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SURVEY PAPER

PROFESSOR RUDOLF GORENFLO AND HIS CONTRIBUTION TO FRACTIONAL CALCULUS

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Abstract

Dedicated to Professor Rudolf Gorenflo on the occasion of his 80th anniversary

This paper presents a brief overview of the life story and professional career of Prof. R. Gorenflo - a well-known mathematician, an expert in the field of Differential and Integral Equations, Numerical Mathematics, Fractional Calculus and Applied Analysis, an interesting conversational partner, an experienced colleague, and a real friend. Especially his role in the modern Fractional Calculus and its applications is highlighted.

MSC 2010: 26A33

Key Words and Phrases: Rudolf Gorenflo, research career, scientific co-operation, fractional calculus

1. Life story

Rudolf Gorenflo was born on July 31st 1930 in Friedrichstal near Karlsruhe, Germany. He started to be interested in mathematics very early. Being 5 years old and before entering the elementary school he already knew the multiplication table and could apply it. With 10 years Rudolf could perform all operations with the rational numbers and started his own research. He formulated and answered the question regarding the lengths

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of the periods of the decimal fractions for the rational numbers like $\frac{1}{7}$ or $\frac{1}{49}$. Mathematics attracted a lot of his attention. Rudolf was one of the best in mathematics at his school and decided to become a school teacher in mathematics and physics. In the very hard post-war years Rudolf entered the Technical University in Karlsruhe. In 1950 he started to learn there mathematics and physics but then decided to become a mathematician. Still he took part in several courses in theoretical physics and engineering sciences - an experience that helped him as a mathematician to always choose a right strategy for solving difficult problems. According to Rudolf Gorenflo, his most exciting experience in mathematics was to learn that a certain type of difference schemes for the heat conduction equation can be interpreted from the random walk viewpoint - of course, this connection reflects also his own "dual nature" as a mathematician and a physicist. In 1956 Rudolf Gorenflo completed his diploma thesis entitled "Meromorphic functions of finite order" (scientific adviser Prof. H. Wittich) and received a diploma in mathematics. In the same year he entered a doctoral course at the Technical University in Karlsruhe. Under supervision of Prof. H. Wittich he worked on applications of complex analysis to the theory of entire functions including the Mittag-Leffler function. His doctoral thesis entitled "On the Wiman-Valiron comparison method for power series and its application in the theory of entire transcendental functions" was successfully defended in 1960. Simultaneously with his research activities Rudolf Gorenflo began to collect his first teaching experiences. In 1957 he started to work as a scientific assistant at the Technical University in Karlsruhe - this was a beginning of a long way to his position as a full professor at the Free University of Berlin many years later. But before becoming completely focusing on teaching and research in mathematics he wanted to see what they are good for and applied for a position in the industry. In 1961-1962 he worked for the Standard Electric Lorenz Company in Stuttgart in the department of informatics. Beginning from 1962 Rudolf Gorenflo dealt mainly with physics combined with mathematical modelling, numerics and computer simulations at the Max-Plank Institute for Plasma Physics in Garching near Munich. His abilities and desire for teaching were approved in 1970 at the Technical University in Aachen with his habilitation in mathematics. In Aachen he has got also the first professorship (1971–1973).

In 1973 Rudolf Gorenflo became a full Professor at the Free University of Berlin, where he is doing research and teaching for already 37 years. He was leader of three FU-sponsored research projects: Modelling and Discretization (1983 - 1988), Regularization (1989 - 1994), Convolutions (1995 - 2003). His valuable contributions, results and achievements both in research, in teaching, as well as in social life at the Free University and

for the international scientific community cannot be overestimated; in the next sections some of them are shortly described.

2. Scientific interests and results

During his research career Prof. R. Gorenflo dealt with and obtained valuable results in several areas of Mathematics. Especially his contributions to the Fractional Calculus can be evaluated to be innovative, valuable, and important for the further development of this research area. In the book entitled "Abel Integral Equations: Analysis and Applications" written jointly by him and Sergio Vessella, published by Springer-Verlag, Berlin, in 1991 in the series Lecture Notes in Mathematics the authors in fact pushed forward a modern approach to the Fractional Calculus, i.e. not just deducing some theoretical results, but applying them to model systems, processes and phenomena of the real world. These ideas were the driving force for his research and academic career. To understand them, let us have a short look at the origins of his scientific activities.

In his student years Rudolf Gorenflo was mainly interested in the theory of functions of one complex variable, in particular in value distribution theory and growth properties. Both his diploma thesis and his doctoral thesis were devoted to this theoretical topic.

