Research Article

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On a program manifold’s stability of one contour automatic control systems

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Abstract: Methodology of analysis of stability is expounded to the one contour systems automatic control feedback in the presence of non-linearities. The methodology is based on the use of the simplest mathematical models of the nonlinear controllable systems. Stability of program manifolds of one contour automatic control systems is investigated. The sufficient conditions of program manifold’s absolute stability of one contour automatic control systems are obtained. The Hurwitz’s angle of absolute stability was determined. The sufficient conditions of program manifold’s absolute stability of control systems by the course of plane in the mode of autopilot are obtained by means Lyapunov’s second method.

Keywords: absolute stability, one contour systems, automatic control, program manifold, Lyapunov function

1 Introduction

The problem of construction of all set of the systems of differential equations represented by an integral curve, as well as problems of construction of systems of differential equations in different aspects were considered by N.P. Erugin, A.S. Galiullin, I.A. Mukhametzhanov, R.G. Mukharlyamov and others. It should be noted that in the process of solution of these problems, being one of basic problems of theory of stability, the problems of construction of the steady systems on a given program manifold grew into an independent theory. A great number of works is devoted to the construction of systems of equations on the given program manifold, possessing properties of stability, optimality and establishment of estimations of indexes' quality of transient in the neighborhood of a program manifold and to solving of various inverse problems of dynamics (see [1, 11]). The detailed reviews of these works are shown in [2, 4, 9, 11]. The large number of the nonlinear systems of automatic control can be presented as one contour system consisting of two parts: linear part and nonlinear element [12]. The analysis of such systems shows that at the nonlinear elements of certain kind, transients in them can be similar, that allows to generalize the results of private experiment.

The given program Ω(t) is exactly realized only if the initial values of the state vector satisfy the condition ω(t0, x0) = 0. However, this condition cannot always be exactly satisfied. Therefore, in the construction of systems of program motion, the requirement of the stability of the program manifold Ω(t) with respect to the vector function ω should also be taken into account.

The study of these problems is due to the existence of a number of the inverse dynamics problems.

In this paper we study the problem of stability of one contour automatic control system in the neighborhood of a program manifold.

Let the differential equation

\[ \dot{\eta} = -Q\eta, \] (1)

where \( \eta \) is \( n \)-dimension vector, \( Q(n \times n) \) is some matrix, possesses by a \( (n-s) \)-dimension smooth manifold \( \Omega(t) \), with given the vector equation

\[ \omega(t, x) = 0, \] (2)

where \( \omega \) is \( s \leq n \) dimension vector.

On the basis of the criterion that \( \Omega(t) \) is integral for the system (1), we have

\[ \dot{\omega} = \frac{\partial \omega}{\partial t} + \frac{\partial \omega}{\partial \eta} \dot{\eta} = F(t, \eta, \omega). \] (3)

The purpose is the construction of one contour systems on the given manifold.

We will notice that at the construction of the systems, except a condition (3), one of the basic requirements is the conditions of stability of the manifold \( \Omega(t) \) in relation to some given function. The given program \( \Omega(t) \) is exactly realized only if the initial values of the state vector satisfy the condition \( \omega(t, x) = 0 \). However, this condition cannot always be exactly satisfied. Therefore, in the construction of
systems of program motion, the requirement of the stability of the program manifold \( \Omega(t) \) with respect to the vector function \( \omega \) should also be taken into account.

Together with equation (1), we will consider the system of kind

\[
\begin{align*}
\dot{\eta} &= -Q\eta - k\sigma, \\
\dot{\sigma} &= l^T P^T \omega - \rho_{n+1}\sigma,
\end{align*}
\]

where \( \eta \), \( \kappa \) and \( l \) are \( n \)-dimensions vector-columns:

\[
\eta^T = \|\eta_1, \ldots, \eta_n\|, \quad \kappa = \|\kappa_1, 0, \ldots, 0\|;
\]

and \( P(s \times n), Q(n \times n) \) are constant matrices:

\[
Q = \begin{bmatrix}
\rho_1 & 0 & 0 & \cdots & 0 & 0 \\
-\alpha_1 & \rho_2 & 0 & \cdots & 0 & 0 \\
0 & -\alpha_2 & \rho_3 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & -\alpha_n & \rho_n
\end{bmatrix}.
\]

Nonlinear characteristic \( f(\sigma) \) is continuous on \( \sigma \) and satisfies the conditions

\[
f(0) = 0; \quad k_1\sigma^2 \leq f(\sigma) \sigma \leq k_2\sigma^2 \quad \forall \sigma \neq 0.
\]

where \( k_1 \) and \( k_2 \) are positive constants.

