A NOTE ON FACE COLORING ENTIRE WEIGHTINGS OF PLANE GRAPHS

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Abstract

Given a weighting of all elements of a 2-connected plane graph \( G = (V,E,F) \), let \( f(\alpha) \) denote the sum of the weights of the edges and vertices incident with the face \( \alpha \) and also the weight of \( \alpha \). Such an entire weighting is a proper face colouring provided that \( f(\alpha) \neq f(\beta) \) for every two faces \( \alpha \) and \( \beta \) sharing an edge. We show that for every 2-connected plane graph there is a proper face-colouring entire weighting with weights 1 through 4. For some families we improved 4 to 3.

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1. Introduction

In the last years several papers appeared that study various colourings defined by weightings (labellings) of elements of the graph. First such a colouring was introduced by Karoński, Łuczak and Thomason [14]. Let \( G \) be a graph. Given a weighting of the edge set of \( G \), let \( f(v) \) denote the sum of the weights of the edges incident to \( v \) for each \( v \in V(G) \). A weighting is irregular if the resulting vertex weighting \( f \) is injective, and the minimum \( k \) such that this can be done with weights 1 to \( k \) is the irregularity strength of the graph, see [9, 11]. A weaker condition is to require \( f(u) \neq f(v) \) only when \( u \) and \( v \) are adjacent; we call such a weighting a proper vertex-colouring edge-weighting, since the resulting \( f \) is a proper vertex colouring.

Karoński et al. posed the following conjecture.

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**Conjecture 1** ([14], 2004). *Every connected graph with at least three vertices has a proper vertex-colouring edge-weighting from \{1, 2, 3\}.*

Conjecture 1 is true for 3-colourable graphs [14]. Regardless of chromatic number, there is a fixed bound \(k\) such that colours 1 to \(k\) always suffice.

In [1] it was shown that \(k = 30\) suffices. This was reduced to 16 in [2] and to 13 in [16]. Currently, the best known result is \(k = 5\) by Kalkowski, Karoński and Pfender [13].

If each vertex is also given a weight forming total weighting, the sum at a vertex includes the weight of the vertex, and the vertex weighting \(f\) is injective then we obtain the total vertex irregular weighting first introduced by Bača, Jendrol’, Miller and Ryan [5] in 2007. The minimum \(k\) such that this can be done with weights 1 to \(k\) is the *total vertex irregularity strength*. A weaker condition, to require \(f(u) \neq f(v)\) only when \(u\) and \(v\) are adjacent, leads to a *proper vertex-colouring total-weighting*. Using this definition and motivated by the above mentioned papers, Przybyło and Woźniak [15] posed the following 1, 2-conjecture.

**Conjecture 2** ([15], 2010). *Every connected graph has a proper vertex-colouring total-weighting from \{1, 2\}.*

Przybyło and Woźniak [15] showed that 1,2-conjecture is true for 3-colourable graphs; that colours 1 through 11 always suffice for a proper total vertex irregularity weighting and that colours 1 through \(1 + \lfloor \chi(G)/2 \rfloor\) suffice. The breakthrough by Kalkowski [12] is that every graph has a proper vertex-colouring total-weighting with vertex weights in \{1, 2\} and the edge weights in \{1, 2, 3\}.

Motivated by the above mentioned conjectures, papers [7, 8] and mainly by the paper of Wang and Zhu [17] we introduce in this note a concept of the entire weighting for 2-connected plane graphs. If each element of a plane graph \(G = (V, E, F)\) is given a weight forming entire weighting, then let \(f(\alpha)\) of the face \(\alpha \in F(G)\) denote the sum of the weights of the edges and the weights of the vertices incident with \(\alpha\) and the weight of \(\alpha\). A weighting is the *face irregular entire weighting* if the resulting face-weighting \(f\) is injective, and the minimum \(k\) such that this can be done with weights 1 through \(k\) is the *entire face irregularity strength*, see Bača et al. [4]. A weaker requirement is that \(f(\alpha) \neq f(\beta)\) only when faces \(\alpha\) and \(\beta\) share an edge (i.e. are adjacent). Call such a weighting the *proper face-colouring entire \(k\)-weighting* provided that it is done with weights 1 to \(k\).

In this note we discuss the problem of finding the minimum \(k\) such that for every 2-connected plane graph \(G\) there exists a proper face-colouring entire \(k\)-weighting. We show that \(k \leq 4\) in general and that for some families of 2-connected plane graphs \(k \leq 3\). At the end we state a conjecture concerning this minimum \(k\).
2. Results

Let $G = (V, E, F)$ be a 2-connected plane graph with $V = V(G)$, $E = E(G)$ and $F = F(G)$ denoting the vertex set, the edge set and the face set, respectively. For a face $\alpha$ let $V(\alpha)$ and $E(\alpha)$ be the set of vertices and the set of edges incident with the face $\alpha$. For an integer $k$ let $w : V(G) \cup E(G) \cup F(G) \rightarrow \{1, 2, \ldots, k\}$ be an integer weighting. Let $f(\alpha) = w(\alpha) + \sum_{uv \in E(\alpha)} w(uv) + \sum_{v \in V(\alpha)} w(v)$ be the colour of the face $\alpha$. The weighting $w$ is called a proper face-colouring entire $k$-weighting, if $f(\alpha) \neq f(\beta)$ for adjacent faces $\alpha, \beta$.

Let $G^* = (F^*, E^*, V^*)$ be the dual of a 2-connected plane graph $G$. One of main results of this note is the following theorem.

**Theorem 3.** For every 2-connected plane graph $G = (V, E, F)$ there is a proper face-colouring entire $\chi^*$-weighting, where $\chi(G^*) = \chi^*$ denotes the chromatic number of the dual $G^*$ of $G$.