But already during his time at the Standard Electric Lorenz Company in Stuttgart he had to learn how to apply mathematics to industrial problems. His main duties there were to design and to perform computer simulations, mainly of waiting lines in communication networks. Rudolf Gorenflo had to study Monte-Carlo simulation method and queueing theory and to apply them. At this time, he entered the world of randomness, which is one of the beloved areas of his research interests up to now.

At the Max Planck Institute for Plasma-Physics in Garching near Munich (1962 - 1970) Rudolf Gorenflo could use his knowledge of complex analysis and of electronic computing machines to investigate several practical problems. There he dealt with plane problems of magneto-hydrostatics and magneto-hydrodynamics, the Monte Carlo simulations of particle flights in a rarefied gas, the theory and application of difference schemes to ordinary and partial differential equations (e.g. for large scale computation of toroidal magnetic fields and systems of diffusion, convection, and reaction), and with evaluations of spectroscopic measurements (in the rotationally symmetric case via Abel integral equations of the first kind). The latter activity aroused his interest in ill-posed problems.

Moreover, at the Max Planck Institute for Plasma-Physics Rudolf Gorenflo was responsible for consulting and assisting physicists and engineers in the mathematical and numerical treatment of their problems. This was

the time, he perfected his ability to communicate with applied scientists, to speak with them and to understand their requirements - a quality that many of the colleagues dealing with the applications of Fractional Calculus are thankful to him for. A special experience was his collaboration with a group from the neighbouring Max Plank Institute for Extraterrestical Physics in the development of instruments for measuring the flux of neutrons in the solar wind.

During his time at the University in Aachen (1970 - 1973) and then at the University in Berlin (since 1973) Prof. R. Gorenflo's primary aim was to share his understanding of the applied mathematics with his students and colleagues. His main research interests were divided between integral equations and the neighboring subject of inverse and ill-posed problems, difference schemes for parabolic differential equations, and different topics in Fractional Calculus. By dealing with any of these topics, his interest was motivated by real applications, by the desire to understand the nature of certain physical processes and to describe them mathematically.

Within his research in the field of integral equations, Prof. R. Gorenflo considered them from different viewpoints, firstly, as a tool for modeling processes with a "memory", second, as a deep mathematical theory, and third, as an application of the developed numerical methods to the investigation of the qualitative behavior of their solutions. Exactly this combination of the different viewpoints led to a series of deep results presented in his book with Sergio Vessella about Abel Integral Equations and in his book with D.D. Ang, V.K. Le, and D.D. Trong "Moment Theory and Some Inverse Problems in Potential Theory and Heat Conduction", Lecture Notes in Mathematics 1792 (2002), Springer-Verlag, Heidelberg. Prof. R. Gorenflo also contributed (in collaboration with co-authors) to the theory of the inverse problems in heat conduction, asymptotic properties of singular values, nonlinear inverse problems, problems of recovery of a function from its moments.

Within his research in the theory of difference schemes for parabolic differential equations Prof. R. Gorenflo always tried to use his physical intuition. In particular, his main intention was to develop and investigate difference scheme that imitate essential properties of the diffusion processes that were modelled, namely properties of conservation of mass or energy, preservation of non-negativity, properties of growth and decay. He also tried to see his research objects from different viewpoints that he learned during his academic and industrial career. In this way, the difference schemes he developed allow a double interpretation, namely as that of a process of discrete redistribution of an extensive quantity on the grid-points, and as that of a random walk discrete in space and time, of a particle wandering

according to a diffusion process described by the parabolic equation at hand.

While studying integral equations of Abel type and some neighboring problems, Prof. R. Gorenflo started to be interested in the theory of special functions that is helpful for the solution of integral equation in closed form and in the Fractional Calculus and allows a very natural formulation and solution of Abel type integral equation. He furthermore continued to work on ill-posed problems, inversion of fractional integration being a specific example.

These interests were also stimulated by research visits to Prof. R. Rutman at the University of Massachusetts in Dartmouth (USA) in 1992 and 1994, to Prof. B. Rubin in Jerusalem 1993 and 1995, and to Prof. M. Yamamoto in Tokyo 1995. Then by his participation in an international workshop on Fractals and Fractional Differentiation organized by A. Le Méhaute and A. Oustaloup in Bordeaux (France) in summer of 1994. In Bordeaux he made acquaintance with Prof. F. Mainardi finding in him a collaborator, a co-author, and a friend for many years. As a matter of fact they started joint research with the initial value problems for the basic differential equations of fractional order (fractional relaxation and oscillation). This collaboration was then re-inforced during the work on a joint and two individual lecture notes for a Course organized by A. Carpinteri and F. Mainardi at the International Centre for Mechanical Sciences, Udine, Italy in September 1996 to explore the possible connections between Fractals and Fractional Calculus in Continuum Mechanics.

