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An iterative method to calculate the thermal characteristics of the rock mass with inaccurate initial data

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Abstract: The paper discusses the coefficient inverse problem for one-dimensional heat equation with inaccurate initial data. A conjugate difference problem is developed on difference level. The problem is solved by method of interval analysis. Condition of applicability of Thomas method and its computational convergence are obtained. Estimates of the interval width of solutions of difference problems and functions of Thomas method are also gained.

Keywords: inverse problem; difference scheme; inaccurate data; estimation; width of intervals; convergence

1 Introduction

The aim of the work is to find an effective coefficient of thermal conductivity of materials. Methods of finding thermal conductivity have been studied by many researchers, as an example we refer to works [1–5]. However, when it comes to practical application of the developed methods, the new circumstances are found out that are not taken into account in the theoretical development of methods of calculation of thermal properties of materials. For example, composites consisting of a matrix and inclusions of various shapes are widely used as structural and functional materials in various instrumentation devices. Significant number of works are devoted to the research of thermal conductivity of composites. However, calculation formulas in these studies were obtained, as a rule, either as a result of processing of experimental data in relation to specific materials, or by setting a priori distribution of temperature and heat flow in models of the structure of heterogeneous bodies [6]. Material of different inclusions in the composite can have different thermal conductivities [7–11]. In this case, when assessing the effective thermal conductivity, composite is considered as multi-phase [12, 13].

Thermal properties of soils are key elements in determining speed of movement and haloe forms of thawing. The thermal conductivity of unfrozen and frozen ground is one of the main parameters that is difficult to determine by indirect method. This parameter is influenced by many factors: genesis, structure of ice, size and configuration of soil particles. Therefore, the actual thermal conductivity may differ significantly from the calculated one [14]. For most materials the thermal conductivity is weakly dependent on temperature, and in the range of natural or calculated temperatures close to zero, may vary up to 30% or more [15].

One of limitations of using devices to work with the frozen soil is insufficient limit of thermal conductivity measurement, focused either on working with insulating materials, whose thermal conductivity is significantly lower than the thermal conductivity of the soil, or with building materials [16]. Furthermore, a large number of currently used devices make use of nonstationary method of determining the thermal conductivity, which involves heating and subsequent cooling of the sample, and the resulting speed of propagation of the thermal wave is interpreted by certain thermal conductivity value. This method reduces time needed for measurement and is applicable to materials whose thermal conductivity does not depend on the temperature. Using this method to determine the thermal conductivity of frozen soils can lead to incorrect values because of the wide temperature range in the sample and the phase transition of water in the ground due to excessive heat exposure of device. Consequently, the thermal conductivity of soil is determined in a state which does not
exist in nature, and the value may differ up to 30% or more from thermal conductivity under natural conditions [16].

Therefore, in view of above arguments, accurate, well-defined models and algorithms are not suited to solve problems that, by their nature, are too complex and multi-sided. The most relevant and promising for studying complex thermodynamic systems, under condition of uncertainty, are interval methods. Such methods allow to take into account the uncertainty and inaccuracy in coefficients and parameters of such systems.

2 Formulation of the problem

To develop a method to calculate the thermal conductivity of the rock, mass the law of heat transfer in a dispersion medium will be used, which is also called heat equation [17], of the following form:

$$C \cdot \frac{\partial T}{\partial \tau} = \text{div}(\lambda \nabla \theta)$$

where \( \theta(x, y, z, \tau) \) – temperature of the rock mass, \( C, \gamma, \lambda \)– coefficient of heat capacity, specific mass and heat conductivity coefficient of the rock mass, respectively. Validity of the above equation was repeatedly verified by experiments [18].

In most cases of the practice, it is worthwhile to consider one-dimensional heat conductivity equations. If the axis Oz is pointed up, the one-dimensional heat conductivity equation is written in the following form:

$$C \cdot \gamma \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial z} \left( \lambda \frac{\partial T}{\partial z} \right)$$

(1)

where \( T(z, \tau) \) – absolute temperature of the ground at time \( \tau \) at the point \( z \in (0, \infty) \). We are interested in the unique solution of equation (1), so we will set the initial-boundary conditions. In this case, the initial condition of the rock temperature is set at time \( \tau = 0 \), i.e.

$$T \big|_{\tau=0} = T_0(z), \quad 0 \leq z \leq H,$$

(2)

where \( H \) – depth of the rock.

On the surface of the earth when \( z = H \) and on the lower boundary of the rock, the following conditions are set

$$\lambda \frac{\partial T}{\partial z} \big|_{z=H} = - \sigma (T - T_2(\tau)) \big|_{z=H}, \quad T \big|_{z=0} = T_0(\tau),$$

(3)

where \( 0 \leq \tau \leq \tau_{\text{max}}, T_2(\tau) \) – absolute air temperature at the earth’s surface.

To find \( \lambda \) one more condition is required. Usually measured value of temperature at the earth’s surface is given [19]:

$$T_g(\tau), \quad 0 \leq \tau \leq \tau_{\text{max}}$$

(4)

It is known that the source of changes in rock temperature (if there are no internal heat sources) is ambient temperature (air). Due to weather conditions, climate and wind speed it is not possible to measure exact values of \( T_2(\tau) \)and \( T_g(\tau) \). Therefore, the measured values are inaccurate data. Hence, the problem of finding the thermal conductivity of the rock can be solved by methods of interval mathematics.

