GRAPHIC SPLITTING OF COGRAPHIC MATROIDS

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Abstract

In this paper, we obtain a forbidden minor characterization of a cographic matroid \( M \) for which the splitting matroid \( M_{x,y} \) is graphic for every pair \( x, y \) of elements of \( M \).

Keywords: binary matroid, graphic matroid, cographic matroid, minor, splitting operation.

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1. Introduction

Fleischner [3] introduced the idea of splitting a vertex of degree at least three in a connected graph and used the operation to characterize Eulerian graphs. Figure 1 shows the graph \( G_{x,y} \) that is obtained from \( G \) by splitting away the edges \( x \) and \( y \) from the vertex \( v \).

Welsh [11] proved that a binary matroid is Eulerian if and only if its dual is bipartite.

It is easy to see that a binary matroid \( M \) is Eulerian if and only if the sum of columns of \( A \) is zero, where \( A \) is a matrix over \( GF(2) \) that represents \( M \). Raghunathan et al. [7] proved that a binary matroid \( M \) is Eulerian if and only if \( M_{x,y} \) is Eulerian for every pair of elements \( x \) and \( y \).
The matroid notations and terminology used here will follow Oxley [6]. We adopt the convention that every graph mentioned in this paper is loopless and coloopless.

Raghunathan et al. [7] extended the splitting operation from graphs to binary matroids as follows:

**Definition 1.1.** Let $M = M[A]$ be a binary matroid and suppose $x, y \in E(M)$. Let $A_{x,y}$ be the matrix obtained from $A$ by adjoining the row that is zero everywhere except for the entries of 1 in the columns labelled by $x$ and $y$. The splitting matroid $M_{x,y}$ is defined to be the vector matroid of the matrix $A_{x,y}$.

**Example 1.2.** Consider the Fano matroid $F_7 = M$ on the set $E = \{1, 2, 3, 4, 5, 6, 7\}$. Let $A$ denote the standard matrix representation with respect to the basis $B = \{1, 2, 3\}$ of $M$ over $GF(2)$, so that

\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}.
\]

Then splitting of $M$ by the pair 2 and 4, i.e. the matroid $M_{2,4}$, is represented by the matrix

\[
A_{2,4} = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}.
\]

Let $M(G)$ and $M^*(G)$ denote the cycle matroid and the cocycle matroid, respectively of a graph $G$. Various properties of a splitting matroid are obtained in [1, 2, 5, 7, 8, 9] and [10].

The splitting operation on a graphic matroid may not yield a graphic matroid. Shikare and Waphare [10] characterized graphic matroids whose splitting matroids for every pair of elements are also graphic. Also, cographicness of a matroid may not be preserved under the splitting operation. Borse, Shikare, and Dalvi [2] obtained a forbidden-minor characterization for this class.

![Figure 2](image-url)
Further, the splitting operation on a cographic matroid may not yield a graphic matroid. In this paper, we characterize those cographic matroids $M$ for which $M_{x,y}$ is graphic for every pair $x, y \in E(M)$. The following is the main theorem.

**Theorem 1.3.** The splitting operation, by any pair of elements, on a cographic matroid yields a graphic matroid if and only if it has no minor isomorphic to any of the cycle matroids $M(G_1)$ and $M(G_2)$, where $G_1$ and $G_2$ are the graphs depicted in Figure 2.

2. **Graphic Splitting of Cographic Matroids**

Firstly, we give some results which are used in the proof of the main result.

**Lemma 2.1** [7]. Let $M = (S, C)$ be a binary matroid on a set $S$ together with the set $C$ of circuits. Then $M_{x,y} = (S, C')$ with $C' = C_0 \cup C_1$, where $C_0 = \{ C \in C : x, y \in C \text{ or } x \notin C, y \notin C \}$; and $C_1 = \{ C_1 \cup C_2 : C_1, C_2 \in C, x \in C_1, y \in C_2, C_1 \cap C_2 = \emptyset \text{ and } C_1 \cup C_2 \text{ contains no member of } C_0 \}$. 

