SMALLEST REGULAR GRAPHS OF GIVEN DEGREE AND DIAMETER

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Abstract

In this note we present a sharp lower bound on the number of vertices in a regular graph of given degree and diameter.

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1. Introduction

The degree/diameter problem consists in determination of the largest order $N(d, k)$ of a graph with (maximum) degree $d$ and diameter $k$. An upper bound for $N(d, k)$ is the Moore bound $M(d, k) = 1 + d + d(d - 1) + \cdots + d(d - 1)^{k-1}$ and graphs achieving this bound are called Moore graphs. As shown in [1, 3, 5], Moore graphs exist only when $d = 2$ or $k = 1$ or when $k = 2$ and the degree is either 3 or 7 or possibly 57. For all other pairs $(d, k)$ we have $N(d, k) \leq M(d, k) - 2$, see [2, 4]. Recently, there are plenty of papers dealing with the degree/diameter problem, some of them constructing “large” graphs of given degree and diameter, which increases the lower bound for $N(d, k)$ for special pairs $(d, k)$, other decreasing $N(d, k)$ for special classes of graphs. For a nice survey see [7].

In this note we consider the inverse of degree/diameter problem. Since usually the degree/diameter problem is formulated for regular graphs (although some authors require only that $d$ is the maximum degree), we ask what is the minimum
order \( n(d, k) \) of a regular graph of degree \( d \) and diameter \( k \). In this note we answer this question completely.

We start with some notation. Let \( G \) be a graph, \( G = (V(G), E(G)) \). For two of its vertices, say \( x \) and \( y \), by \( \text{dist}_G(x, y) \) we denote their distance in \( G \). By \( N_i(x) \) we denote the set of vertices that are at distance \( i \) from \( x \). As usual, \( N_1(x) \) is often abbreviated to \( N(x) \). The longest distance in \( G \) is the diameter \( \text{diam}(G) \).

The complete graph on \( n \) vertices is denoted by \( K_n \) and the discrete graph on \( n \) vertices (the complement of \( K_n \)) is denoted by \( D_n \). If \( G \) is a graph, then by \( G^{(−1)} \) (and \( G^{(−2)} \)) we denote a graph obtained from \( G \) by removing all the edges of one 1-factor (one 2-factor).

If \( G \) and \( H \) are graphs, then \( G + H \) denotes the join of \( G \) and \( H \), that is, a graph obtained from the disjoint union of \( G \) and \( H \) by adding all edges \( xy \), where \( x \in V(G) \) and \( y \in V(H) \). The sequential join of graphs \( G_1, G_2, \ldots, G_r \) is denoted by \( G_1 + G_2 + \cdots + G_r \) and is defined by

\[
G_1 + G_2 + \cdots + G_r = (G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{r−1} + G_r).
\]

Thus, one can obtain \( G_1 + G_2 + \cdots + G_r \) from the disjoint union \( G_1 \cup G_2 \cup \cdots \cup G_r \) by adding all edges \( xy \) where \( x \in V(G_i) \) and \( y \in V(G_{i+1}) \) for \( i = 1, 2, \ldots, r−1 \). To simplify the expressions, instead of

\[
\underbrace{\cdots + G + G + \cdots + G}_{\text{k times}} + \cdots
\]

we write

\[
\cdots + (G)_k + \cdots
\]

Finally, denote by \( G \div H \) a graph obtained from the disjoint union of \( G \) and \( H \) by adding all edges of one 1-factor, every edge of which joins a vertex of \( G \) with a vertex of \( H \). Obviously, \( G \div H \) is defined only if \( |V(G)| = |V(H)| \).

Analogously as in the case of join, by \( G_1 \div G_2 \div \cdots \div G_r \) we denote the graph \( (G_1 \div G_2) \cup (G_2 \div G_3) \cup \cdots \cup (G_{r−1} \div G_r) \). We can form also more complicated expressions using both \( + \) and \( \div \). In such a way, \( K_1 + D_2 \div D_2 \div K_2 \) is a cycle of length 7; see Figure 1.

![Figure 1. The graph K_1 + D_2 \div D_2 \div K_2.](image)
2. Results

For small diameters we have the following statement.

**Proposition 1.** Let $d \geq 2$. We have

(i) $n(d, 1) = d + 1$;

(ii) if $d$ is even, then $n(d, 2) = d + 2$;

(iii) if $d$ is odd, then $n(d, 2) = d + 3$;

(iv) $n(d, 3) = 2d + 2$.