Beginning from this time, the main research activities and achievements of Prof. R. Gorenflo are devoted to Fractional Calculus. First he began working on ordinary fractional differential equations and related special functions, and later extended his interests to many other actual research topics of Fractional Calculus, its applications and generalizations. Among other things, he investigated special functions of FC like the Mittag-Leffler and the Wright functions both from the analytical and numerical viewpoints, developed numerical methods for the integral and differential equations of the fractional order, constructed operational calculi for FC operators, considered the initial- and boundary-value problems for fractional differential equations. In all cases he was interested not just in the underlying mathematics but in the applications of the results, too. In particular, in his works and in the works of his co-authors some important models for the adequate description of anomalous physical processes like slow and fast relaxation, oscillation, and diffusion in terms of fractional differential equations were investigated. Nowadays we can already speak about a wide collection of FC models in physics, engineering, medicine, and other sciences

even including finance. Without any doubts, several pioneering works in this area were done by Prof. R. Gorenflo and his co-authors.

During the last years, in collaboration with Prof. F. Mainardi and his students and other colleagues, Prof. R. Gorenflo started to investigate the partial fractional equations (fractional in time or in space or in both time and space) that are suitable for modeling non-classical diffusion processes. In the framework of this collaboration, various types of random walk models were devised and analyzed. Prof. R. Gorenflo himself considers this activity to be a fascinating generalization of his earlier investigations of difference schemes conserving mass or non-negativity (or energy) in classical diffusion processes. He often said to Mainardi that he had discovered in Paris a new random walk model after a glass of good red wine produced in Bordeaux!

In the framework of the recent 4th IFAC Workshop on Fractional Differentiation and Its Applications (FDA'10) Prof. R. Gorenflo was honored by the international FC community for his valuable contributions to this field. The special session "Fractional Calculus: Basic Theory and Neighbouring Fields" of the FDA'10 was completely dedicated to Prof. R. Gorenflo on the occasion of his 80th anniversary. Moreover, he received an FDA'10 award in the category "Professional Life Achievements" that was designed to recognize and appreciate excellence in Fractional Calculus.

3. Scientific collaborations

The important role of Prof. R. Gorenflo in Fractional Calculus is not restricted to his own contributions to this field. He always paid a lot of attention to the work with young scientists, supported them and shared with them his ideas, hints, and suggestions. Ten of his students defended their PhD thesis: Matthias Blumenfeld (at Free University of Berlin, 1983) Dinh Nho Hao (from Vietnam, at Free University of Berlin, 1991), Christine Kutsche (at Free University of Berlin, 1994), Katrin Bühring (at Free University of Berlin, 1995, supervised jointly with Prof. S. Prößdorf), Holger Stephan (at Free University of Berlin, 1995, supervised jointly with Prof. H. Gajewski), Evelyn Buckwar (at Free University of Berlin, 1996), Gabriela Witte (at Free University of Berlin, 1997), Talaat El Danaf (from Egypt, at El-Menoufia University, Egypt, 1998), Alain Nkamnang (from Cameroon, at Free University of Berlin, 1999), Entsar Ahmed Abdalla Abdel-Rehim (from Egypt, at Free University of Berlin, 2004). During periodical visits to the University of Bologna (Italy) he had the opportunity to be co-supervisor for master theses in Physics of three students of Prof. Mainardi: Gianni Pagnini (2000), Alessandro Vivoli (2002) and Claudia Manzoni (2002).

Prof. R. Gorenflo was and is ready to communicate with young scientists both in person, say during the conferences or workshops, or via electronic communication media helping them in finding suitable literature, interesting research topics, or with ideas how to solve this or that problem. Even nowadays, as a Professor emeritus, he has been a supervisor of two diploma theses in Mathematics (concerning topics of FC) at the Free University of Berlin written by Honorata Inga Zadworna and Christian I. Schulz.

During his time in Aachen and in Berlin he has given courses and held seminars in several fields, e.g. Numerical Mathematics, Introduction to Computer Science, Basic Analysis, Ill-posed Problems, Integral Equations, Integral Transforms, Functional Analysis. He was furthermore engaged in courses for non-mathematicians: e.g. Differential Geometry for students of geodesy, Mathematics for physicists, Mathematics for biologists and mineralogists, Statistics for biologists.