**Proof.** It is easy to see that there exists a proper face colouring $\varphi : F(G) \rightarrow \{1, 2, \ldots, \chi^*\}$. Because of the Four Colour Theorem, $\chi^* \leq 4$, see [3]. Now we associate the following weighting $w$ with elements of $G$: put $w(v) = 2$ for every vertex $v \in V(G)$, $w(e) = 2$ for every edge $e \in E(G)$ and $w(\alpha) = \varphi(\alpha)$ for every face $\alpha \in F(G)$.

Next we have to show that for every two faces $\alpha$ and $\beta$ sharing an edge $f(\alpha) \neq f(\beta)$. To this end suppose that $\alpha$ is an $i$-gon and $\beta$ is a $j$-gon, $j \geq i \geq 2$. If $i < j$, then $f(\alpha) = \varphi(\alpha) + 4i \leq 4(i + 1) < 4j + \varphi(\beta) = f(\beta)$. If $i = j$, because $\varphi(\alpha) \neq \varphi(\beta)$, we immediately have $f(\alpha) \neq f(\beta)$.

**Corollary 4.** Every 2-connected plane graph has a proper face-colouring entire 4-weighting.

Grötzsch [10] (see also [6]) proved that every triangle-free planar graph is 3-colorable. This implies:

**Theorem 5.** Every 2-connected plane graph $G$ whose dual $G^*$ is triangle-free has a proper face-colouring entire 3-weighting.

**Theorem 6.** Every 2-connected plane graph all faces of which are $m$-gons, $m \in \{3, 4, 5\}$, has a proper face-colouring entire 3-weighting.

**Proof.** Let $G = (V, E, F)$ be a 2-connected plane graph and let $G^* = (F^*, E^*, V^*)$ be the dual of $G$. By Kalkowski [12] there is a proper vertex-colouring total-weighting from $\{1, 2, 3\}$. Let $w^*$ be this weighting and let $f^*(\alpha^*) = w^*(\alpha^*) + \sum_{e \in E(\alpha)} w^*(e^*)$.

Define an entire weighting $w$ of $G$ from $\{1, 2, 3\}$ as follows: $w(v) = 2$ for every $v \in V(G)$, $w(\alpha) = w^*(\alpha^*)$ for every face $\alpha \in F(G)$ and $w(e) = w^*(e^*)$ for
every edge \( e \in E(G) \). Then the colour \( f(\alpha) \) of the face \( \alpha \in F(G) \) is defined as
\[
f(\alpha) = w(\alpha) + \sum_{e \in E(\alpha)} w(e) + \sum_{v \in V(\alpha)} w(v) = w^*(\alpha^*) + \sum_{e \in E(\alpha)} w^*(e^*) + 2m \]
\[
= f^*(\alpha^*) + 2m.
\]
But \( f^*(\alpha^*) \neq f^*(\beta^*) \) if \( \alpha^* \beta^* \) is an edge of \( G^* \). This implies \( f(\alpha) \neq f(\beta) \) for adjacent faces \( \alpha \) and \( \beta \) because in this case \( f^*(\alpha^*) \neq f^*(\beta^*) \).

An Eulerian plane graph \( G \) is a connected one each vertex of which has an even degree. It is well known that chromatic number \( \chi(G^*) = 2 \). Using Theorem 3 and this fact we obtain:

**Theorem 7.** Every 2-connected Eulerian plane graph has a proper face-colouring entire 2-weighting.

We expect that any 2-connected plane graph has a proper face-colouring entire 3-weighting. But unfortunately at this moment, we are not able to prove it. We can prove the following:

**Theorem 8.** Every 2-connected cubic plane graph has a proper face-colouring entire 3-weighting.

**Proof.** Let \( G = (V, E, F) \) be a 2-connected cubic plane graph. Proof consists of two main parts. In the first part we associate each face \( \alpha \) with colours
\[
f(\alpha) = w(\alpha) + \sum_{e \in E(\alpha)} w(e) + \sum_{v \in V(\alpha)} w(v) \]
using weighting \( w \) as in the proof of Theorem 3. This weighting \( w \) uses labels from \( \{1, 2, 3, 4\} \) and has property that \( f(\alpha) \neq f(\beta) \) whenever \( \alpha \) and \( \beta \) share an edge in common. Note that the weights 4 are used only on some faces.

In the second part this weighting will be locally changed keeping the colours of faces fixed. The main aim is to delete (lowered) label 4 from faces of \( G \).

We proceed as follows: For every face \( \alpha \) which \( w(\alpha) = 4 \) we choose a vertex \( z \in V(\alpha) \) and two edges \( e_1 \) and \( e_2 \) incident with \( \alpha \) and with \( z \). Next we locally change the weighting \( w \) to the new weighting \( \tilde{w} \) so that \( \tilde{w}(\alpha) = w(\alpha) - 1 = 3 \),
\[
\tilde{w}(z) = w(z) - 1 = 1, \quad \tilde{w}(e_i) = w(e_i) + 1 = 3, \quad i = 1, 2,
\]
for all quadruples \( \alpha, z, e_1, e_2 \) with \( w(\alpha) = 4 \). For all other elements \( x \) of \( G \) we put \( \tilde{w}(x) = w(x) \). It is easy to see that \( \tilde{w}(y) \leq 3 \) for all elements \( y \) of \( G \) and that the colours of all faces of \( G \) are not changed.

We even strongly believe that the following is true.

**Conjecture 9.** Every 2-connected plane graph has a proper face-colouring entire 2-weighting.

**References**

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