We will notice that a function \( f(\sigma) \) is essentially a control function on deviation from the program and at \( \omega = \sigma = 0 \), and by virtue of conditions (6), the system (4) takes the form of (1). Consequently \( \Omega(t) \) is integral for the system (4) also.

If

\[
\begin{align*}
\eta_i &= \eta_i, \quad \sigma = \eta_{n+1}, \quad \rho_i = T_{i1}^{-1} > 0 \land \\
\alpha_i &= \gamma_i T_{i1}^{-1} > 0 \quad \forall i \in [n+1],
\end{align*}
\]

then we will get one contour system consisting of \( n + 1 \) aperiodic links, closed by a nonlinear feedback connection. Aperiodic links can serve engines of different types (electric, hydraulic, pneumatic etc.), electric generator of direct-current, reservoir with gas etc. [13]. Usually processes in such system are described by equations [14]:

\[
\begin{align*}
(T_1 p + 1) x_1 &= -\gamma_1 f(x_n), \quad \forall i \in [n+1], \\
(T_p + 1) x_i &= \gamma_i x_{i-1},
\end{align*}
\]

where \( p = \frac{d}{dt} \); \( T_1, \gamma_i \) are positive constants:

\[
T_1 > 0 \land \gamma_i > 0 \quad \forall i \in [n+1].
\]

We will suppose that \( F = -A\omega \), \((-A(s \times s)) \) is Hurwitz matrix, then differentiating a manifold (2), by virtue of the system (4), we will get

\[
\dot{\omega} = -A\omega - \frac{\partial \omega}{\partial \eta}\kappa\sigma,
\]

\[
\dot{\sigma} = l^T P^T \omega - \rho_{n+1}\sigma.
\]

**Definition 1.** A program manifold \( \Omega(t) \) is called absolutely stable with respect to a vector-function \( \omega \), if it is asymptotically stable on the whole on the solutions of the system (10) at any \( \omega(t_0, \eta_0) \) and function \( f(\sigma) \) satisfying conditions (6).

We will consider the linearized system

\[
\dot{\omega} = -A\omega - bh\sigma,
\]

\[
l^T = \|0, \ldots, 0, a_{n+1}\|, \quad c = Pl,
\]

which is derived from (10) \( f(\sigma) = ha, \quad b = \frac{\partial \omega}{\partial \eta}\kappa, \quad c = Pl \).

We will suppose that it is asymptotically stable at \( \forall h : k_1 - \varepsilon \leq h \leq k_2 + \varepsilon, \)

\[
\text{where } \varepsilon \text{ is how pleasingly small positive constant.}
\]

As we, whether a program manifold \( \Omega(t) \) of the system (10) will be asymptotically stable on the whole with respect to a vector-function \( \omega \) for any function \( \forall f(\sigma) \in C[k_1, k_2] \).

2 Necessary conditions of stability and determination of Hurwitz's angle

**Theorem 1.** For absolute stability of program manifold \( \Omega(t) \) of the system (10) with respect to a vector-function \( \omega \) in Hurwitz's angle \([k_1, k_2]\), it is necessary that the system (1) was asymptotically stable at the condition (12).