Of course, Prof. R. Gorenflo helped and assisted not only his students and young scientists but his colleagues, too. There is one of his phrases known, namely, "to work together". At the Department of Mathematics and Informatics of the Free University of Berlin, there were always scientists invited by him. They were supported by the FU Berlin, DAAD Program, the Alexander von Humboldt Foundation, DFG and other special funds. All of these guests were working jointly with Prof. R. Gorenflo. They came from different places around the world from Japan and Vietnam in the far East, to Canada in the West, and to Egypt and Cameroon in the South.

Prof. R. Gorenflo had special relations with colleagues from the former Soviet Union (Azerbaijan, Belarus, Russia, Uzbekistan). He invited them to visit his university, supported them both from the scientific and administrative viewpoints, and helped with ideas and hints in their research work. Many of them including two authors of this article were introduced by Prof. R. Gorenflo into the beautiful world of Fractional Calculus and its applications and are very thankful to him for this. At the Department of Mathematics and Informatics of the Free University there even was a joke that Prof. R. Gorenflo is "the most Russian German" at the Free University.

One other aspect of his contribution to mathematics in general and to FC in particular is his participation in a huge number of conferences all over the world and in his talks given there. He likes traveling, entering new mathematical subjects, and discovering new countries and cultures. At the important conference on Fractional Calculus in Tokyo, organized by Prof. K. Nishimoto, he met many of the leading people of FC and special functions. Many times he visited Hanoi and Ho Chi Minh City for joint

research on inverse and ill-posed problems. It should be especially noted that due to his joint work with many people Prof. R. Gorenflo acquired many friends who are grateful to him for his activities, constant interest to all new matters, optimism and deep knowledge in many areas of life and science.

His friends wish him to be healthy, to keep for a long time all his nice features and love to Fractional Calculus. Many happy returns!

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List of selected publications

This list should provide the reader with an impression about the wide range of the scientific interests of Prof. R. Gorenflo. A fairly complete list of his publications can be found at the Web-site

`http://www.fracalmo.org/gorenflo/index.htm`

Books

1. Gorenflo R., Vessella S. *Abel Integral Equations. Analysis and Applications*. Lecture Notes in Mathematics, 1461. – Springer-Verlag, Berlin, 1991. – viii+215 pp. ISBN: 3-540-53668-X.

2. Ang, Dang Dinh, Gorenflo R., Le Vy Khoi, Trong, Dang Duc. *Moment Theory and Some Inverse Problems in Potential Theory and Heat Conduction*. Lecture Notes in Mathematics, 1792. – Springer-Verlag, Berlin, 2002. – viii+183 pp. ISBN: 3-540-44006-2.

Books' editor

1. Gorenflo R. and Hoffmann K.-H. (Editors). *Applied Nonlinear Functional Analysis* (Variational Methods and Ill-posed Problems). Workshop at Free University of Berlin, September 1981. Verlag Peter Lang, Frankfurt-am-Main and Bern 1983. Volume 25 of the Series Methoden und Verfahren der mathematischen Physik.
2. Vogel A., Gorenflo R., Kummer B., Ofoegbu Ch.O. (Editors). *Inverse Modeling in Exploration Geophysics*. Proc. of the 6th International Mathematical Geophysics Seminar held at the Free University of Berlin, Feb 3 - 6, 1988. Verlag Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1989.
3. Vogel A., Ofoegbu Ch.O., Gorenflo R., Ursin B. (Editors). *Geophysical Data Inversion, Methods and Applications*. Proc. of the 7th International Geophysics Seminar held at the Free University of Berlin, Feb 8 - 11, 1989. Verlag Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1990.
4. Vogel A., Sarwar Abu K.M., Gorenflo R., Kounchev O.I. (Editors). *Theory and Practice of Geophysical Data Inversion*. Proc. of the 8th International Mathematical Geophysics Seminar on Model Optimization in Exploration Geophysics 1990. Verlag Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1992.
5. Vogel A., Gorenflo R., Webers W., Rutman R.S. (Editors). *Geophysical Data Interpretation by Inverse Modeling*. Proc. of the Ninth International Seminar on Model Optimization in Exploration Geophysics, Berlin 1991. Verlag Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1993.
6. Anger G., Gorenflo R., Jochmann H., Moritz H., Webers W. (Editors). *Inverse Problems: Principles and Applications in Geophysics, Technology, and Medicine*. Proc. of the International Conference held in Potsdam, August 30 - September 3, 1993. Akademie Verlag GmbH, Berlin 1993. Volume 74 of the Series Mathematical Research.
7. Ang D.D., Gorenflo R., Rutman R.S., Van T.D., Yamamoto M. (Editors). *Inverse Problems and Applications to Geophysics, Industry, Medicine and Technology*. Proc. of the International Workshop on

Inverse Problems 17 - 19 January 1995 Ho Chi Minh City. Publications of the Ho Chi Minh City Mathematical Society Vol 2, 1995.
8. Gorenflo R. and Navarro M.P. (Editors). *Proceedings: International Conference on Inverse Problems and Applications*, February 23-27, 1998, University of the Philippines - Diliman, Quezon City. Special Issue, August 1998, Matimyas Matematika 21 (Official Journal of the Mathematical Society of the Philippines). ISSN 0115-6926, 191 pages. Printed by Institute of Science and Mathematics Education Development, University of the Philippines, Diliman, Quezon City.

