + 1 \). Thus, \( |V(G)| = |N_0(x)| + |N_1(x)| + |N_2(x)| + |N_3(x)| \geq 2d + 2 \), and so \( n(d,3) \geq 2d + 2 \). On the other hand, denote by \( K_n \) a complete bipartite graph on \( 2n \) vertices in which the two partite sets have \( n \) vertices each. Then \( K_{d+1}^{(-1)} \) is a \( d \)-regular graph of diameter 3 on \( 2d + 2 \) vertices, which shows \( n(d,3) \leq 2d + 2 \).

Now we turn our attention to larger diameters. Since there are only two 2-regular graphs of diameter \( k \), namely the cycle on \( 2k \) vertices and the cycle on \( 2k + 1 \) vertices, we have the following trivial observation.

**Proposition 2.** If \( k \geq 4 \), then \( n(2,k) = 2k \).

For larger degrees we have a slightly different bound.

**Theorem 3.** Let \( k = 3j + t \), where \( k \geq 4 \) and \( 0 \leq t \leq 2 \), and let \( d \geq 3 \). Then \( n(d,k) = (d + 1)(j + 1) + t + \delta \), where \( \delta = 1 \) if either \( d \) is odd and \( t = 1 \) or \( d \) is even and \( t = 2 \). Otherwise \( \delta = 0 \).
**Proof.** First we prove a lower bound for \( n(d, k) \). Let \( G \) be a regular graph of degree \( d \) and diameter \( k \) and let \( x, y \in V(G) \) such that \( \text{dist}_G(x, y) = k \). Denote \( n_i = |N_i(x)| \). Since \( x \in N_0(x) \), we have \( \{x\} \cup N(x) \subseteq N_0(x) \cup N_1(x) \). Thus, \( n_0 + n_1 \geq d + 1 \). Analogously \( n_{k-1} + n_k \geq d + 1 \) since \( y \in N_k(x) \). Further, for every \( i, 1 \leq i \leq j-1 \), we have \( n_{3i-1} + n_{3i} + n_{3i+1} \geq d + 1 \) since for \( z_i \in N_{3i}(x) \) it holds \( \{z_i\} \cup N(z_i) \subseteq N_{3i-1}(x) \cup N_{3i}(x) \cup N_{3i+1}(x) \). Finally, if \( t \geq 1 \) then \( n_{k-1-t} \geq 1 \) where \( 1 \leq t \leq t \). Summing up all these inequalities we get

\[
|V(G)| = \sum_{i=0}^{k} n_i \geq (d+1)(j+1) + t.
\]

If \( t = 2 \) then we use \( n_{k-3} \geq 1 \) and \( n_{k-2} \geq 1 \). But if \( d \) is even then \( G \) cannot have a bridge, and so \( n_{k-3} + n_{k-2} \geq 3 \). Thus, we get \( |V(G)| = \sum_{i=0}^{k} n_i \geq (d+1)(j+1) + t + 1 \) in this case.

Similarly, if \( t = 1 \) and \( d \) is odd then \( (d+1)(j+1) + t \) is an odd number. But a regular graph of odd degree cannot have an odd number of vertices, and so \( |V(G)| = \sum_{i=0}^{k} n_i \geq (d+1)(j+1) + t + 1 \) also in this case.

To prove the upper bound we construct extremal graphs, that is, regular graphs of degree \( d \) and diameter \( k \) on \( n(d, k) \) vertices. First we define an extremal graph \( G \) for odd \( d \). The case \( k = 4 \) is treated separately. If \( d = 3 \) then one extremal graph \( G \) is on Figure 2. For \( d \geq 5 \) we set \( G = K_2 + K_{d-1}^{(-2)} + D_2 \sqcup D_2 + K_{d-1} \).