Characteristic equation of the system (11) looks like

\[
\Delta(h, \lambda) = Q + \lambda E - \frac{hb}{c^TP\rho_{n+1} + \lambda} = \sum_{m=0}^{s+1} a_s(h) \lambda^{s+1-m} = 0.
\]

We compose the Hurwitz matrix by means of coefficients of characteristic equation

\[
\Gamma_m(h) =
\begin{bmatrix}
\begin{array}{cccccc}
    & a_1(h) & 1 & 0 & \cdots & 0 \\
    & a_2(h) & a_1(h) & 0 & \cdots & 0 \\
    & a_3(h) & a_2(h) & a_1(h) & \cdots & 0 \\
    & \cdots & \cdots & \cdots & \cdots & \cdots \\
    & a_{m-1}(h) & a_{m-2}(h) & a_{m-3}(h) & \cdots & 0 \\
    & \cdots & \cdots & \cdots & \cdots & \cdots \\
    & a_m(h) & 0 & \cdots & \cdots & \cdots \\
\end{array}
\end{bmatrix}
\end{bmatrix}_{\forall m_1 \in [1, s]}
\]
We will introduce the following denotations:

\[ \xi_m = \inf \xi_m(h) \quad \forall m^{s+1}, \quad (15) \]

\[ \xi_m(h) = \det \Gamma_m(h) \quad \forall m^{1s+1}, \quad (16) \]

Then, by virtue of the Hurwitz theorem for asymptotic stability of program manifold \( \Omega(t) \) with respect to a vector-function \( \omega \) on the whole it is necessary the inequality

\[ \xi_m > 0 \quad \forall m^{s+1}. \quad (17) \]

Thus a Hurwitz’s angle is determined as crossing on a line

\[ [k_1 - \varepsilon, k_2 + \varepsilon] = \bigcap_{m=1}^{s+1} \xi_m(h) > 0 \quad \forall h. \quad (18) \]

Thus, for determination of a Hurwitz’s angle it is necessary calculations of polynomials \( \xi_m(h) \).

### 3 Algorithm of calculation of coefficients \( a_m(h) \)

We will define

\[ \det \| Q + \lambda E \| = \sum_{r=0}^{n-1} q_r \lambda^{n-r} \quad (q_0 = 1); \quad (19) \]

\[ \det \| Q + \lambda E \| (\rho_n + \lambda) = \sum_{s=0}^{n} a_m \lambda^{n-m} \quad (a_0 = 1); \quad (20) \]

\[ Q(\lambda) = \sum_{\gamma=0}^{n-2} Q_m \lambda^{n-2-\gamma}, \quad (21) \]

where \( Q(\lambda) \) is the added matrix for \( \| Q + \lambda E \| \), coefficients \( q_m \) and \( a_m \) are determined as:

\[ q_1 = \sum_{i=1}^{n-1} \rho_i; \quad q_2 = \rho_1 \rho_2 + \cdots + \rho_{n-2} \rho_{n-1}; \cdots; \]

\[ q_{n-1} = \prod_{i=1}^{n-1} \rho_i; \]

\[ a_1 = \sum_{i=1}^{n} \rho_i; \quad a_2 = \rho_1 \rho_2 + \cdots + \rho_{n-1} \rho_n; \cdots; \]

\[ a_n = \prod_{i=1}^{n} \rho_i = T^{-1}, \quad T = \prod_{i=1}^{n} T_i. \]

and a multiplier \( \Delta(h, \lambda) \) from (13) we will transform to the kind

\[ \Delta(h, \lambda) = \| Q + \lambda E \| (\rho_n + \lambda) + h T^T (Q + \lambda E)^{-1} = \quad (22) \]

= \| Q + \lambda E \| (\rho_n + \lambda) + h T^T (Q + \lambda E)^{-1} b. \]

By definition of the revers matrix, we have

\[ \| Q + \lambda E \| = \Delta(h, \lambda) E, \quad (23) \]

where \( E \) is unit matrix of dimension \( (n - 1) \times (n - 1) \).

On the basis of formulas (20) - (22) from (23) after equating the coefficients at identical degrees \( \lambda \), we will get a recurrent formula for determination of matrix-coefficients:

\[ Q_m = -Q_{m-2} + q_mE \quad \forall m^{n-2} \quad (24) \]

\[ (Q_{-1} = 0; \quad Q_0 = E), \text{what is more} \]

\[ QQ_{n-2} = q_{n-2}E. \quad (25) \]

On the basis of (13), (22) and (23) we will determine all the coefficients of the polynomial \( \Delta(h, \lambda) \):

\[ a_m(h) = a_m + h d_m \quad \forall m_1, \quad (26) \]

\[ d \delta = T Q s-2 \quad (Q_{-2} = Q_{-1} = 0; \quad Q_0 = E). \quad (27) \]