**Lemma 2.2** [5, 10]. Let $x$ and $y$ be elements of a binary matroid $M$ and let $r(M)$ denote the rank of $M$. Then the following statements hold.

(i) $M_{x,y} = M$ if and only if $x$ and $y$ are in series in $M$ or both $x$ and $y$ are coloops in $M$;

(ii) $r(M_{x,y}) = r(M) + 1$ if and only if $M \neq M_{x,y}$,

(iii) if $x_1, x_2$ are in series in $M$, then they are in series in $M_{x,y}$.

(iv) If $C^*$ is a cocircuit of $M$ containing $x, y$ with $|C^*| \geq 3$, then $C^* - \{x, y\}$ is a cocircuit of $M_{x,y}$; and

(v) $M_{x,y}/\{x\}$ is Eulerian if and only if $M$ is Eulerian.

**Theorem 2.3** [6]. A binary matroid is graphic if and only if it has no minor isomorphic to $F_7$, $F_7^*$, $M^*(K_5)$ or $M^*(K_{3,3})$.

**Theorem 2.4** [6]. A binary matroid is cographic if and only if it has no minor isomorphic to $F_7$, $F_7^*$, $M(K_5)$ or $M(K_{3,3})$.

**Notation.** For the sake of convenience, let $\mathcal{F} = \{ F_7, F_7^*, M^*(K_5), M^*(K_{3,3}) \}$.

**Lemma 2.5.** Let $M$ be a cographic matroid and let $x, y \in E(M)$ such that $M_{x,y}$ is not graphic. Then there is a minor $N$ of $M$ with $\{x, y\} \subset E(N)$ such that $N_{x,y}/\{x\} \cong F$ or $N_{x,y}/\{x, y\} \cong F$ for some $F \in \mathcal{F}$ and further, $N$ has no non-trivial series class except possibly a series class which contains $x$ and $y$.

**Proof.** As in the proof of Theorem 2.3 in [10], there exists a minor $N$ of $M$ such that $N_{x,y}/\{x\} \cong F$ or $N_{x,y}/\{x, y\} \cong F$ for some $F \in \mathcal{F}$. If $x$ and $y$ are not in...
series in $N$, then $N$ has no non-trivial series class. Suppose $x$ and $y$ are in series in $N$. Then, $N = N_{x,y}$. Since $F$ does not have any 2-cocircuit, every 2-cocircuit of $N$ must contain $x$ or $y$. Hence $N$ has at most one non-trivial series class.

**Definition 2.6.** Let $M$ be a cographic matroid and let $F \in \mathcal{F}$. We say that $M$ is minimal with respect to $F$ if there exist two elements $x$ and $y$ of $M$ such that $M_{x,y}/\{x\} \cong F$ or $M_{x,y}/\{x, y\} \cong F$ and further, $M$ has no non-trivial series class except possibly a series class which contains $x$ and $y$.

**Corollary 2.7.** Let $M$ be a cographic matroid. For any $x, y \in E(M)$, the matroid $M_{x,y}$ is graphic if and only if $M$ has no minor isomorphic to a minimal matroid with respect to any $F \in \mathcal{F}$.

**Proof.** The proof follows from Lemma 2.2 and Lemma 2.5.

**Lemma 2.8.** Let $M$ be a minimal matroid with respect to $F$ for some $F \in \mathcal{F}$ and let $x, y$ be two elements of $M$ such that either $M_{x,y}/\{x\} \cong F$ or $M_{x,y}/\{x, y\} \cong F$. Then

(i) $M$ has neither loops nor coloops,

(ii) if $M_{x,y}/\{x, y\} \cong F$ or $M_{x,y}/\{x\} \cong M^*(K_5)$, then $M$ has at most one 2-circuit.

**Proof.** The proof follows from Lemmas 2.1, 2.2 and the fact that $F$ does not contain loops, coloops and 2-circuits.

**Lemma 2.9** [10]. A graph is minimal with respect to the matroid $F_7$ or $F_7^*$ if and only if it is isomorphic to one of the three graphs $G_1, G_2$ and $G_3$ in Figure 3.