![Figure 2. An extremal graph for \( d = 3 \) and \( k = 4 \).](image)

Recall that \( k = 3j + t \). To cover the remaining diameters, that is, 5, 6, 7, \ldots, in the next we assume \( j \geq 1 \) if \( t = 2 \), and \( j \geq 2 \) if \( t = 0 \) or \( t = 1 \):

- \( G = K_2 + K_{d-1}^{(-1)} + (K_1 + K_1 + K_{d-1})_{j-1} + K_1 + K_1 + K_{d-1}^{(-1)} + K_2 \), if \( t = 2 \);
- \( G = K_2 + K_{d-1}^{(-1)} + (K_1 + K_1 + K_{d-1})_{j-2} + K_1 + K_1 + K_{d-1}^{(-1)} + K_2 \), if \( t = 0 \);
- \( G = K_2 + K_{d-1}^{(-1)} + (K_1 + K_1 + K_{d-1})_{j-2} + K_1 + K_1 + K_{d-1}^{(-1)} + D_2 \sqcup D_2 + K_{d-1} \), if \( t = 1 \).

Now we define an extremal graph \( G \) for even \( d \). To cover all possible diameters, that is, 4, 5, 6, \ldots, in the next we assume \( j \geq 1 \) if \( t = 1 \) or \( t = 2 \), and \( j \geq 2 \) if \( t = 0 \):

- \( G = K_3 + K_{d-2}^{(-1)} + (K_1 + K_2 + K_{d-2})_{j-1} + K_1 + D_2 + K_{d-1} \), if \( t = 1 \);
- \( G = K_3 + K_{d-2}^{(-1)} + (K_1 + K_2 + K_{d-2})_{j-1} + K_1 + K_2 + K_{d-2}^{(-2)} + K_3 \), if \( t = 2 \);
- \( G = K_3 + K_{d-2}^{(-1)} + (K_1 + K_2 + K_{d-2})_{j-1} + K_1 + K_2 + K_{d-2}^{(-2)} + K_3 + K_{d-2}^{(-3)} + K_4 \), if \( t = 3 \);
- \( G = K_3 + K_{d-2}^{(-1)} + (K_1 + K_2 + K_{d-2})_{j-1} + K_1 + K_2 + K_{d-2}^{(-2)} + K_3 + K_{d-2}^{(-3)} + K_4 + K_{d-3}^{(-4)} + K_5 \), if \( t = 4 \);
- \( G = K_3 + K_{d-2}^{(-1)} + (K_1 + K_2 + K_{d-2})_{j-1} + K_1 + K_2 + K_{d-2}^{(-2)} + K_3 + K_{d-2}^{(-3)} + K_4 + K_{d-3}^{(-4)} + K_5 + K_{d-4}^{(-5)} + K_6 \), if \( t = 5 \).

... (continued)
\[ G = K_3 + K_{d-2}^{(-1)} + (K_1 + K_2 + K_{d-2})_{j-2} + K_1 + K_2 + K_{d-2} \div K_{d-2}^{(-1)} + K_3, \text{ if } t = 0. \]

Observe that in all these graphs, whenever we removed a 1-factor out of \( K_q \),
then the number of vertices \( q \) was even. Obviously, in each case \( G \) has diameter \( k \) and it is a matter of routine to check that \( G \) is a regular graph of degree \( d \).
(For example, a vertex in the last copy of \( K_{d-2}^{(-1)} \) in the last graph is joined to
1 vertex of \( K_{d-2} \), \( d-4 \) vertices of \( K_{d-2}^{(-1)} \) and to 3 vertices of \( K_3 \), so its degree
is \( 1 + d - 4 + 3 = d \).) Also, in each of these cases the number of vertices of \( G \)
tends the bound of the theorem. To verify this statement it suffices to check
the number of vertices for the smallest admissible values of \( j \) since in each case
in the brackets we have exactly \( d + 1 \) vertices.

By Proposition 2, if \( d = 2 \) then \( n(d, k) = dk \). However, for higher degrees we get
\[ n(d, k) \sim \frac{1}{3}dk. \] Denote by \( n_{vt}(d, k) \) the minimum number of vertices in a vertex-
transitive \( d \)-regular graph with diameter \( k \). As shown in [6], for \( k \geq 4 \) and “large”
\( d \) we have \( n_{vt}(d, k) \sim \frac{1}{2}dk \), and so \( n_{vt}(d, k) = 2n(d, k) \) in this case. On the other
hand, since the extremal graphs constructed in the proof of Proposition 1 are vertex-transitive, we have \( n_{vt}(d, k) = n(d, k) \) when \( k \leq 3 \).

References


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