Thus,

$$T_2(\tau) \in \left[ T_{2L}(\tau), T_{2U}(\tau) \right], \quad T_g(\tau) \in \left[ T_{gL}(\tau), T_{gU}(\tau) \right].$$

The above problem will be solved by the method of finite differences.

Grid method. In our opinion, the superiority of the difference problem over the continuous case was proved in the studies [20, 21]. Therefore, the problem is solved in discrete space. So, segment \((0, H)\) is divided into \( N \) equal parts with the step \( \Delta z = H/N \), and the segment \((0, \tau_{\text{max}})\) is divided into \( m \) equal parts with the step \( \Delta \tau = \tau_{\text{max}}/m \). In the resulting discrete area \( \Theta^m_\tau = \{(z_i, \tau_j), \; z_i = i \cdot \Delta z, \; \tau_j = j \cdot \Delta \tau\}, \; i = 0, 1, \ldots, N; \; j = 0, 1, \ldots, m \), we study the next discrete problem

$$Y^{i+1}_z C \gamma = (\lambda Y^i_z)z,$$

(5)

where \( i = 0, 1, \ldots, N; \; j = 0, 1, \ldots, m \).

$$Y^0_z = T_0(z_i), \; Y^1_z = T_1,$$

(6)

where \( i = 0, 1, \ldots, N; \; j = 0, 1, \ldots, m \)

$$\lambda \cdot \frac{Y^{i+1}_z - Y^{i-1}_z}{\Delta z} = -\sigma (Y^{i+1}_z - Y^{i-1}_z).$$

(7)

In addition, measured temperature of the rock mass on the earth’s surface is given (4). It is required to determine the thermal conductivity coefficient of material of mass \( \lambda \).

In the system (5) – (7) expression \( Y^i_z \in \left[ Y^i_{zL}, Y^i_{zU} \right] \) – an approximate value of the absolute temperature of the rock; \( T^i_2 \in \left[ T^i_{2L}, T^i_{2U} \right] \) – air temperature at the earth’s surface; \( T^0_z \in \left[ T^0_z, T^1_z \right] \) – the initial distribution of the rock temperature.

Thermal conductivity coefficient of the rock will be sought from the minimum of the functional

$$J(\lambda) = \sum_{j=0}^{m-1} \left( Y^1_z - Y^0_z \right)^2.$$
3 Conjugate difference problem

The main idea of this study is that using the solution of the conjugate problem of the difference scheme (5) - (7), the thermal conductivity coefficient is determined by iterative interval method. The convergence of the computing process is controlled by a sufficiently small function $\beta_n$. For this, an initial approximation $\lambda_0$, $n = 0$ is given. Next approximation $\lambda_{n+1}$ is determined from the monotony (decrease) of the functional (8). And the monotony of functional is achieved by the control parameter $\beta_n$. Solution of the system (5) - (7) for $\lambda_n$ and $\lambda$ is denoted by $Y_i^{j+1}(n)$ and $Y_i^{j+1}(n + 1)$. Then for the difference of these values $\Delta \lambda = \lambda_{n+1} - \lambda_n$, $\Delta Y_i^{j+1} = Y_i^{j+1}(n + 1) - Y_i^{j+1}(n)$, the following problem is stated

$$C \cdot \gamma \cdot Y_i^{j+1} = (\Delta \lambda \cdot Y_i^{j+1}(n + 1) + \lambda_n \cdot \Delta Y_i^{j+1})_\sigma,$$

where $i = 0, N - 1; j = 0, m - 1$

where $i = 0, 1, \ldots, N, j = 0, 1, \ldots, m - 1$

$$\Delta Y_i^{0} = 0, \quad \Delta Y_i^{j+1} = 0,$$

(10)

(11)

In order to get a conjugate problem on discrete level, multiply (9) by $U_i^{j} \Delta t \cdot \Delta z$ and sum over $i$ and $j$ in discrete area $Q_M^N$.

$$\sum_{i=1}^{N-1} \sum_{j=0}^{M-1} C \cdot \gamma \cdot \Delta Y_i^{j+1} U_i^{j} \Delta t \Delta z = \sum_{i=1}^{N-1} \sum_{j=0}^{M-1} (\Delta \lambda \cdot Y_i^{j+1}(n + 1) + \lambda_n \cdot \Delta Y_i^{j+1})_\sigma \Delta z \Delta t.$$

Using equation $Y_i^{j+1} \cdot U_i^{j} = (Y_i^{j+1} \cdot U_i^{j})_\sigma - Y_i^{j+1} \cdot U_i^{j+1}$ and Lemma (5) (21) we deduce, that

$$\sum_{i=1}^{N-1} \sum_{j=0}^{M-1} C \cdot \gamma \cdot (\Delta \lambda \cdot Y_i^{j+1} U_i^{j} - \Delta Y_i^{j+1} U_i^{j})_\sigma \Delta z -$$

$$- \sum_{i=1}^{N-1} \sum_{j=0}^{M-1} C \cdot \gamma \cdot \Delta Y_i^{j+1} \cdot \Delta U_i^{j+1} \Delta t \Delta z =$$

$$\sum_{j=0}^{m-1} \sum_{i=1}^{N-1} \sigma \cdot \Delta Y_i^{j+1} \cdot U_i^{j} \Delta t - \sum_{j=0}^{m-1} U_i^{j} \cdot \lambda_1 \cdot Y_i^{j+1} \Delta t -$$

$$\sum_{j=0}^{m-1} \sum_{i=1}^{N} (\Delta \lambda Y_i^{j+1}(n + 1) + \lambda_n \cdot \Delta Y_i^{j+1})_\sigma U_i^{j+1} \Delta t \Delta z.$$