**Proof.** The case $k = 1$ is obvious since $K_{d+1}$ is the unique graph of diameter 1 and degree $d$.

Let $k = 2$. Let $G$ be a $d$-regular graph of diameter 2, and let $x, y \in V(G)$ such that $\text{dist}_G(x, y) = 2$. Then $\{x\} \cup N(x) = N_0(x) \cup N_1(x)$, which gives $|N_0(x)| + |N_1(x)| = d + 1$. Since $y \in N_2(x)$, we have $|V(G)| = |N_0(x)| + |N_1(x)| + |N_2(x)| \geq d + 2$, which gives $n(d, 2) \geq d + 2$. However, if $d$ is odd then $|V(G)|$ cannot be odd and so $n(d, 2) \geq d + 3$ in this case. If $d$ is even then $K_{d+1}^{(-1)}$ is a $d$-regular graph of diameter 2 on $d + 2$ vertices, which shows $n(d, 2) \leq d + 2$; while if $d$ is odd then $K_{d+3}^{(-1)}$ is a $d$-regular graph of diameter 2 on $d + 3$ vertices, which shows $n(d, 2) \leq d + 3$.

Finally, let $k = 3$. Analogously as above, let $G$ be a $d$-regular graph of diameter 3, and let $x, y \in V(G)$ such that $\text{dist}_G(x, y) = 3$. Then $\{x\} \cup N(x) = N_0(x) \cup N_1(x)$, which gives $|N_0(x)| + |N_1(x)| = d + 1$, and $\{y\} \cup N(y) \subseteq N_2(x) \cup N_3(x)$, which gives $|N_2(x)| + |N_3(x)| \geq d + 1$. Thus, $|V(G)| = |N_0(x)| + |N_1(x)| + |N_2(x)| + |N_3(x)| \geq 2d + 2$, and so $n(d, 3) \geq 2d + 2$. On the other hand, denote by $K_{n,n}$ a complete bipartite graph on $2n$ vertices in which the two partite sets have $n$ vertices each. Then $K_{d+1, d+1}^{(-1)}$ is a $d$-regular graph of diameter 3 on $2d + 2$ vertices, which shows $n(d, 3) \leq 2d + 2$.

Now we turn our attention to larger diameters. Since there are only two 2-regular graphs of diameter $k$, namely the cycle on $2k$ vertices and the cycle on $2k + 1$ vertices, we have the following trivial observation.

**Proposition 2.** If $k \geq 4$, then $n(2, k) = 2k$.

For larger degrees we have a slightly different bound.

**Theorem 3.** Let $k = 3j + t$, where $k \geq 4$ and $0 \leq t \leq 2$, and let $d \geq 3$. Then $n(d, k) = (d + 1)(j + 1) + t + \delta$, where $\delta = 1$ if either $d$ is odd and $t = 1$ or $d$ is even and $t = 2$. Otherwise $\delta = 0$. 
Proof. First we prove a lower bound for $n(d,k)$. Let $G$ be a regular graph of degree $d$ and diameter $k$ and let $x,y \in V(G)$ such that $\text{dist}_G(x,y) = k$. Denote $n_i = |N_i(x)|$. Since $x \in N_0(x)$, we have $\{x\} \cup N(x) \subseteq N_0(x) \cup N_1(x)$. Thus, $n_0 + n_1 \geq d + 1$. Analogously $n_{k-1} + n_k \geq d + 1$ since $y \in N_k(x)$. Further, for every $i$, $1 \leq i \leq j-1$, we have $n_{3i-1} + n_{3i} + n_{3i+1} \geq d + 1$ since for $z_i \in N_{3i}(x)$ it holds $\{z_i\} \cup N(z_i) \subseteq N_{3i-1}(x) \cup N_{3i}(x) \cup N_{3i+1}(x)$. Finally, if $t \geq 1$ then $n_{k-1-t} \geq 1$ where $1 \leq t \leq t$. Summing up all these inequalities we get

$$|V(G)| = \sum_{i=0}^{k} n_i \geq (d + 1)(j + 1) + t.$$  

If $t = 2$ then we use $n_{k-3} \geq 1$ and $n_{k-2} \geq 1$. But if $d$ is even then $G$ cannot have a bridge, and so $n_{k-3} + n_{k-2} \geq 3$. Thus, we get $|V(G)| = \sum_{i=0}^{k} n_i \geq (d + 1)(j + 1) + t + 1$ in this case.