![Figure 3](image)

**Lemma 2.10** [10]. A graph is minimal with respect to the matroid $M^*(K_{3,3})$ if and only if it is isomorphic to one of the four graphs $G_4, G_5, G_6$ and $G_7$ presented in Figure 4.
Lemma 2.11 [10]. A graph is minimal with respect to the matroid $M^*(K_5)$ if and only if it is isomorphic to $G_8$ and $G_9$ presented in Figure 5.

Lemma 2.12. Let $M$ be a cographic matroid. Then $M$ is minimal with respect to the matroid $F_7$ or $F_7^*$ if and only if $M$ is isomorphic to one of the cycle matroids $M(G_1)$, $M(G_2)$ and $M(G_3)$, where $G_1$, $G_2$ and $G_3$ are the graphs in Figure 6.

Proof. From the matrix representation it follows that $M(G_1)_{x,y}/\{x\} \cong F_7$, $M(G_2)_{x,y}/\{x,y\} \cong F_7$ and $M(G_3)_{x,y}/\{x\} \cong F_7^*$. Therefore, $M(G_1)$, $M(G_2)$ and $M(G_3)$ are minimal with respect to $F_7$ or $F_7^*$.

Conversely, suppose $M$ is minimal with respect to $F_7$ or $F_7^*$. Then there exist elements $x$, $y$ such that $M_{x,y}/\{x\} \cong F_7$, or $M_{x,y}/\{x,y\} \cong F_7$, or $M_{x,y}/\{x\} \cong F_7^*$ or $M_{x,y}/\{x,y\} \cong F_7^*$. Suppose $x$ and $y$ are in series. Then, by Lemma 2.2(i), $M = M_{x,y}$. Therefore, $M$ has $F_7$ or $F_7^*$ as a minor, which is a contradiction to Theorem 2.4. Hence $x$ and $y$ are not in series in $M$. Thus, no two elements of $M$ are in series in $M$. Now, the proof follows from Lemma 2.9.
Lemma 2.13. Let $M$ be a cographic matroid. Then $M$ is minimal with respect to the matroid $M^*(K_{3,3})$ or $M^*(K_5)$ if and only if $M$ is isomorphic to one of $M(G_i)$ for $i = 5, 6, 7, 12$ and to one of $M^*(G_j)$ for $j = 4, 8, 9, 10, 11, 13, 14, 15$, where the graphs $G_i$’s and $G_j$’s are shown in Figure 7.

Proof. From the matrix representation, it follows that

- $M^*(G_4)_{x,y}/\{x\} \cong M^*(K_{3,3})$, $M(G_5)_{x,y}/\{x\} \cong M^*(K_{3,3})$,
- $M(G_6)_{x,y}/\{x,y\} \cong M^*(K_{3,3})$, $M(G_7)_{x,y}/\{x,y\} \cong M^*(K_{3,3})$,
- $M^*(G_8)_{x,y}/\{x,y\} \cong M^*(K_{3,3})$, $M^*(G_9)_{x,y}/\{x,y\} \cong M^*(K_{3,3})$,
- $M^*(G_{10})_{x,y}/\{x,y\} \cong M^*(K_{3,3})$, $M^*(G_{11})_{x,y}/\{x,y\} \cong M^*(K_{3,3})$,
- $M^*(G_{12})_{x,y}/\{x\} \cong M^*(K_5)$, $M^*(G_{13})_{x,y}/\{x\} \cong M^*(K_5)$,
- $M^*(G_{14})_{x,y}/\{x\} \cong M^*(K_5)$ and $M^*(G_{15})_{x,y}/\{x,y\} \cong M^*(K_5)$.

Therefore, $M(G_i)$ for $i = 5, 6, 7, 12$ and $M^*(G_j)$ for $j = 4, 8, 9, 10, 11, 13, 14, 15$ are minimal with respect to the matroid $M^*(K_{3,3})$ or $M^*(K_5)$.