Selected articles

1. Gorenflo R. *Über die Wiman-Valironsche Potenzreihenvergleichsmethode und ihre Anwendung in der Theorie der ganzen transzendenten Funktionen.* 92 pages. Dissertation. Technische Hochschule Karlsruhe, 1960. Supervisor: Hans Wittich.
2. Gorenflo R. Über die singulären Stellen der Lösungen nichtlinearer Differential gleichungen. *Archiv der Mathematik* **12** (1961), 188 - 192.
3. Gorenflo R. Über ganze transzendente Funktionen von regelmäßigem Wachstum. *Mathematische Annalen* **146** (1962), 226 - 231.
4. Gorenflo R. and Kovetz Y. Solution of an Abel type integral equation in the pre sence of noise by quadratic programming. *Numerische Mathematik* **8** (1966), 392 - 406.
5. Gorenflo R., Pacco M.G. and Scherzer B. M. U. Monte Carlo simulation of a Knudsen gas flow with due allowance for the sojourn time. *Zeitschrift fur Angewandte Physik* **22** (1967), 500 - 505.
6. Gorenflo R. Differenzenverfahren vom Irrfahrttypus für die Differentialgleichung von Fokker-Planck-Kolmogorov. *Zeitschrift für Angewandte Mathematik und Mechanik* **48** (1968), T 69 - T 72.
7. Göllnitz H., Heidbreder E., Pinkau K., Reppin C., Schönfelder V. and Gorenflo R. Design of a neutron scattering chamber using Monte Carlo calculations. *Nuclear Instruments and Methods* **74** (1969), 109 - 122.
8. Gorenflo R. Diskrete Diffusionsmodelle und monotone Differenzenschemata für parabolische Differentialgleichungen. *Methoden und Verfahren der Mathematischen Physik* **1** (1969), 143 - 162.
9. Croci M.G., Gorenflo R. and Hertweck F. Calculation of the current flow directions in a channel with staggered electrodes under MHD generator conditions. *Methoden und Verfahren der Mathematischen Physik* **6** (1972), 1 - 39.

10. Gorenflo R. and Niedack M. Conservative difference schemes for diffusion problems with boundary and interface conditions. *Computing* **25** (1980), 299 - 316.
11. Gorenflo R. and Kuban A. Numerische Simulation von Diffusions prozessen mit nichtnegativitätserhaltenden konservativen Differenzenverfahren. *Zeitschrift für Naturforschung* **37a** (1982), 759 - 768.
12. Gorenflo R. Analysis of parabolic difference schemes by Gerschgorin's method. *Annales Polonici Mathematici XLII* (1983), 83 - 91.
13. Gorenflo R. Historische, heuristische und numerische Meditationen über den Logarithmus. *Didaktik der Mathematik* **13** (1985), 272 - 284.
14. Tran Duc Van, Gorenflo R., Le Van Hap. Sobolev-Orlicz spaces of infinite order and nonlinear differential equations. *Analysis* **10** (1990), 231 - 245.
15. Dinh Nho Ho, Gorenflo R. A noncharacteristic Cauchy problem for the heat equation. *Acta Applicandae Mathematicae* **24** (1991), 1 - 27.
16. Gorenflo R., Pfeiffer A. On analysis and discretization of nonlinear Abel integral equations of first kind. *Acta Mathematica Vietnamica* **16** (1991), 211 - 262.
17. Tran Duc Van, Ha Huy Bang, Gorenflo R. On Sobolev-Orlicz spaces of infinite order for a full Euclidean plane. *Analysis* **11** (1991), 67 - 81.
18. Gorenflo R. and Yakubovich S. B. On the integral convolution for inverse G-transforms. *Acta Mathematica Vietnamica* **19** (1994), 145 - 154.
19. Gorenflo R. and Rutman, R. On ultraslow and intermediate processes. In: P. Rusev, I. Dimovski, V. Kiryakova (Editors), *Transform Methods and Special Functions* (Proc. International Workshop,12 - 17 August 1994, Sofia, Bulgaria), Science Culture Technology Publishing (SCTP), Singapore, 1995, pp. 61 - 81.
20. Gorenflo R. and Vu Kim Tuan. Singular value decomposition of fractional integration operators in L^2 spaces with weights. *Journal of Inverse and Ill-Posed Problems* **3** (1995), 1 - 9.
21. Kilbas A., Saigo M. and Gorenflo R. On asymptotic solution of nonlinear Abel-Volterra integral equations with quasipolynomial free term. *Journal of Fractional Calculus* **8** (1995), 75 - 93.
22. Kilbas A.A. and Gorenflo R. Asymptotic solution of a nonlinear Abel-Volterra integral equation of second kind. *Journal of Fractional Calculus* **8** (1995), 103 - 117.

