TETRAVALENT ARC-TRANSITIVE GRAPHS OF ORDER $3p^2$

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Abstract

Let $s$ be a positive integer. A graph is $s$-transitive if its automorphism group is transitive on $s$-arcs but not on $(s+1)$-arcs. Let $p$ be a prime. In this article a complete classification of tetravalent $s$-transitive graphs of order $3p^2$ is given.

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1. Introduction

In this paper we consider undirected finite connected graphs without loops or multiple edges. For a graph $X$ we use $V(X)$, $E(X)$ and $\text{Aut}(X)$ to denote its vertex set, edge set and its full automorphism group, respectively. For $u,v \in V(X)$, $\{u,v\}$ is the edge incident to $u$ and $v$ in $X$, and $N(u)$ is the neighborhood of $u$ in $X$, that is, the set of vertices adjacent to $u$ in $X$. A graph $X$ is locally primitive if for any vertex $v \in V(X)$, the stabilizer $\text{Aut}(X)_v$ of $v$ in $\text{Aut}(X)$ is primitive on $N(v)$. An $s$-arc in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of the graph such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. For a subgroup $G \leq \text{Aut}(X)$, a graph $X$ is said to be $(G,s)$-arc-transitive or $(G,s)$-regular if $G$ acts transitively or regularly on the set of $s$-arcs of $X$, respectively. A $(G,s)$-arc-transitive graph is said to be $(G,s)$-transitive if it is not $(G,s+1)$-arc-transitive. In particular, an $(\text{Aut}(X),s)$-arc-transitive, $(\text{Aut}(X),s)$-regular or $(\text{Aut}(X),s)$-transitive graph is simply called an $s$-arc-transitive, $s$-regular or $s$-transitive graph, respectively. Note that 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric. A graph is edge-transitive if $\text{Aut}(X)$ is transitive on $E(X)$. 
Edge-transitive graphs or s-transitive graphs of small valencies have received considerable attention in the literature. For instance, Tutte [29] initiated the investigation of cubic s-transitive graphs by proving that there exist no cubic s-transitive graphs for \( s \geq 6 \), and later much subsequent work was done along this line (see [7, 8, 9, 10, 11, 12, 13, 14, 24]). Gardiner and Praeger [15, 16] generally explored the tetravalent symmetric graphs by considering their automorphism groups. Recently, Li et al. [22] classified all vertex-primitive symmetric graphs of valency 3 or 4. Moreover, Weiss [31] proved that if \( X \) is s-transitive, then \( s \in \{1, 2, 3, 4, 5, 7\} \). Let \( p \) be a prime. Conder [6] showed that for a fixed integer \( n \) and any integer \( s > 1 \), there are only finitely many cubic s-transitive graphs of order \( np \). Li [20] generalized this result to connected symmetric graphs of any valency, and he also posed the following problem: for small values \( n \) and \( k \), classify vertex-transitive locally primitive graphs of order \( np \) and valency \( k \).

In this paper we classify all symmetric graphs of order \( np \) and valency \( k \) for certain values of \( n \) and \( k \). The classification of s-transitive graphs of order \( np \) and of valency 3 or 4 can be obtained from [4, 5, 30], where \( 1 \leq n \leq 3 \). Feng et al. [10, 12, 13] classified cubic s-transitive graphs of order \( np \) with \( n = 4, 6, 8 \) or 10. Recently, Zhou and Feng [35, 36] classified tetravalent s-transitive graphs of order \( 4p \) or \( 2p^2 \). Also Ghasemi and Zhou [18] classified tetravalent s-transitive graphs of order \( 4p^2 \). In this paper, we prove that there are no tetravalent s-transitive graphs of order \( 3p^2 \), for \( s > 1 \).

2. Preliminaries

In this section, we introduce some notation and definitions as well as some preliminary results which will be used later in the paper.

For a regular graph \( X \), use \( d(X) \) to represent the valency of \( X \), and for any subset \( B \) of \( V(X) \), the subgraph of \( X \) induced by \( B \) will be denoted by \( [B] \).