Considering initial-boundary conditions (10), (11) and equation

$$Y_i^{j+1}(n + 1) = Y_i^{j+1}(n) + \Delta Y_i^{j+1},$$

Additionally, we set the following initial-boundary conditions for $U_i^{j}$:

$$U_i^{n} = 0, \quad i = 0, 1, \ldots, m; U_i^{j, 0} = 0, \quad 1, \ldots, m - 1.$$

(12)

Thus,

$$- \sum_{j=0}^{m-1} \sum_{i=1}^{N-1} c_\gamma \Delta Y_i^{j+1} U_i^{j+1} \Delta t \Delta z = - \sum_{j=0}^{m-1} \sigma \Delta Y_i^{j+1} U_i^{j+1} \Delta t -$$

$$\sum_{j=0}^{m-1} \sum_{i=1}^{N} \Delta \lambda Y_i^{j+1}(n + 1) U_i^{j+1} \Delta z \Delta t - \sum_{j=0}^{m-1} \sum_{i=1}^{N} \lambda_n \Delta Y_i^{j+1} U_i^{j+1} \Delta z \Delta t.$$

To sum

$$\sum_{j=0}^{m-1} \sum_{i=1}^{N} \lambda_n \Delta Y_i^{j+1} U_i^{j+1} \Delta z \Delta t$$

we apply formula of summation by parts (20), then

$$- \sum_{j=0}^{m-1} \sum_{i=1}^{N-1} c_\gamma \Delta Y_i^{j+1} U_i^{j+1} \Delta t \Delta z = - \sum_{j=0}^{m-1} \Delta Y_i^{j+1}(n + 1) U_i^{j} \Delta z \Delta t -$$

$$\sum_{j=0}^{m-1} \sum_{i=1}^{N} \lambda_n \Delta Y_i^{j+1} U_i^{j} \Delta z \Delta t -$$

$$- \sum_{j=0}^{m-1} \sum_{i=1}^{N} \Delta Y_i^{j+1} \cdot \lambda_1 U_i^{j+1} \Delta t +$$

$$\sum_{j=0}^{m-1} \sum_{i=1}^{N} \Delta Y_i^{j+1}(\lambda_n U_i^{n}) \Delta z \Delta t.$$

Considering boundary condition (12) and regrouping terms, we conclude that:

$$- \sum_{j=0}^{m-1} \sum_{i=1}^{N-1} \Delta Y_i^{j+1}(c_\gamma U_i^{j} + (\lambda_n U_i^{j})_\sigma) \Delta t \Delta z =$$

$$- \sum_{j=0}^{m-1} \sum_{i=1}^{N} \lambda_n U_i^{j} + \sigma U_i^{j} \Delta t$$

$$- \sum_{j=0}^{m-1} \sum_{i=1}^{N} \lambda_n \Delta Y_i^{j+1}(n + 1) U_i^{j+1} \Delta z \Delta t.$$

Discrete function $U_i^{j}$ is chosen so that the following equalities are valid:

$$c_\gamma U_i^{j+1} + (\lambda_n U_i^{j})_\sigma = 0,$$

where $i = 1, 2, \ldots, N - 1; j = 0, 1, 2, \ldots, m - 1.$

$$\lambda_n U_i^{j} + \sigma U_i^{j} = 2(Y_i^{j+1} - T_i^{j+1}), \quad j = 0, 1, \ldots, m - 1.$$

(14)

Then an important formula for further research is derived as:

$$2 \sum_{j=0}^{m-1} (Y_i^{j+1} - T_i^{j+1}) \Delta Y_i^{j+1} \Delta t$$

(15)
During the derivation of the formula (15) conjugate difference problem was obtained. Combining (12) - (13) into integral unit, the conjugate problem is written as:

$$A_\gamma U_j^{i+1} + (\lambda_n U_j^i) = 0,$$

where $i = 1, 2, \cdots, N_j; j = m - 1, m - 2, \cdots, 0$.

During this derivation, the first sum is a small integral unit, the conjugate problem is written as:

$$\lambda_n U_j^i + \sigma U_N^i = 2(Y_j^{i+1} - T_g^{i+1}), j = m - 1, m - 2, \cdots, 0; (17)$$

$$U_n^0 = 0, i = 0, 1, \cdots, N; U_N^0 = 0, j = m - 1, \cdots, 0.$$

(18)