Thus, a Hurwitz’s angle (18) is determined by virtue of correlations (14) - (16) and (26). From (4) and (5) we will get the simplified expression:

\[ lT Q (\lambda) b = \| Q, \ldots , 0, a_n\| \times \| q_j \| \times \| \alpha_1, 0, \ldots , 0 \|^T \quad (28) \]

\[ = \prod_{i=1}^{n} \alpha_i. \]

Consequently, we have

\[ d_1 = d_2 = \ldots = d_{n-1} = 0; \quad (29) \]

\[ d_n = \prod_{i=1}^{n} \alpha_i = \frac{\gamma}{T} > 0; \]

\[ \gamma = \prod_{i=1}^{n} \gamma_i > 0 \wedge T > 0. \quad (30) \]

From (26) taking into account inequalities (9) (29), and (30) we obtained

\[ a_\delta(h) = a_\delta > 0 \quad \forall s_1^{n-1}; \quad (31) \]

\[ a_n(h) = a_n + h \cdot \gamma/T = T^{-1}(1 + h \gamma) > 0. \]

As it is seen from inequalities (31), the necessary coefficient conditions of absolute stability are performed, if constants to time \( T_i \) and \( \gamma_i \) are positive. It remains only to require the inequality (17) by virtue of correlations (31).
4 Algebraic criterion of stability

For the system (10), we build the Lyapunov function of the form
\[ V = \omega^T L \omega + \beta \int_0^\sigma f(\sigma) d\sigma, \]  
where \( L = L^T > 0 \); \( \beta \) is parameter, and we find
\[ -\dot{V} = \omega^T C \omega + 2 \omega^T g \beta + \beta \rho_{n+1} \sigma f, \]  
where \( C \) is symmetrical three diagonal matrix
\[ C = A^T L + LA, \quad g = Lb - (\beta/2) c^T. \]  
Supposing \( h(\sigma) = f(\sigma)/\sigma \) from (33), we get
\[ -\dot{V} > 0, \quad -\dot{V} = \omega^T C \omega + 2 \omega^T g h(\sigma) + \beta \rho_{n+1} h(a) \sigma^2. \]  
For that \( -\dot{V} > 0 \), it is necessary and sufficiently performing of a generalized Silvester’s inequality:
\[ G[h(\sigma)] = \begin{pmatrix} C & gh(\sigma) \\ g^T h(\sigma) & \beta \rho_{n+1} h(\sigma) \end{pmatrix} \geq \varepsilon > 0, \]  
where \( \varepsilon \) is a small enough positive number. As is generally known, if \( -A \) is a Hurwitz’s matrix, then \( C \geq 0 \), consequently, for performing of condition (36), it is enough to require that the inequality
\[ \beta \rho_{n+1} - h g^T C^{-1} g \geq \varepsilon \]  
subject to condition \( h(\sigma) > \varepsilon \). We will set a matrix \( C = \|c_{ij}\|_1 > 0 \) so that hold next correlations:
\[ \det \|c_{ij}\|_1 > \varepsilon. \]  
Then sufficient conditions of absolute stability \( C \geq 0 \) is equivalent to the necessary conditions (17).