23. Vu Kim Tuan and Gorenflo R. Asymptotics of singular values of Volterra integral operators. *Numerical Functional Analysis and Applications* **17** (1996), 453 - 461.
24. Gorenflo R., Medvedeva T. and Rubin B. Locally controllable regularization of Abel integral equations via operator complementation. *Journal of Inverse and Ill-posed Problems* **5** (1997), 427 - 436.
25. Gorenflo R. and Samko S. On the dependence of asymptotics of s-numbers of fractional integration operators on weight functions. *Integral Transforms and Special Functions* **5** (1997), 191 - 212.
26. Gorenflo R. and Mainardi F. Fractional calculus: integral and differential equations of fractional order. In: A. Carpinteri and F. Mainardi (Editors): *Fractals and Fractional Calculus in Continuum Mechanics*. Springer Verlag, Wien and New York, 1997, pp. 223 - 276.
27. Gorenflo R. Fractional calculus: Some numerical methods. In: A. Carpinteri and F. Mainardi (Editors): *Fractals and Fractional Calculus in Continuum Mechanics*. Springer Verlag, Wien and New York, 1997, pp. 277 - 290.
28. Gorenflo R. and Luchko Yu. An operational method for solving Abel integral equations of second kind. *Integral Transforms and Special Functions* **5** (1997), 47 - 58.
29. Gorenflo R, Luchko Yu. and Srivastava H.M. Operational method for solving integral equations with Gauss' hypergeometric function as a kernel. *International Journal of Mathematical and Statistical Sciences* **6** (1997), 179 - 200.
30. Gorenflo R., Mainardi F. and Srivastava H.M. Special functions in fractional relaxation-oscillation and fractional diffusion-wave phenomena, In: D. Bainov (Editor), *Proc. VIII International Colloquium on Differential Equations, Plovdiv 1997*, VSP (International Science Publishers), Utrecht, 1998, pp. 195 - 202.
31. Gorenflo R., Kilbas A.A. and Rogosin S.V. On the generalized Mittag-Leffler type functions. *Integral Transforms and Special Functions* **7** (1998), 215 - 224.
32. Gorenflo, R. and Mainardi, F. Fractional calculus and stable probability distributions. *Archives of Mechanics* **50** No 3 (1998), 377-388.
33. Luchko, Yu. and Gorenflo, R. Scale-invariant solutions of a partial differential equation of fractional order. *Fractional Calculus and Applied Analysis* **1** No 1 (1998), 63 - 78.
34. Gorenflo, R. and Mainardi, F. Random walk models for space-fractional diffusion processes, *Fractional Calculus and Applied Analysis* **1** No 2 (1998), 167 - 190.

35. Ang D. D., Gorenflo R., Trong D.D. A multidimensional Hausdorff moment problem: Regularization by finite moments. *Zeitschrift für Analysis und ihre Anwendungen* **18** (1999), 13 - 25.
36. Fleischer G., Gorenflo R., Hofmann B. On the autoconvolution equation and total variation constraints. *Zeitschrift für Angewandte Mathematik und Mechanik* **79** (1999) 3, 149 - 159.
37. Tautenhahn U. and Gorenflo R. On optimal regularization methods for fractional differentiation. *Journal for Analysis and its Applications* **18** (1999), 449 - 467.
38. Gorenflo R. and Yamamoto M. Operator-theoretic treatment of linear Abel integral equations of first kind. *Japan Journal of Industrial and Applied Mathematics* **16** (1999), 137 - 161.
39. Gorenflo R., De Fabritiis G. and Mainardi F. Discrete random walk models for symmetric Lévy-Feller diffusion processes. *Physica A* **269** No 1 (1999), 79 - 89.
40. Gorenflo R. and Mainardi F. Approximation of Levy-Feller diffusion by random walk. *Journal for Analysis and its Applications* **18** (1999), 231 - 246.
41. Gorenflo R., Luchko Yu. and Zabrejko, P.P. On solvability of linear fractional differential equations in Banach spaces. *Fractional Calculus and Applied Analysis* **2** (1999), 163 - 176.
42. Gorenflo R., Luchko Yu. and Mainardi F. Analytical properties and applications of the Wright function. *Fractional Calculus and Applied Analysis* **2** No 4 (1999), 383 - 414.
43. Gorenflo R. and Luchko Yu. An operational method for solving fractional differential equations with the Caputo derivatives. *Acta Mathematica Vietnamica* **24** (1999), 207-233.
44. Gorenflo R., Luchko Yu. and Mainardi F. Wright functions as scale-invariant solutions of the diffusion-wave equation. *Journal of Computational and Applied Mathematics* **118** (2000), 175 -191.
45. Mainardi F. and Gorenflo R. On Mittag-Leffler type functions in fractional evolution processes. *Journal of Computational and Applied Mathematics* **118** (2000), 283 - 299.
46. Gorenflo R., Luchko Yu., Umarov S.R. On some boundary value problems for pseudo-differential equations with boundary operators of fractional order. *Fractional Calculus and Applied Analysis* **3** (2000), 453 - 468.
47. Gorenflo R., Iskenderov A. and Luchko Yu. Mapping between solutions of fractional diffusion-wave equations. *Fractional Calculus and Applied Analysis* **3** (2000), 75 - 86.