Conversely, suppose that $M$ is a minimal matroid with respect to the matroid $M^*(K_{3,3})$ or $M^*(K_5)$. Then there exist elements $x$ and $y$ of $M$ such that $M_{x,y}/\{x\} \cong M^*(K_{3,3})$ or $M_{x,y}/\{x,y\} \cong M^*(K_{3,3})$ or $M_{x,y}/\{x\} \cong M^*(K_5)$ or $M_{x,y}/\{x,y\} \cong M^*(K_5)$.

Suppose $x$ and $y$ are in series in $M$. Then, by Lemma 2.2(i), $M = M_{x,y}$. Hence $M/\{x\} \cong M^*(K_{3,3})$ or $M/\{x,y\} \cong M^*(K_{3,3})$ or $M/\{x\} \cong M^*(K_5)$ or $M/\{x,y\} \cong M^*(K_5)$; i.e. $M^* \setminus \{x\} \cong M(K_{3,3})$ or $M^* \setminus \{x,y\} \cong M(K_{3,3})$ or $M^* \setminus \{x\} \cong M(K_5)$ or $M^* \setminus \{x,y\} \cong M(K_5)$. Since $x$ and $y$ are in parallel in $M^*$, it follows that $M \cong M^*(G_i)$ for $i = 4, 8, 9, 13, 15$. 

Figure 7
Now, suppose \( x \) and \( y \) are not in series in \( M \). Then \( M \neq M_{x,y} \). By Lemma 2.2(ii), \( r(M_{x,y}) = r(M) + 1 \).

**Case (i).** \( M_{x,y}/\{x\} \cong M^* (K_{3,3}) \). We claim that \( M \) is graphic. By Theorems 2.3 and 2.4, it suffices to prove that \( M \) does not have any of the matroids \( F_7 \), \( F_7^* \), \( M^* (K_{3,3}) \) and \( M^* (K_5) \) as a minor. As \( M \) is cographic, \( F_7 \) and \( F_7^* \) are excluded minors for \( M \). Further, \(|E(M)| = 10\) and, by Lemma 2.2(ii), \( r(M) = r(M_{x,y}) - 1 = r(M_{x,y}/\{x\}) = r(M^*(K_{3,3})) = 4 \). Hence \( M \) cannot have a minor isomorphic to \( M^*(K_5) \). Assume that \( M \) has a minor isomorphic to \( M^*(K_{3,3}) \). There exists an element \( q \) in \( M \) such that \( M \setminus q \cong M^*(K_{3,3}) \). Therefore \( M^*/q \cong M(K_{3,3}) \). Since \( M^*(K_{3,3}) \) is Eulerian, by Lemma 2.2(v), \( M \) is Eulerian and hence \( M^* \) is bipartite. By Lemma 2.8(i), \( q \) is neither a loop nor a coloop. Hence there exists a circuit \( C \) in \( M^* \) containing \( q \). Since \( C \) is an even circuit, \( C/q \) is an odd circuit in \( M^*/q \cong M(K_{3,3}) \), a contradiction. Thus \( M \) is graphic. Hence \( M \cong M(G) \), where \( G \) is a planar graph. It follows from the proof of Lemma 2.10 that \( M \cong M(G_3) \) of Figure 7.

**Case (ii).** \( M_{x,y}/\{x,y\} \cong M^* (K_{3,3}) \). If \( M \) is graphic, then by Lemma 2.10, \( M \cong M(G_6) \) or \( M(G_7) \) of Figure 7. Suppose that \( M \) is not graphic. As \( M \) is cographic, \( M \cong M^*(G) \) for some graph \( G \). Further, \( G \) has 7 vertices and 11 edges because \( r(M^*) = 6 \). As \(|E(M^*(K_{3,3}))| = 9\), \( r(M^*(K_{3,3})) = 4 \), \( M \setminus \{p\}/\{q\} \cong M^*(K_{3,3}) \) for some elements \( p,q \) of \( M \). Therefore \( M^*/\{p\} \setminus \{q\} \cong M(K_{3,3}) \). Since \( M \) has no 2-cocircuit, \( G \) is simple. Further, \( G \) is non-planar. By Lemma 2.8(ii), \( M \) has at most one 2-circuit and hence \( G \) has at most one vertex of degree 2. Therefore, the degree sequence of \( G \) is \((4,3,3,3,3,3,3)\), \((4,4,3,3,3,3,2)\) or \((5,3,3,3,3,3,2)\).