We will define now, that
\[ \inf (\beta \rho_{n+1} - h g^T C^{-1} g) = g_s. \]  
A parameter \( \beta \) is determined from a condition (39). Putting the value of \( g \) in (39), we get a quadratic equation in relation to this parameter
\[ \tau_1 \beta^2 - 2 \tau_2 \beta + \tau_3 \neq 0, \]  
where
\[ \tau_1 = \frac{H}{4} (C^T C^{-1} C) > 0; \]  
\[ \tau_2 = \frac{1}{2} (L c^T C^{-1} b + \rho_{n+1}). \]  
For existing of \( \beta > 0 \), it is necessary and sufficiently performing of inequality
\[ \tau_2 > 0 \land \tau_2^2 - \tau_1 \tau_3 > 0. \]  
We suppose, that \( L = diag \|L_1, \ldots, L_s\| > 0, \)
\[ A = \begin{pmatrix} \rho_1 & 0 & 0 & \cdots & 0 & 0 \\ -\rho_2 & \rho_2 & 0 & \cdots & 0 & 0 \\ 0 & -\rho_3 & \rho_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\rho_s & \rho_s \end{pmatrix}, \]  
then we have
\[ C = A^T L + LA \]
\[ \begin{pmatrix} c_{11} & -c_{21} & 0 & \cdots & 0 & 0 \\ -c_{21} & c_{22} & -c_{32} & \cdots & 0 & 0 \\ 0 & -c_{32} & c_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{s-1,s-1} & -c_{s,s-1} \\ 0 & 0 & 0 & \cdots & -c_{s,s-1} & c_{ss} \end{pmatrix}. \]  
Here
\[ c_{ii} = 2 \rho_i L_i; \quad c_{ij} = \alpha_i L_j. \]  
We put
\[ c_{11} = 2 \rho_1 L_1; \quad \left| c_{ij}\right|_1^2 = \begin{pmatrix} g_1 & -\alpha_2 L_1 \\ -\alpha_2 L_1 & 2 \rho_2 L_2 \end{pmatrix} = g_2; \]
\[ \left| c_{ij}\right|_1^3 = \begin{pmatrix} g_1 & -\alpha_2 L_1 & 0 \\ -\alpha_2 L_1 & 2 \rho_2 L_2 & -\alpha_3 L_2 \\ 0 & -\alpha_3 L_2 & 2 \rho_3 L_3 \end{pmatrix} = g_3; \]  
\[ \left| c_{ij}\right|_1^s = g_s. \]  
From here, a recurrent formula is easily set for the coefficients of the matrix \( L \):
\[ L_i = \left( g_i + \alpha_i L_i \right) / 2 \rho_i L_{i-1} \quad \forall i \]  
\( (g_1 = 0; \quad g_0 = 1). \)
where $g_i$ determine by the formula (45).

We calculate a vector $g$ from (34) taking into account (45). We have

$$g^T = \left\| a_1H_1, 0, \ldots, 0, -\alpha_0\beta H_{n-1} 2^{-1} \right\|.$$ (47)

**Theorem 3.** Let a function (32), $L$ is diagonal, and a matrix $A$ has a structure (43) and condition (42) is valid. Then a matrix $C$ is determined from (44), (45) and the program manifold (2) of system (10) is absolute stable with respect to vector-function $\omega$ in angle (18).

## 5 Program manifold’s absolute stability of control systems by the course of plane

We will consider now the equations of motion of control system the course of plane in the mode of autopilot [15]:

$$\begin{cases} T^2 \ddot{\psi} + U \dot{\psi} + k \psi = 0; & \dot{\mu} = f^T (\sigma) \\ \sigma = a \omega + E \dot{\omega} + G^2 \dot{\omega} - l^{-1} \mu \end{cases},$$ (48)

that possesses by smooth manifold $\Omega (t)$, given by equation

$$\omega (\tau, \psi) = 0.$$ (49)

Here constant $T^2$ characterizes the inertness of regulating object, $U$ is the natural damping, $k$ characterizes the action of restorative force, $a$, $E$, $G^2$, $l$ are constants of regulator.

**Statement of the problem:** to determine the condition of stability for the program manifold (49).

Differentiating a program manifold (49) on $\tau$, by virtue of the system (48) we get

$$\dot{\omega} = \frac{\partial^2 \omega}{\partial \tau^2} + 2 \frac{\partial \omega}{\partial \tau} \dot{\psi} + \frac{\partial^2 \omega}{\partial \tau^2 \partial \psi} \dot{\psi}^2 + \frac{\partial \omega}{\partial \psi} \left[ - \frac{U}{T^2} \dot{\psi} - \mu \right].$$ (50)

We suppose, that [7]

$$\begin{cases} \frac{\partial^2 \omega}{\partial \tau^2} + 2 \frac{\partial \omega}{\partial \tau} \dot{\psi} + \frac{\partial^2 \omega}{\partial \tau^2 \partial \psi} \dot{\psi}^2 + \frac{\partial \omega}{\partial \psi} \left( - \frac{U}{T^2} \dot{\psi} - k \right) = F(\tau, \psi, \omega, \dot{\omega}), \\
F(\tau, \psi, \omega, \dot{\omega}) = -\alpha_1 \dot{\omega} - \alpha_2 \omega. 
\end{cases}$$ (51)