48. Scalas E., Gorenflo R. and Mainardi F. Fractional calculus and continuous-time finance. *Physica A* **284** (2000), 376 - 384.
49. Mainardi F., Raberto M., Gorenflo R., and Scalas E. Fractional Calculus and continuous-time finance II: The waiting-time distribution. *Physica A* **287** (2000), 468 - 481.
50 Gorenflo R., Mainardi F, Scalas, E. and Raberto M. Fractional calculus and continuous-time finance III: the diffusion limit. In: M. Kohlmann and S. Tang (Editors): *Mathematical Finance*, Birkhäuser Verlag, Basel-Boston-Berlin, 2001, pp. 171 - 180.
51. Gorenflo R. The tomato salad problem in spherical stereology. In: S.I. Kabanikhin and V.G. Romanov (Editors): *Ill-Posed Problems and Inverse Problems*, pp. 117-134. VSP, AH Zeist, The Netherlands, 2002, pp. 117 - 134.
52. Gorenflo R., Loutchko J. and Luchko Yu. Computation of the Mittag-Leffler function and its derivative. *Fractional Calculus and Applied Analysis* **5** (2002), 491 - 518.
53 Chechkin A.V., Gorenflo R. and Sokolov I.M. Retarding sub- and accelerating super-diffusion governed by distributed order fractional diffusion equations. *Physical Review E* **66** (2002), 046129/1–7.
54. Gorenflo R., Mainardi F., Moretti D., Pagnini G. and Paradisi P. Fractional diffusion: probability distributions and random walk models. *Physica A* **305** (1/2) (2002), 106 - 112.
55. Gorenflo R., Mainardi F., Moretti D. and Paradisi P. Time-fractional diffusion: a discrete random walk approach. *Nonlinear Dynamics* **29** No 1-4 (2002), 129 - 143.
56. Gorenflo R, Mainardi F., Moretti D., Pagnini G. and Paradisi P. Discrete random walk models for space-time fractional diffusion, *Chemical Physics* **284** No 1/2 (2002), 521 - 544.
57. Gorenflo R. and Vivoli A. Fully discrete random walks for space-time fractional diffusion equations. *Signal Processing* **83** No.11 (2003), 2411 - 2420.
58. Chechkin A.V., Gorenflo R., Sokolov I.M., Gonchar V.Yu. Distributed order time-fractional diffusion equation. *Fractional Calculus and Applied Analysis* **6** (2003), 259 - 279.
59. Gorenflo R. and Mainardi F. Fractional diffusion processes: probability distributions and continuous time random walk, In: G. Rangarajan and M. Ding (Editors), *Processes with Long Range Correlations*, Springer-Verlag, Berlin 2003, pp. 148 - 166. [Lecture Notes in Physics, No. 621]