For a positive integer \( n \), denote by \( Z_n \) the cyclic group of order \( n \) as well as the ring of integers modulo \( n \), by \( Z_n^* \) the multiplicative group of \( Z_n \) consisting of numbers coprime to \( n \), by \( D_{2n} \) the dihedral group of order \( 2n \), and by \( C_n \) and \( K_n \) the cycle and the complete graph of order \( n \), respectively. We call \( C_n \) an \( n \)-cycle.

Let \( G \) be a permutation group on a set \( \Omega \) and \( \alpha \in \Omega \). Denote by \( G_\alpha \) the stabilizer of \( \alpha \) in \( G \), that is, the subgroup of \( G \) fixing the point \( \alpha \). We say that \( G \) is semiregular on \( \Omega \) if \( G_\alpha = 1 \) for every \( \alpha \in \Omega \) and regular if \( G \) is transitive and semiregular. For any \( g \in G \), \( g \) is said to be semiregular if \( \langle g \rangle \) is semiregular. The following proposition gives a characterization for Cayley graphs in terms of their automorphism groups.

**Proposition 2.1** (Lemma 16.3 [2]). A graph \( X \) is isomorphic to a Cayley graph on a group \( G \) if and only if its automorphism group has a subgroup isomorphic
to \( G \), acting regularly on the vertex set of \( X \).

Let \( X \) be a connected symmetric graph and let \( G \leq \text{Aut}(X) \) be arc-transitive on \( X \). For a normal subgroup \( N \) of \( G \), the quotient graph \( X_N \) of \( X \) relative to the orbits of \( N \) on \( V(X) \) and with two orbits adjacent if there is an edge in \( X \) between those two orbits. If \( X_N \) and \( X \) have the same valency, then \( X \) is called a normal cover of \( X_N \). Let \( X \) be a connected tetravalent symmetric graph and \( N \) an elementary abelian \( p \)-group. A classification of connected tetravalent symmetric graphs was obtained when \( N \) has at most two orbits in [15] and a characterization of such graphs was given when \( X_N \) is a cycle in [16].

The following proposition is due to Praeger et al. (refer to Theorem 1.1 [15] and [27]).

**Proposition 2.2.** Let \( X \) be a connected tetravalent \((G,1)\)-arc-transitive graph. For each normal subgroup \( N \) of \( G \), one of the following holds.

1. \( N \) is transitive on \( V(X) \),
2. \( X \) is bipartite and \( N \) acts transitively on each part of the bipartition,
3. \( N \) has \( r \geq 3 \) orbits on \( V(X) \), the quotient graph \( X_N \) is a cycle of length \( r \), and \( G \) induces the full automorphism group \( D_{2r} \) on \( X_N \),
4. \( N \) has \( r \geq 5 \) orbits on \( V(X) \), \( N \) acts semiregularly on \( V(X) \), the quotient graph \( X_N \) is a connected tetravalent \( G/N \)-symmetric graph, and \( X \) is a \( G \)-normal cover of \( X_N \).

Moreover, if \( X \) is also \((G,2)\)-arc-transitive, then case (3) cannot happen.

The following proposition characterizes the vertex stabilizer of the connected tetravalent \( s \)-transitive graphs, which can be deduced from Lemma 2.5 [23], or Proposition 2.8 [22], or Theorem 2.2 [21].

**Proposition 2.3.** Let \( X \) be a connected tetravalent \((G,s)\)-transitive graph. Let \( G_v \) be the stabilizer of a vertex \( v \in V(X) \) in \( G \). Then \( s = 1, 2, 3, 4 \) or 7. Furthermore, either \( G_v \) is a 2-group for \( s = 1 \), or \( G_v \) is isomorphic to \( A_4 \) or \( S_4 \) for \( s = 2 \); \( A_4 \times \mathbb{Z}_3 \), \( \mathbb{Z}_3 \times S_4 \), \( S_3 \times S_4 \) for \( s = 3 \); \( \mathbb{Z}_3^3 \times \text{GL}(2,3) \) for \( s = 4 \); or \( [3^5] \times \text{GL}(2,3) \) for \( s = 7 \), where \( [3^5] \) represents an arbitrary group of order \( 3^5 \).