Then we have

$$\dot{\omega} + \alpha_1 \dot{\omega} + \alpha_2 \omega + \frac{\partial \omega}{\partial \psi} \mu = 0$$

$$\dot{\mu} = f^T (\sigma)$$

$$\sigma = a \omega + E \dot{\omega} + G^2 \dot{\omega} - l^{-1} \mu.$$ (52)

Let $\frac{\partial \omega}{\partial \psi} = h$ is const. Introducing denotations

$$\begin{cases} \omega = \eta_1; & \dot{\omega} = \sqrt{\eta_2}; \mu = i \xi; \ t = \frac{\sqrt{\eta_2}}{\eta_1}; \ r = i \\
i = \frac{\sqrt{\eta_2}}{i \tau + \eta_2}; & \varphi (\sigma) = \frac{\eta_3}{\eta_2}; \ a_2 = -\frac{\eta_4}{\eta_2} \\
p_1 = a - a_2 G^2; \ p_2 = \left( E - a_1 G^2 \right) \sqrt{r} \end{cases}.$$ (53)

we reduce the system (52) to the normalized form in dimensionless variables

$$\begin{cases} \ddot{\eta} = -A \eta - \delta \xi \\
\ddot{\xi} = \varphi (\sigma) \\
\ddot{\sigma} = p^T \eta - \xi \end{cases}$$ (54)

where

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}; \ A = \begin{bmatrix} 0 & -1 \\ a_2 & a_1 \end{bmatrix}; \ \delta = 0; \ c = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}; \ \varphi (\sigma) \in C(0, k).$$

We transform the system (52) by means of substitute

$$z_1 = \eta_2; \ z_2 = -a_2 \eta_1 - \eta_2 - \xi,$$ to the form

$$\begin{cases} \dot{z} = -A z - \delta \varphi (\sigma) \\
\dot{\xi} = \varphi (\sigma) \\
\dot{\sigma} = c^T z - \gamma \xi \end{cases}$$ (55)

Here

$$c^T = \left[ \frac{p_1 a_1}{a_2} + p_2, \ \frac{p_1}{a_2} \right]; \ \gamma = 1 + \frac{p_1}{a_2}.$$ (56)

A matrix $L$ determine as the following [4]:

$$l_1 = c_2 + \frac{c_1 a_1}{2 a_2} + \frac{c_2 a_1 + c_1}{2 a_1} \frac{c_1}{a_2}; \ l_2 = \frac{c_1}{2 a_1} \frac{c_1}{a_2}; \ l_3 = \frac{c_2}{2 a_1} + \frac{c_1}{2 a_1} \frac{c_1}{a_2} a_2.$$ (57)

**Theorem 4.** Let $-A$ be a Hurwitz’s matrix.

Then for absolute stability of program manifold (49) of the system (48) it is sufficiently performing of the following conditions

$$\gamma > 0 \land c > 0,$$ (58)

$$\frac{c_2}{a_1} + \frac{c_1}{a_1 a_2} + \frac{p_1}{a_2} = 0 \land \frac{c_1}{a_2} - \frac{p_1 a_1}{a_2} + p_2 = 0.$$ (59)

**Theorem 5.** For absolute stability of program manifold (49) of the system (48) on the angle $(0, k)$ it is sufficiently performing of the following inequality

$$\pi (\mu, q) = k^{-1} + \gamma q + c^T \left( A^T + \mu E \right)^{-1} (A + qg) \delta > 0$$

$$\forall \mu \geq 0.$$ (60)
6 Conclusion

The conditions of stability of the one contour automatic control systems are obtained in the neighborhood of program manifold by means of construction of Lyapunov function. These results may be use for construction of stable one contour automatic control systems with respect to zero positions of equilibrium and will be developed for one contour systems with nonlinearity on type backlash, by the zone of insensitivity, satiation.

References