4 Iterative scheme of calculation of heat conductivity coefficient

For two distinct $\lambda_n$ and $\lambda_{n+1}$ from formula (8) is written in the form

$$f(\lambda_{n+1}) = \sum_{j=0}^{m-1} (Y_j^{i+1}(n + 1) - T_g^{i+1})^2 \Delta t,$$

$$f(\lambda_n) = \sum_{j=0}^{m-1} (Y_j^{i+1}(n) - T_g^{i+1})^2 \Delta t.$$

Subtracting these quantities from each other we have

$$f(\lambda_{n+1}) - f(\lambda_n) = 2 \sum_{j=0}^{m-1} \Delta Y_j^{i+1}(Y_j^{i+1}(n) - T_g^{i+1}) \Delta t +$$

$$+ \sum_{j=0}^{m-1} \Delta Y_j^{i+1}^2 \Delta t.$$

Taking into account the relation (15), we deduce that:

$$f(\lambda_{n+1}) - f(\lambda_n) = - \sum_{j=0}^{m-1} \sum_{i=0}^{m-1} \Delta \lambda Y_j^{i+1} U_j^{i+1} \Delta t \Delta z -$$

$$- \sum_{j=0}^{m-1} \sum_{i=0}^{m-1} \Delta \lambda U_j^{i+1} \Delta t \Delta z + \sum_{j=0}^{m-1} (\Delta Y_j^{i+1})^2 \Delta t.$$

Firstly, consider the case when $\lambda = const$. On the right hand side of the equation (19), the first sum is a small quantity of the first order, the second and third sums are small quantities of the second order. It is therefore expected that the sign of the value $f(\lambda_{n+1}) - f(\lambda_n)$ is determined by the sign of the first sum. To ensure the monotony of the functional (8) we set that

$$\Delta \lambda = \beta(n) \sum_{j=0}^{m-1} \sum_{i=0}^{m-1} Y_j^{i+1} U_j^{i+1} \Delta t \Delta z,$$

where $\beta(n)$ - sufficiently small positive function. It is control, which is used to achieve the monotony of functional and convergence of the iterative process. Considering the equation (20), the relation (19) is written in the form:

$$f(\lambda_{n+1}) - f(\lambda_n) = - \beta(n) \sum_{j=0}^{m-1} \sum_{i=0}^{m-1} Y_j^{i+1} U_j^{i+1} \Delta t \Delta z -$$

$$- \beta(n) \sum_{j=0}^{m-1} \sum_{i=0}^{m-1} Y_j^{i+1} U_j^{i+1} \Delta t \Delta z +$$

$$+ \sum_{j=0}^{m-1} (\Delta Y_j^{i+1})^2 \Delta t.$$

5 Conditional stability of the solution of the coefficient inverse problem with inaccurate data for the equation of heat conductivity

5.1 Difference problems with inaccurate data

We investigate the stability of the calculation scheme. To solve the problem (5) - (8) the scalar sweep method is used. In this case, the sweep method is written as [21]:

Step 1. At first, the boundary conditions of sweep method are calculated

$$a_{N-1} = \frac{1}{1 + \frac{E}{a}} \beta_{N-1} = \frac{E}{1 + \frac{E}{a}} \beta_{N-1}, (22)$$

where $j = 0, 1, \cdots, m - 1. E = \frac{\Delta t}{\beta_{N-1}}$.

Step 2. For every $j = 0, 1, \cdots, m - 1$. Approximate values of the temperature are determined in the following way. Firstly, using values

$$a_{i-1} = \frac{C_i}{B_i - A_i a_i} \beta_{i-1} = \frac{A_i \beta_{i-1} + F_i}{B_i - A_i a_i}$$

(23)

when $i = N - 1, N - 2, \cdots, 1$.

Here $A_i, B_i, C_i$ and $F_i$ are parameters of the three-point difference scheme

$$A_i Y_{i+1}^{j+1} - B_i Y_i^{j+1} + C_i Y_{i-1}^{j+1} + F_i = 0,$$

(24)

where

$$A_i = \frac{\Delta t \lambda}{(\Delta z)^2}, C_i = A_i, B_i = A_i + B_i + 1, F_i = Y_i^j.$$
Step 3. For every \( j = 0, 1, \ldots, m - 1 \) approximate values \( Y_{i}^{j+1} \) are calculated by the formula
\[
Y_{i}^{j+1} = a_{i} Y_{i}^{j+1} + \beta_{i}, \quad i = 0, N - 1, \quad Y_{0}^{j+1} = T_{1}(\tau_{j+1})
\] (25)

It was proved that the measured values \( T_{a}(\tau) \) and \( T_{g}(\tau) \) are interval values. Because of this, the whole solution of the system (5) - (8) and (15) - (17) becomes interval values. That is, the next inclusions are valid
\[
Y_{i}^{j+1} \in [\underline{Y}_{i}^{j+1}, \overline{Y}_{i}^{j+1}], \quad U_{i}^{j} \in [\underline{U}_{i}^{j}, \overline{U}_{i}^{j}]
\] (26)

Theorem 1. Let
\[
Y_{i}^{j+1} = a_{i} Y_{i}^{j+1} + \beta_{i}, \quad i = 0, N - 1, \quad j = 0, 1, \ldots, m - 1,
\]
besides \( a_{i}, \beta_{i} \) are calculated by the formulas (22) and (23). Assume that all expressions are valid, then (26) has an inclusion point.