Let \( X \) be a tetravalent one-regular graph of order \( 3p^2 \). If \( p \leq 13 \), then \( |V(X)| = 12, 27, 75, 147, 363, \) or 507. Now, a complete census of the tetravalent arc-transitive graphs of order at most 640 has been recently obtained by Potočnik, Spiga and Verret [25, 26]. Therefore, a quick inspection through this list (with the invaluable help of magma (see [3])) gives the number of tetravalent one-regular graphs in the case \( p \leq 13 \). The following Proposition can be extracted from Theorem 3.4 [17].
Proposition 2.4. Let \( p \) be a prime and \( p > 3 \). A tetravalent graph \( X \) of order \( 3p^2 \) is 1-regular if and only if one of the following holds:
(i) \( X \) is a Cayley graph over \( \langle x, y | x^p = y^{6p} = [x, y] = 1 \rangle \), with connection set \( \{y, y^{-1}, xy, x^{-1}y^{-1}\} \),
(ii) \( X \) is a connected arc-transitive circulant graph with respect to every connection set \( S \),
(iii) \( X \) is one of the graphs described in Lemma 8.4 [16].

Proposition 2.5 (Theorem 1.2 [16]). Let \( X \) be a connected tetravalent symmetric graph of order \( 3p^2 \) where \( p > 5 \) is a prime. Let \( A = \text{Aut}(X) \) and let \( N = \mathbb{Z}_p^2 \) be a minimal normal subgroup of \( A \). Let \( K \) denote the kernel of \( G \) acting on \( N \)-orbits. If the quotient graph \( X_N \) is a 3-cycle, then \( K_v \cong \mathbb{Z}_2 \), and \( X \) is one-regular.

Finally in the following example we introduce \( G(3p, r) \), which was first defined in [5].

Example 2.6. For each positive divisor \( r \) of \( p - 1 \) we use \( H_r \) to denote the unique subgroup of \( \text{Aut}(\mathbb{Z}_p) \) of order \( r \), which is isomorphic to \( \mathbb{Z}_r \). Define a graph \( G(3p, r) \) by \( V(G(3p, r)) = \{x_i | i \in \mathbb{Z}_3, x \in \mathbb{Z}_p\} \), and \( E(G(3p, r)) = \{x_iy_i+1 | i \in \mathbb{Z}_3, y \in \mathbb{Z}_p, y-x \in H_r\} \). Then \( G(3p, r) \) is a connected symmetric graph of order \( 3p \) and valency \( 2r \). Also \( \text{Aut}(G(3p, p-1)) \cong S_p \times S_3 \). For \( r \neq p - 1 \), \( \text{Aut}(G(3p, r)) \) is isomorphic to \( (\mathbb{Z}_p.H_r).S_3 \) and acts regularly on the arc set, where \( X.Y \) denotes an extension of \( X \) by \( Y \).

3. Main Results

In this section, we classify tetravalent \( s \)-transitive graphs of order \( 3p^2 \) for each prime \( p \). To do so, we need the following lemmas.

Lemma 3.1. Let \( p \) be a prime and let \( n > 1 \) be an integer. Let \( X \) be a connected tetravalent graph of order \( 3p^n \). If \( G \leq \text{Aut}(X) \) is transitive on the arc set of \( X \), then every minimal normal subgroup of \( G \) is solvable.

Proof. Let \( v \in V(X) \). Since \( G \) is arc-transitive on \( X \), by Proposition 2.3, \( G_v \) either is a 2-group or has order dividing \( 2^4 \cdot 3^6 \). It follows that \( |G| \) is divisible by \( 2^4 \cdot 3^7 \cdot p^n \) or \( 2^6 \cdot 3 \cdot p^n \) for some integer \( m \). Let \( N \) be a minimal normal subgroup of \( G \).