60. Mainardi F., Pagnini G. and Gorenflo R., Mellin transform and subordination laws in fractional diffusion processes, *Fractional Calculus and Applied Analysis* **6** No. 4 (2003), 441 - 459.
61. Mainardi F., Gorenflo R. and Scalas E. A fractional generalization of the Poisson processes. *Vietnam Journal of Mathematics* **32** SI (2004), 53 - 64 .
62. Gorenflo R., Vivoli A. and Mainardi F. Discrete and continuous random walk models for space-time fractional diffusion. *Nonlinear Dynamics* **38** (2004), 101 - 116.
63. Scalas E., Gorenflo R. and Mainardi F. Uncoupled continuous-time random walks: analytic solution and limiting behaviour of the master equation. *Physical Review E* **69** (2004), 011107/1–8.
64. Umarov S. and Gorenflo R. Cauchy and nonlocal multi-point problems for distributed order pseudo-differential equations, Part One. *Zeitschrift für Analysis und ihre Anwendungen* **24** No 3 (2005), 449 - 466.
65. Mainardi F., Gorenflo R. and Vivoli A. Renewal Processes of Mittag-Leffler and Wright type. *Fractional Calculus and Applied Analysis* **8** (2005), 7 - 38.
66. Umarov S. and Gorenflo R. On multi-dimensional random walk models approximating symmetric space-fractional diffusion processes. *Fractional Calculus and Applied Analysis* **8** (2005), 73 - 86.
67. Gorenflo R. and Mainardi F. Simply and multiply scaled diffusion limits for continuous time random walks, *IOP Journal of Physics: Conference Series* **7** (2005), 1 - 16.
68. Scalas E., Gorenflo R., Luckock H., Mainardi F., Mantelli M. and Raberto M. On the intertrade waiting-time distribution. *Finance Letters* **3** No 1 (2005), 38 - 43.
69. Mainardi F., Vivoli A. and Gorenflo R. Continuous time random walk and time fractional diffusion: a numerical comparison between the fundamental solutions. *Fluctuation and Noise Letters* **5** No 2 (2005), L291 - L297.
70. Gorenflo R. and Mainardi F. Fractional relaxation of distributed order. In: M. Novak (Editor): *Complexus Mundi: Emergent Patterns in Nature* World Scientific, Singapore, 2006, pp. 33 - 42.
71. Mainardi F., Gorenflo R. and Vivoli A. Beyond the Poisson renewal process: a tutorial survey. *Journal of Computational and Applied Mathematics* **205** (2007), 725 - 735.
72. Mainardi F., Pagnini G. and Gorenflo R. Some aspects of fractional diffusion equations of single and distributed order, *Applied Mathematics and Computation* **187** (2007), 295 - 305.

73 Gorenflo R., Mainardi F. and Vivoli A. Continuous time random walk and parametric subordination in fractional diffusion. *Chaos, Solitons and Fractals* **34** (2007), 87 - 103.
74. Gorenflo R. and Abdel-Rehim E.A. Convergence of the Grünwald-Letnikov scheme for time-fractional diffusion. *J. Comput. Appl. Math.* **205** No 2 (2007), 871–881.
75. Mainardi F., Mura A., Gorenflo R. and Stojanović, M. The two forms of fractional relaxation of distributed order. *J. Vib. Control* **13** No 9/10 (2007), 1249–1268.
76. Mainardi F. and Gorenflo R. Time-fractional derivatives in relaxation processes: a tutorial survey. *Fractional Calculus and Applied Analysis* **10** No 3 (2007), 269–308.
77. Chechkin, A.V., Gonchar, V.Yu., Gorenflo, R., Korabel, N. and Sokolov, I.M. Generalized fractional diffusion equations for accelerating subdiffusion and truncated Lévy flights. *Phys. Rev. E* **78** (2008), 021111/1–13.
78. Gorenflo R. and Mainardi F. Continuous time random walk, Mittag-Leffler waiting time and fractional diffusion: mathematical aspects, Chap. 4 In R. Klages, G. Radons and I.M. Sokolov (Editors): ' *Anomalous Transport: Foundations and Applications*, Wiley-VCH, Weinheim, Germany, 2008, pp. 93 - 127.
79. Mainardi F., Mura A., Pagnini G. and Gorenflo R. Time-fractional diffusion of distributed order, *Journal of Vibration and Control* **14** (2008), 1267–1290.
80. Gorenflo R. and Mainardi F. Some recent advances in theory and simulation of fractional diffusion processes. *J. Comput. Appl. Math.* **229** (2) (2009), 400–415.
81. Stojanovic M. and Gorenflo R. Nonlinear two-term time-fractional diffusion-wave problem. *Nonlinear Analysis: Real World Applications* **11** (2010), 3512–3523.
82. Gorenflo R. Mitttag-Leffler waiting time, power laws, rarefaction, continuous time random walk, diffusion limit. Pages 1-22 in: Proceedings of the National Workshop on Fractional Calculus and Statistical Distributions, November 25-27, 2009, Publication No.41 (July 2010) of Centre for Mathematical Sciences Pala Campus, Pala/Kerala/India, edited by S.S. Pai, N. Sebastian, S.S. Nair, Dh.P. Joseph, D. Kumar.