Proof. We use the fundamental theorem of interval arithmetic [22] and method of mathematical induction. If found \( a_{N-1}, \beta_{N-1} \) by the formula (22), then according to the fundamental theorem of interval mathematics
\[
a_{N-1} \in [\alpha_{N-1}, \overline{\alpha_{N-1}}], \quad \beta_{N-1} \in [\beta_{N-1}, \overline{\beta_{N-1}}].
\]

Suppose that
\[
a_{i-1} \in [\alpha_{i-1}, \overline{\alpha_{i-1}}], \quad \beta_{i-1} \in [\beta_{i-1}, \overline{\beta_{i-1}}].
\]

Then from the formula (23) follows that
\[
a_{i} \in [\alpha_{i}, \overline{\alpha_{i}}], \quad \beta_{i} \in [\beta_{i}, \overline{\beta_{i}}].
\] (27)

Hence, by induction, we see that (27) holds for all \( i \). The first inclusion of the system (26) follows from (25) on the basis of the theory of interval mathematics. The second inclusion of the system (26) is proved similarly. Formula (22) implies the inclusion \( \lambda_{n} \in [\underline{\lambda_{n}}, \overline{\lambda_{n}}], \) \( n = 1, 2, 3, \ldots \). Referring to the formula (24) we conclude that
\[
A_{i} = C_{i} \in [\underline{A_{i}}, \overline{A_{i}}], \quad B_{i} \in [\underline{B_{i}}, \overline{B_{i}}], \quad F_{i} \in [\underline{F_{i}}, \overline{F_{i}}].
\]

Thus, the formula (22) - (25) give an interval variant of the sweep method. To implement it, the expression in the denominator of (23) should not be an interval containing zero.

5.2 Evaluation of width of the solution of differential problem with inaccurate data

This section presents a theorem, which gives sufficient conditions for realizability of the sweep interval and the condition that the absolute value of the intervals \( A_{i} \) does not exceed one. Here, as in above, we will use some definitions of the interval arithmetic [22].

**Theorem 2.** If conditions (22) and (24) are valid, then from (23) follows inequality \( |a_{i-1}| < 1 \).

Proof. From formula (22) we deduce that
\[
|a_{N-1}| = 1/(1 + E) = (1 + \sigma \Delta \mu (1/z_{n})) < 1.
\]

That is, \( 0 < a_{N-1} < 1 \). We use mathematical induction. Let \( 0 < a_{i} < 1 \). We modify
\[
a_{i-1} = C_{i} \quad B_{i} - A_{i} \quad a_{i} = C_{i} \quad A_{i} + C_{i} + 1 - A_{i} \quad a_{j}
\]
\[
= C_{i} \cdot (1 - a_{i}) + C_{i} + 1.
\]

Considering inequality \( 1 - a_{i} > 0 \) we have
\[
a_{i-1} < C_{i} \quad C_{i} + 1 = [C_{i}, C_{i}] \cdot \left[\frac{1}{C_{i} + 1}, \frac{1}{C_{i} + 1}\right]
\]
\[
= \left[\frac{C_{i}}{C_{i} + 1}, \frac{C_{i}}{C_{i} + 1}\right].
\]

However, \( C_{i} = \frac{\Delta t}{(\Delta z)^{2}} \cdot \frac{1}{C_{i}} \), thus
\[
0 < a_{i-1} < \frac{\Delta t}{(\Delta z)^{2}} \cdot \frac{1}{C_{i}} \left[\frac{\lambda_{n}}{1 + \frac{\Delta t}{(\Delta z)^{2}} \cdot \frac{1}{C_{i}}}, \frac{\lambda_{n}}{1 + \frac{\Delta t}{(\Delta z)^{2}} \cdot \frac{1}{C_{i}}}\right].
\]

From here, it follows that the next equation is valid
\[
\frac{\Delta t}{(\Delta z)^{2}} \cdot \frac{\lambda_{n}}{C_{i}} \cdot \frac{1}{1 + \frac{\Delta t}{(\Delta z)^{2}} \cdot \frac{1}{C_{i}}} < 1
\]
or
\[
\frac{\Delta t}{(\Delta z)^{2}} \cdot \frac{\lambda_{n} - \lambda_{n}}{C_{i}} \cdot \frac{\lambda_{n}}{C_{i}} < 1
\] (28)

That is, \( \Delta t \) and \( \Delta z \) are chosen such, that the relation (28) is valid. Then equation \( 0 < a_{i} < 1, \quad i = N - 2, N - 3, \ldots, 0 \), holds. On the basis of this theorem, it can be argued that the conditions (22), (24) are analogous to a well conditionality [23].

We now estimate the width \( \omega(Y_{i}^{j+1}) \) of interval solution of the system (5) - (8) depending on the width of coefficients and free terms of the system (5).

At first, we estimate the width of the coefficient \( a_{i} \). For this we use the terms of the following theorem.

**Theorem 3.** If (28) and
\[
s(c) < m < \frac{1}{2}, \quad w(c) \leq q < \frac{1}{4}, \quad w(a_{N-1}) < q_{1} < 1
\] are valid, then an inequality
\[
0 < w(a_{i}) \leq \frac{q}{1 - m} + m^{2(i-1)} q_{1}
\]
is also valid.
Proof. From both sides of the equality sign (23) we take function $\omega$, then

$$
\omega(a_{i-1}) = \omega\left(\frac{1}{B_i - A_i \cdot a_i}\right) = \omega(C_i) \cdot s\left(\frac{1}{B_i - A_i \cdot a_i}\right) + \omega\left(\frac{1}{B_i - A_i \cdot a_i}\right) \cdot s(C_i).
$$