Suppose that \( N \) is non-solvable. Then \( p > 3 \) because a \( \{2,3\} \)-group is solvable by a theorem of Burnside Theorem 8.5.3 [28]. Since \( N \) is minimal, it is a product of isomorphic non-abelian simple groups. Since \( |N| \) is divisible by \( 2^4 \cdot 3^7 \cdot p^n \), or \( 2^6 \cdot 3 \cdot p^n \) by [19], pp.12–14, each direct factor of \( N \) is one of the following: \( A_5, A_6, PSL(2, 7), PSL(2, 8), PSL(2, 17), PSL(3, 3), PSU(3, 3) \) or \( PSU(4, 2) \).

An inspection of the orders of such groups gives \( n = 2 \) and \( |N| \) is divisible by \( 2^4 \cdot 3^7 \cdot p^n \). It follows that \( X \) is \( (G, 2) \)-arc transitive and we have \( N \cong A_5 \times A_5 \). Then \( p = 5 \) and
$|X| = 75$. However, from [32] we know that all tetravalent arc-transitive graphs of order 75 are 1-transitive, a contradiction.

**Lemma 3.2.** Let $X$ be a connected tetravalent $G$-arc-transitive graph of order $3p^2$, where $p > 13$. Assume that $G$ has a normal subgroup $N$ of prime order. If $N$ has at least three orbits on $V(X)$, then either $X_N$ is of valency 4 or $G$ is regular on the arcs of $X$.

**Proof.** By our assumption $N$ has at least three orbits on $V(X)$. If $N$ has $r \geq 5$ orbits on $V(X)$, then by Proposition 2.2, $X_N$ has valency 4 and $X$ is a normal cover of $X_N$. Thus we may suppose that $N$ has $r \geq 3$ orbits. Thus $d(X_N) = 2$ and $|X_N| = 3p$ or $|X_N| = p^2$.

First suppose that $|X_N| = 3p$. Thus $X_N \cong C_{3p}$ and hence $G/K \cong \text{Aut}(C_{3p}) \cong D_{6p}$. Let $\Delta$ and $\Delta'$ be two adjacent orbits of $N$ in $V(X)$. Then the subgraph $[\Delta \cup \Delta']$ of $X$ induced by $\Delta \cup \Delta'$ has valency 2. Since $p > 13$, one has $|\Delta \cup \Delta'| \cong C_{2p}$. The subgroup $K^*$ of $K$ fixing $\Delta$ pointwise also fixes $\Delta'$ pointwise. The connectivity of $X$ and the transitivity of $G/K$ on $V(X_N)$ imply that $K^* = 1$, and consequently, $K \leq \text{Aut}([\Delta \cup \Delta']) \cong D_{4p}$. Since $K$ fixes $\Delta$, one has $|K| \leq 2p$. It follows that $|G| = |G/K||K| \leq 12p^2$, and hence $G$ is regular on the arcs of $X$.

Now suppose that $|X_N| = p^2$. Thus $X_N \cong C_{p^2}$. It follows that $G/K \cong D_{2p^2}$. Let $\Delta$ and $\Delta'$ be two adjacent orbits of $N$ in $V(X)$. Then the subgraph $[\Delta \cup \Delta']$ of $X$ induced by $\Delta \cup \Delta'$ has valency 2. Clearly, we have $|\Delta \cup \Delta'| \cong C_6$. The subgroup $K^*$ of $K$ fixing $\Delta$ pointwise also fixes $\Delta'$ pointwise. The connectivity of $X$ and the transitivity of $G/K$ on $V(X_N)$ imply that $K^* = 1$, and consequently, $K \leq \text{Aut}([\Delta \cup \Delta']) \cong D_{12}$. Since $K$ fixes $\Delta$, one has $|K| \leq 6$. It follows that $|G| = |G/K||K| \leq 12p^2$, and hence $G$ is regular on the arcs of $X$. Now the proof is complete.

**Theorem 3.3.** Let $p$ be a prime and let $X$ be a connected tetravalent graph of order $3p^2$. Then $X$ is $s$-transitive for some positive integer $s$ if and only if it is isomorphic to one of the graphs in Proposition 2.4.