Similarly, if $t = 1$ and $d$ is odd then $(d + 1)(j + 1) + t$ is an odd number. But a regular graph of odd degree cannot have an odd number of vertices, and so $|V(G)| = \sum_{i=0}^{k} n_i \geq (d + 1)(j + 1) + t + 1$ also in this case.

To prove the upper bound we construct extremal graphs, that is, regular graphs of degree $d$ and diameter $k$ on $n(d,k)$ vertices. First we define an extremal graph $G$ for odd $d$. The case $k = 4$ is treated separately. If $d = 3$ then one extremal graph $G$ is on Figure 2. For $d \geq 5$ we set $G = K_2 + K_{d-1}^{(2)} + D_2 \cup D_2 + K_{d-1}$.

![Figure 2. An extremal graph for $d = 3$ and $k = 4$.](image)

Recall that $k = 3j + t$. To cover the remaining diameters, that is, 5, 6, 7, \ldots, in the next we assume $j \geq 1$ if $t = 2$, and $j \geq 2$ if $t = 0$ or $t = 1$:

- $G = K_2 + K_{d-1}^{(1)} + (K_1 + K_1 + K_{d-1})_{j-1} + K_1 + K_1 + K_{d-1}^{(1)} + K_2$, if $t = 2$;
- $G = K_2 + K_{d-1}^{(1)} + (K_1 + K_1 + K_{d-1})_{j-2} + K_1 + K_1 + K_{d-1} + K_{d-1}^{(1)} + K_2$, if $t = 0$;
- $G = K_2 + K_{d-1}^{(1)} + (K_1 + K_1 + K_{d-1})_{j-2} + K_1 + K_1 + K_{d-1} + D_2 \cup D_2 + K_{d-1}$, if $t = 1$.

Now we define an extremal graph $G$ for even $d$. To cover all possible diameters, that is, 4, 5, 6, \ldots, in the next we assume $j \geq 1$ if $t = 1$ or $t = 2$, and $j \geq 2$ if $t = 0$:

- $G = K_3 + K_{d-2}^{(1)} + (K_1 + K_2 + K_{d-2})_{j-1} + K_1 + D_2 + K_{d-1}$, if $t = 1$;
- $G = K_3 + K_{d-2}^{(1)} + (K_1 + K_2 + K_{d-2})_{j-1} + K_1 + K_2 + K_{d-2}^{(2)} + K_3$, if $t = 2$;
\[ G = K_3 + K_{d-2}^{(-1)} + (K_1 + K_2 + K_{d-2})j - 2 + K_1 + K_2 + K_{d-2} \div K_{d-2}^{(-1)} + K_3, \text{ if } t = 0. \]

Observe that in all these graphs, whenever we removed a 1-factor out of \( K_q \), then the number of vertices \( q \) was even. Obviously, in each case \( G \) has diameter \( k \) and it is a matter of routine to check that \( G \) is a regular graph of degree \( d \).

(For example, a vertex in the last copy of \( K_{d-2}^{(-1)} \) in the last graph is joined to 1 vertex of \( K_{d-2} \), \( d-4 \) vertices of \( K_{d-2}^{(-1)} \) and to 3 vertices of \( K_3 \), so its degree is \( 1 + d - 4 + 3 = d \).) Also, in each of these cases the number of vertices of \( G \) attains the bound of the theorem. To verify this statement it suffices to check the number of vertices for the smallest admissible values of \( j \) since in each case in the brackets we have exactly \( d + 1 \) vertices.

By Proposition 2, if \( d = 2 \) then \( n(d, k) = dk \). However, for higher degrees we get \( n(d, k) \sim \frac{1}{3} dk \). Denote by \( n_{VT}(d, k) \) the minimum number of vertices in a vertex-transitive \( d \)-regular graph with diameter \( k \). As shown in [6], for \( k \geq 4 \) and “large” \( d \) we have \( n_{VT}(d, k) \sim \frac{1}{3} dk \), and so \( n_{VT}(d, k) = 2n(d, k) \) in this case. On the other hand, since the extremal graphs constructed in the proof of Proposition 1 are vertex-transitive, we have \( n_{VT}(d, k) = n(d, k) \) when \( k \leq 3 \).

References


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