In our case $A_i = C_i$, $B_i = 2A + 1$. Therefore,

$$
\omega(a_{i-1}) = \omega(C_i) \cdot s\left(\frac{1}{A + 1 + A(1 - a_i)}\right) + \omega\left(\frac{1}{A + 1 + A(1 - a_i)}\right) \cdot s(C_i). \tag{29}
$$

We introduce notation $B_i - a_i A_i = 2A + 1 - a_i A_i = K_i$, then

$$
w(c) = \frac{\Delta t}{(\Delta z)^2} \frac{w(\lambda_n)}{c^2} = q \cdot S\left(\frac{1}{B_i - a_i A_i}\right) = \frac{1}{K_i} \cdot \frac{1}{K_i} = \frac{1}{K_i} \cdot \frac{1}{K_i} = \frac{K_i + K_i}{2K_i K_i} = \frac{2A + 1 - a_i A_i}{2K_i K_i} = \frac{2A + 1 - a_i A_i}{2K_i K_i} = \frac{2A + 1 - a_i A_i}{K_i K_i} = \frac{2A + 1 + a_i A_i}{K_i K_i} = \frac{K_i K_i}{K_i K_i} = \frac{S(K_i)}{K_i K_i}.
$$

On the other hand,

$$
w\left(\frac{1}{K_i}\right) = \frac{1}{K_i} - \frac{1}{K_i} = \frac{2A + 1 - a_i A_i}{K_i K_i} = \frac{2A + 1 + a_i A_i}{K_i K_i} = \frac{K_i K_i}{K_i K_i} = \frac{S(K_i)}{K_i K_i}.
$$

Considering this, equality (30) is transformed into the next form

$$
w(a_{i-1}) = w(c) \frac{S(K_i)}{K_i K_i} + S(c)S(2 - a_i) \cdot w(c) \frac{S(K_i)}{K_i K_i} + \frac{S^2(c) w(a_i)}{K_i K_i}.
$$

or

$$
w(a_{i-1}) = w(c) \left[ \frac{S(K_i)}{K_i K_i} + S(c)S(2 - a_i) \right] + \frac{S^2(c) w(a_i)}{K_i K_i} \tag{31}
$$

Steps of the grid $\Delta t$ and $\Delta z$ are chosen in such manner, so that the condition of Courant is valid. Thus, in our case $S(c) \leq m < \frac{1}{2}$. Then the inequality $w(c) \leq q < \frac{1}{2}$ holds.

On the other hand, expression, that is in square brackets, is also estimated above by the number 1, i.e.

$$
\frac{S(K_i)}{K_i K_i} + S(c)S(2 - a_i) \leq 1.
$$

Taking into account all that has been said from (31), the following inequality is deduced:

$$
\begin{cases}
\frac{w(a_{i-1})}{q} < m^2 w(a_i) \\
i = N - 1, N - 2, \ldots 1.
\end{cases} \tag{32}
$$

Consider the case, when $i = N - 1$. Then, in this case

$$
w(a_{N-1}) = w\left(\frac{1}{1 + E}\right) = \frac{1}{1 + E} - \frac{1}{1 + E} = \frac{w(E)}{(1 + E)(1 + E)} = \frac{w(E)}{(1 + E)(1 + E)}
$$

where

$$
E = \frac{\sigma \Delta z}{\lambda_n}, \quad E = \frac{\sigma \Delta z}{\lambda_n}, \quad E = \frac{\sigma \Delta z}{\lambda_n}.
$$

For soil characteristics inequality $4.42 < \sigma < 11.42$, $0.5 < \lambda_n < 1$. is valid. Hence, approximate values $E$ and $E$ make

$$
E = \frac{\sigma \Delta z}{\lambda_n} = 4.42 \cdot 10^{-6} = 0.442 \cdot 10^{-3},
$$

$$
E = \frac{\sigma \Delta z}{\lambda_n} = 22.84 \cdot 10^{-4}.
$$

Consequently, we can write inequality $w(a_{N-1}) = q_1 = 0.65$.

In view of this evaluation from (32) at $i = N - 1$ we have inequality

$$
w(a_{N-2}) < q + m^2 \cdot q_1.
$$
Similarly at $i = N - 2$

$$w(a_{N-2}) < q + m^2 w(a_{N-2}) < q + m^2 (q + m^2 q_1) = q(1 + m^2) + m^4 q_1.$$  

When $i = N - 3$ we have inequality:

$$w(a_{N-3}) < q(1 + m^2 + m^4) + m^6 q_1.$$  

In overall case,

$$w(a_{N-k}) < q(1 + m^2 + m^4 + \ldots + m^{2(k-2)}) + m^{2(k-1)} q_1 = q \left( 1 - \frac{m^{2(k-1)}}{1 - m} \right) + m^{2(k-1)} q_1.$$  

When the value of $k$ increases, the second value in the right hand side of the inequality becomes sufficiently small, and the first value in the limit tends to $\frac{q}{1 - m}$. Therefore, for large values of $k$ we can write estimation $w(a_{N-k}) < \frac{q}{1 - m}$. That is, using $\Delta z$ and $\Delta t$ width of the interval $a_i$ can be controlled.

Note: Above we proved that inequality $a_i < 1$ is valid. On the other hand $a_i > 0$. Thus from (30) follows inequality $w(a_i) > 0$, $i = N - 2, N - 3, \ldots, 1$.

**Theorem is proved.**

Consider the procedure of estimating parameter $\beta_i$.