**Proof.** Let $X$ be a tetravalent $s$-transitive graph of order $3p^2$ for a positive integer $s$. By [25, 26], we may assume that $p > 13$. If $X$ is one-regular, then $X$ is one of the graphs in Proposition 2.4 and so $s = 1$. In what follows, we assume that $p > 13$ and that $X$ is not one-regular. Set $A = \text{Aut}(X)$ and let $P$ be a Sylow $p$-subgroup. Then $|P| = p^2$ and by Lemma 3.1, $A$ is solvable. First we prove a claim.

**Claim 1.** $P$ is not normal in $A$.

**Proof.** Suppose to, the contrary that $P \triangleleft A$. If $P$ is a minimal normal subgroup of $A$ then by Proposition 2.5, $X$ is one-regular, a contradiction. Suppose that $P$ contains a non-trivial subgroup, say $N$, which is normal in $A$. Consider the
quotient graph $X_N$ of $X$ relative to the orbit set of $N$, and let $K$ be the kernel of $A$ on $V(X_N)$. Since $p > 13$, one has $|X_N| = 3p$. By Lemma 3.2 either $X$ is a normal cover of $X_N$ or $d(X_N) = 2$ and $X$ is one-regular. Since $X$ is not one-regular, we may suppose that $d(X_N) = 4$. By [30], $G(3p, 2)$ is the only tetravalent symmetric graph of order $3p$ (see Example 2.6). Also $|\text{Aut}(G(3p, 2))| = 12p$ and $G(3p, 2)$ is one-regular. Thus $|A/M| = 12p$ and so $|A| = 12p^2$. Thus $X$ is one-regular, a contradiction.

Let $M$ be the maximal normal 2-subgroup of $A$ and assume $|M| > 1$. Consider the quotient graph $X_M$ of $X$ relative to the orbit set of $M$, and let $K$ be the kernel of $A$ acting on $V(X_M)$. Since $p > 13$, every orbit of $M$ has length 2 or 4, a contradiction. So $A$ has no non-trivial normal 2-subgroup.

Now we are ready to complete the proof. Let $M$ be a minimal normal subgroup of $A$. Clearly, $M$ is a 3-group or a $p$-group. First suppose that $M$ is a $p$-group. Thus $|M| = p$ or $p^2$. If $|M| = p^2$, then $M = P$ is a Sylow $p$-subgroup of $A$. By Claim 1, $P$ is not normal in $A$, a contradiction. Suppose that $|M| = p$. By Lemma 3.2 either $X$ is a normal cover of $X_M$ or $d(X_M) = 2$ and $X$ is one-regular. Since $X$ is not one-regular, we may suppose that $d(X_M) = 4$. By [30], $G(3p, 2)$ is the only tetravalent symmetric graph of order $3p$ (see Example 2.6). Also $|\text{Aut}(G(3p, 2))| = 12p$ and $G(3p, 2)$ is one-regular. Thus $|A/M| = 12p$ and so $|A| = 12p^2$. Thus $X$ is one-regular, a contradiction.

Now suppose that $M$ is a 3-group. Thus $|X_M| = p^2$. If $d(X_M) = 4$, then by Proposition 2.5, $K = M$ is semiregular on $V(X_M)$. Therefore $K = M \cong \mathbb{Z}_3$. Since $P > 13$, $PM = P \times M$ is abelian. Clearly, $PM$ is transitive on $V(X)$. Thus $PM$ is regular on $V(X)$, because $|PM| = 3p^2$. Thus $X$ is a Cayley graph on abelian group of order $3p^2$. By Theorem 1.2 [1], $X$ is normal. If $PM$ is cyclic, then by [33] $X$ is one-regular, a contradiction. Thus $PM$ is not cyclic. Now by Proposition 3.3 [34], $X$ is one-regular, a contradiction. If $d(X_M) = 2$, then $X_M \cong C_{p^2}$. By Lemma 3.2, $X$ is one-regular, a contradiction.

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