Consider the degree sequence \((5,3,3,3,3,2)\). A non-planar simple graph with degree sequence \((5,3,3,3,3,2)\) can be obtained from a non-planar simple graph with degree sequence \((4,3,3,3,3,2)\) or \((5,3,3,3,2,2)\) by adding a vertex of degree 2. But there is no non-planar simple graph with any of these two degree sequences see [4]. So, we discard the degree sequence \((5,3,3,3,3,3,2)\).

Since all cocircuits of \( M^*(K_{3,3}) \) are even and \( M \) has no odd cocircuit, the graph \( G \) cannot have an \( i \)-circuit containing both \( x \) and \( y \) for \( i = 3,4,5,7 \).

Now, consider the degree sequence \((4,3,3,3,3,3,3)\). By [10], there is only one non-planar simple graph of degree sequence \((4,3,3,3,3,3,3)\), as shown in Figure 8(iv).
In this graph every pair of edges is contained in an $i$-circuit, for some $i = 3, 4, 5, 7$. Hence we discard this graph.

A non-planar simple graph with degree sequence $(4,4,3,3,3,2)$ can be obtained from a non-planar simple graph with degree sequence $(3,3,3,3,3)$ or $(4,4,3,3,2,2)$ by adding a vertex of degree 2. It follows from [4] that every non-planar simple graph with degree sequence $(4,4,3,3,3,2)$ is isomorphic to one of the first three graphs of Figure 8. Graph (i) is discarded because every pair of edges is contained in an $i$-circuit for some $i = 3, 4, 5, 7$. The remaining two graphs are nothing but the graphs $G_{10}$ and $G_{11}$ in the statement of the lemma.

**Case (iii).** $M_{x,y}/\{x\} \cong M^*(K_5)$. If $M$ is graphic, then, by Lemma 2.11, we get two graphs which one of them is a graph (iv) of Figure 8, which is already discarded. So, $M \cong M(G_{12})$ of Figure 7. Suppose that $M$ is not graphic. As $M$ is cographic, $M = M^*(G)$ for some non-planar graph $G$. Further, $G$ has 6 vertices and 11 edges because $r(M^*) = 5$. By Lemma 2.8(i), $M$ has no loops and coloops and also no two elements of $M$ are in series, $G$ is simple and has minimum degree at least 2. Also, by Lemma 2.8(ii), $G$ has at most one vertex of degree 2. Hence the degree sequence of $G$ is $(4,4,4,4,3,3)$ or $(4,4,4,4,4,2)$. By [4], the graph $G_{14}$ of Figure 7 is the only one non-planar simple graph with the degree sequence $(4,4,4,4,3,3)$. Also, there is only one non-planar simple graph with degree sequence $(4,4,4,4,4,2)$ see [4]. In this graph, any pair of edges are either in a 3-circuit or a 4-circuit. If $G$ is isomorphic to this graph, then $x, y$ belong to a 3-cocircuit or a 4-cocircuit $C^*$ of $M$ and hence $C^* - \{x, y\}$ is a 1-cocircuit or a 2-cocircuit in $M_{x,y}/\{x\}$, a contradiction.

**Case (iv).** $M_{x,y}/\{x, y\} \cong M^*(K_5)$. First we show that $M$ is graphic. Suppose that $M$ is not graphic. Then $M$ has $M^*(K_5)$ or $M^*(K_{3,3})$ as a minor. On the contrary, suppose $M$ has $M^*(K_5)$ or $M^*(K_{3,3})$ as a minor. As $r(M) = 7$ and $|E(M)| = 12$, $M/\{p\}/\{q\} \cong M^*(K_5)$ for some elements $p, q \in E(M)$. This implies that $M^*/\{p\} \setminus \{q\} \cong M(K_5)$. Also, $M/\{n, m, s\} \cong M^*(K_{3,3})$ for some elements $n, m, s \in E(M)$. This implies that $M^*/\{n, m, s\} \cong M(K_{3,3})$. Thus $M \cong M^*(G)$, where $G$ is a non-planar simple graph with 6 vertices and 12 edges. By Lemma 2.8(ii), $G$ has at most one vertex of degree 2. Therefore the degree sequence of $G$ is $(4,4,4,4,4), (5,4,4,4,4), (5,5,4,4,3,3), (5,5,5,3,3,3)$ or $(5,5,4,4,4,2)$. By [4], there is only one non-planar simple graph for each of these sequences, as shown in Figure 9.