**Theorem 4.** If $|\beta_{N-k}| \leq q_s < 1$, then from (21) follows estimations

$$|\beta_{N-k}| \leq \frac{q_s}{1 - q_s} M' \cdot k = 2, 3, \ldots, N.$$  

Here $M' = \max_i |Y_i|$, $0 < q_s < 1, 0 < q_s < 1$. Calculate absolute value of function the beta:

$$|\beta_{i-1}| = \left| \frac{C \beta_i + F_i}{2C - C - \alpha_i} \right| \leq \frac{C \beta_i}{2A + 1 - A \alpha_i} + \frac{|Y_i|}{2A + 1 - A \alpha_i}.$$  

Considering obtained estimations, the following inequality is concluded

$$w \left( \frac{C}{2A + 1 - A \alpha_i} \right) = w \left( \frac{C}{(2 - \alpha_i)A + 1} \right) \leq q_s < 1,$$

$$w \left( \frac{C}{2A + 1 - A \alpha_i} \right) = \frac{1}{2A + 1 - A \alpha_i} w(C) \leq q_s < 1.$$  

Hence,

$$|\beta_{i-1}| \leq q_s |\beta_i| + q_3 |Y_i|.$$  

From equality

$$\beta_{N-1} = \frac{E}{1 + E} T_{i+1}^b$$

follows estimation

$$|\beta_{N-1}| \leq E \cdot T_{i+1}^b = q_s < 1.$$  

For materials we earlier obtained an approximate estimation $E = 0.2284 \cdot 10^{-2}$. Thus from (34) at $i = N - 1$ it follows that

$$|\beta_{i-2}| \leq q_s q_4 + q_3 |Y_{N-1}|.$$  

When $i = N - 2$ we have

$$|\beta_{N-3}| = q_s \left( q_s q_4 + q_3 |Y_{N-1}| \right) + q_3 |Y_{N-2}| = q_s^2 q_4 \left( q_s |Y_{N-1}| + |Y_{N-2}| \right) + q_3 |Y_{N-2}|.$$  

When $i = N - 3$ we deduce estimation

$$|\beta_{N-4}| \leq q_s |\beta_{N-3}| + q_3 |Y_{N-3}| \leq q_s^2 q_4 + \left( q_s^2 |Y_{N-1}| + q_3 |Y_{N-2}| \right) + \left( |Y_{N-3}| \right) q_3.$$  

For any $i = N - k$ we have

$$|\beta_{N-k}| \leq q_s^k q_4 + \left( q_s^2 |Y_{N-1}| + q_3 |Y_{N-2}| + \ldots + |Y_{N-K+1}| q_3 \right) q_3.$$  

Let max $|Y_{i}| = M'$, then $|\beta_{N-k}| \leq q_s^{k-1} q_4 + \frac{q_3}{1 - q_s} M'$ at large values of $k$ follows estimations

$$|\beta_{N-k}| \leq \frac{q_3}{1 - q_s} M', \quad k = 2, \ldots, N$$  

In the view of these relations we have

$$w(\beta_{i-1}) = w \left( \frac{A B_i}{2A + 1 - A \alpha_i} \right) + w \left( \frac{Y_i}{2A + 1 - A \alpha_i} \right) \leq w \left( \frac{A}{2A + 1 - A \alpha_i} |\beta_i| + w(\beta_i) \right) \left( \frac{A}{2A + 1 - A \alpha_i} \right) +$$

$$+ w \left( \frac{Y_i}{2A + 1 - A \alpha_i} \right) \left( \frac{1}{2A + 1 - A \alpha_i} \right) + |Y_i| w \left( \frac{1}{2A + 1 - A \alpha_i} \right).$$  

Assume, that max \(w(Y_{i}''), Y_{i}') = M'$ is given. Then the following inequality is derived

$$w(\beta_{i-1}) \leq q_s w(\beta_i) + q_5 M'.$$

From boundary conditions of the sweep method we deduce estimation

$$w(\beta_{N-k}) = w \left( \frac{E}{1 + E} T_{i+1}^b \right) = q_s T_{i+1}^b = M_1.$$  

Thus, $w(\beta_{i-2}) \leq q_2 \cdot M_1 + q_5 \cdot M'$. 

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From this we derive an estimate

\[ w(\beta_{i-1}) \leq q_2 w(\beta_{i-2}) + q_5 M^i \leq q_2 \left( q_2 M_1 + q_5 M^i \right) + q_5 M^i = q_2^2 M_1 + q_5 (q_2 + 1) M^i. \]

Continuing this process, we find that:

\[ w(\beta_{i-k}) \leq q_2^{k-1} M_1 + q_5 (q_2^{k-2} + q_2^{k-3} + \ldots + 1) M^i. \]

From this we derive an estimate

\[ w(\beta_{N-k}) \leq q_2^{k-1} M_1 + q_5 M^i / (1 - q_2), \quad k = 2, \ldots, N. \tag{36} \]

Consider the estimation procedure of approximate values of temperature.

If theorems 2 – 4 are valid, then the following estimation is valid

\[ w\left(Y_{i+1}^{j+1}\right) \leq q_9 M_0 + \frac{1}{1 - q_9} M_3. \tag{37} \]

**Proof.** From equality \( Y_{i+1}^{j+1} = a_i Y_{i+1}^{j+1} + \beta_i, \quad i = 0, N - 1, \quad Y_{0}^{j+1} = T_1 \) follows relation

\[ Y_{i+1}^{j+1} \leq q \left( Y_{i+1}^{j+1} + q_i M^j \right). \]

When \( i = 0 \) we have

\[ Y_{i+1}^{j+1} \leq q T_1 + q_i M^j. \]

If \( i = 1 \), then

\[ Y_{i+1}^{j+1} \leq q (q T_1 + q_i M^j) + q_i M^j = q^2 T_1 + (q + 1) q_i M^j. \]

For arbitrary \( i \) the following estimation is obtained

\[ Y_{i+1}^{j+1} \leq q^i T_1 + (q^{i+1} + q + \ldots + 1) q_i M^j \]

or

\[ Y_{i+1}^{j+1} \leq q^i T_1 + \frac{q_i M^j}{1 - q}, \quad i = 1, \ldots, N \tag{37} \]

From (37) follows inequality

\[ M^{i+1} \leq q^i T_1 + \frac{q_i M^j}{1 - q}. \]

Let \( j=0 \), then

\[ M^1 \leq q^i T_1 + q_7 M^0. \]

Here \( M^0 = \max_i \left| \theta_0(z_i) \right| \). When \( j=1 \)

\[ M^2 \leq q^i T_1 + (q_7 M^0 + q^i T_1) q_7 = q^2 M^0 + (1 + q_7) q^i T_1. \]

Continuing this process, we conclude that

\[ M^{i+1} \leq q^i T_1 + q_7 M^0 + \frac{q^i T_1}{1 - q_7} = M^2(i) \]

Therefore, from (36) follows the estimate of the Theorem 5. We were able to estimate the maximum absolute value of the temperature through the output data \( \theta_0(z), T_1 \) and \( \Delta t, \Delta z \).

Let us estimate the width of the approximate temperature. For this purpose the following transformations are applied:

\[ w(Y_{i+1}^{j+1}) = w(\alpha_i Y_{i+1}^{j+1} + \beta_i) = w(\alpha_i Y_{i+1}^{j+1}) + w(\beta_i) = w(\alpha_i) S(Y_{i+1}^{j+1}) + S(\alpha_i) w(Y_{i+1}^{j+1}) + w(\beta_i) \]

On the base of above estimates, the next inequality is shown

\[ w(Y_{i+1}^{j+1}) \leq q_9 w(Y_{i+1}^{j+1}) + M_3(i), \quad i = 0, N - 1; \]

\[ w(Y_{0}^{j+1}) = M_0 - \text{given}. \]

When \( i=0 \) we have \( w(Y_{1}^{j+1}) \leq q_9 M_0 + M_3 \).

If \( i=1 \), then

\[ w(Y_{2}^{j+1}) \leq q_9 (q_9 M_0 + M_3) + M_3 = q_9^2 M_0 + M_3 (1 + q_9). \]

For arbitrary \( i \) we deduce inequality

\[ w(Y_{i+1}^{j+1}) \leq q^i T_1 + \frac{1}{1 - q_9} M_3. \]

The evaluation is obtained.

Thus, summarizing the results, we can conclude the following:

1. The object of study is a mathematical model of heat propagation in a metastable rock mass.
2. Input data for the mathematical model are the ambient temperature, the temperature of the rock mass on the earth’s surface, density and heat capacity of the rock mass. It is known that during a measurement of source data, certain measurement errors are allowed. In this paper, the coefficient inverse problem is solved by taking into account errors of initial data. Interval analysis was chosen as a unit of calculation methods for this problem.
3. Proofs of some relations of interval analysis and theory of difference schemes are covered, which are required further study.
4. Conjugate differential problem is derived from the initial difference problem with inaccurate data. An iterative formula is developed to calculate the thermal conductivity of the rock mass with inaccurate initial data.
5. The basic idea is that by using a solution of the conjugate and direct difference problem, the thermal conductivity coefficient is determined by iterative interval method. The convergence of the iterative process, and the monotony of the minimized functional is achieved by the control parameter \( \beta_n \).
6. The difference problem is solved by the Thomas method using interval arithmetic. Computational stability of the method is proved, as well as the stability of the entire computing process, depending on the initial error of the measured data.

7. It is proved that the width of the sweep coefficient \( \alpha \) is controlled through \( \Delta z \) and \( \Delta t \).

8. Practical work is realized: research location is chosen - multilayered ground, measurement operations are conducted - ambient temperature and soil temperature at the surface of the earth (taking into account the errors of instruments), computational experiment is expected by the measured data. In the future, the computational algorithms will be developed for finding the thermal conductivity coefficient of the soil with inaccurate input data. Management of computing process will be implemented with inaccurate input data.

6 Conclusion

In the course of work, the following results were obtained:

- An iterative method is developed to find the thermal conductivity coefficient with inaccurate data for the complex thermodynamic metastable systems.
- Stability of the method of solving the coefficient inverse problem is studied with inaccurate data for the heat conductivity equation.

The obtained results reflect the actual problem of developing elements of the theory of automatic control of interval-specified objects of various classes and the development of interval methods in management theory, which allows to reduce the computational cost. These results can be used for various applications in the management of modern objects of industries in education. Theoretical and practical results will be used in the construction of interval algorithms to study the thermal parameters of metastable rock systems.

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