![Figure 9](image-url)
It follows from the nature of cocircuits of $M^*(K_3)$, that both $x, y$ do not belong to an $i$-circuit for $i = 3, 4$ nor to a $j$-cocircuit for $j = 3, 4, 5, 7$. These conditions are not satisfied by any pair of edges of the first 4 graphs of Figure 9. Hence we discard these graphs. Further, in the graph $(v)$ of Figure 9 each pair of edges belongs to an $i$-circuit for $i = 3, 4$ and to a $j$-cocircuit for $j = 3, 4, 5, 7$, except the pairs $(1,4), (1,11)$ and $(4,7)$. For these pairs, there is a 5-circuit in $M_{x,y}/\{x, y\}$ and hence it cannot be isomorphic to $M^*(K_3)$ since $M^*(K_3)$ has 5 circuits of size 4 and 10 circuits of size 6. Thus $G$ cannot be obtained from this graph. So $M$ does not have $M^*(K_3)$ or $M^*(K_{3,3})$ as a minor. We conclude that $M$ is graphic. Now the proof follows from Lemma 2.11.

Now, we use Lemmas 2.12 and 2.13 to prove Theorem 1.3.

**Proof of Theorem 1.3.** Let $M$ be a cographic matroid. On combining Corollary 2.7 and Lemmas 2.12 and 2.13, it follows that $M_{x,y}$ is graphic for every pair $\{x, y\}$ of elements of $M$ if and only if $M$ has no minor isomorphic to any of the matroids $M(G_i), i = 1, 2, 3, 5, 6, 7, 12$ and $M^*(G_j), j = 4, 8, 9, 10, 11, 13, 14, 15$ where the graphs $G_i$ and $G_j$ are shown in the statements of the Lemmas 2.12 and 2.13. However, we have $M(G_3) \cong M(G_2) \setminus \{e\} \cong M(G_5) \setminus \{2, w\} \cong M(G_6)/\{2\} \setminus \{6, w\} \cong M(G_7)/\{2\} \setminus \{3, 5\} \cong M(G_{12}) \setminus \{1\}/\{v, 2\}; M^*(G_1) \cong M(G_4)/\{x, e\} \cong M(G_5)/\{x, y\} \cong M(G_6)/\{x, y\} \cong M(G_7)/\{2\} \setminus \{3, 5\} \cong M(G_{12}) \setminus \{1\}/\{v, 2\}; M^*(G_3) \cong M(G_{10})/\{9, y\}$ and $M^*(G_5) \cong M(G_{11})/\{6, y\} \setminus \{11\} \cong M(G_{13}) \setminus \{1, 2, x\} \cong M(G_{14})/\{y\} \setminus \{2, 3\} \cong M(G_{15}) \setminus \{e, f, x, y\}$. 

This means that

$M(G_1) \cong M^*(G_4)/\{x, e\} \cong M^*(G_5)/\{x, y, f\} \cong M^*(G_9)/\{x, y, g\}$

$\cong M^*(G_{10})/\{9, y\} \setminus \{11\}$ and $M(G_3) \cong M^*(G_{11})/\{11\} \setminus \{6, y\}$

$\cong M^*(G_{13})/\{1, 2, x\} \cong M^*(G_{14})/\{2, 3\} \setminus \{y\} \cong M^*(G_{15})/\{e, f, x, y\}$. 

Thus, $M_{x,y}$ is graphic if and only if $M$ has no minor isomorphic to any of the matroids $M(G_i)$ for $i = 1, 3$. But the graphs $G_i$ are precisely the graphs given in the statement of the theorem. This completes the proof. 

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**References**


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