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RESEARCH PAPER

ON FRACTIONAL ORDER DERIVATIVES AND DARBOUX PROBLEM FOR IMPLICIT DIFFERENTIAL EQUATIONS

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Abstract

In this paper we prove some relations between the Riemann-Liouville and the Caputo fractional order derivatives, and we investigate the existence and uniqueness of solutions for the initial value problems (IVP for short), for a class of functional hyperbolic differential equations by using some fixed point theorems.

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1. Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus since, starting from some speculations of G.W. Leibniz (1697) and L. Euler (1730), it has been developed up to nowadays (see [24]). The idea of fractional calculus and fractional order differential equations and inclusions has been a subject of interest not only among mathematicians, but also among physicists and engineers. Indeed, we can find

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numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [11, 13, 17, 18, 20, 21, 25]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas *et al.* [16], Miller and Ross [19], Samko *et al.* [23], the papers of Abbas and Benchohra [1, 2], Abbas *et al.* [3], Belarbi *et al.* [4], Benchohra *et al.* [5, 6, 7], Diethelm [10], Kilbas and Marzan [14, 15], Mainardi [17], Podlubny *et al.* [22], Vityuk and Golushkov [27], Yu and Gao [30], Zhang [31] and the references therein.

The Darboux problem for partial hyperbolic differential equations was studied in the papers of Abbas and Benchohra [1, 2], Vityuk [26], Vityuk and Golushkov [27], Vityuk and Mykhailenko [28, 29] and by other authors.

In the present article we are concerning by the connection between the Riemann-Liouville and the Caputo fractional derivatives and we investigate the existence and uniqueness of solutions to fractional order IVP for the system

$$\overline{D}_\theta^r u(x,y) = f(x,y,u(x,y),\overline{D}_\theta^r u(x,y));\ \text{if } (x,y)\in J := [0,a]\times[0,b], \tag{1.1}$$

$$\begin{cases} u(x,0)=\varphi(x);\ x\in[0,a],\\ u(0,y)=\psi(y);\ y\in[0,b],\\ \varphi(0)=\psi(0), \end{cases} \tag{1.2}$$

where $a,b>0,\ \theta=(0,0),\ \overline{D}_\theta^r$ is the mixed regularized derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1],\ f: J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is a given function, $\varphi\in AC([0,a])$ and $\psi\in AC([0,b])$.

We present two results for the problem (1.1)-(1.2), the first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray-Schauder type.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in J}\|w(x,y)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$ and $L^1(J)$ is the space of Lebegue-integrable functions

$w : J \to \mathbb{R}^n$ with the norm

$$\|w\|_1 = \int_0^a \int_0^b \|w(x,y)\| dy dx.$$

3. Some properties of fractional calculus operators

Definition **3.1**. [22, 23] The Riemann-Liouville fractional integral of order $\alpha \in (0, \infty)$ of a function $h \in L^1([0,b])$; $b > 0$ is defined by

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

where $\Gamma(.)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt; \ \xi > 0.$$

Definition **3.2**. [22, 23] The Riemann-Liouville fractional derivative of order $\alpha \in (0,1]$ of a function $h \in L^1([0,b])$ is defined by

$$\begin{aligned} D_0^\alpha h(t) &= \frac{d}{dt} I_0^{1-\alpha} h(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s) ds; \text{ for almost all } t \in [0,b]. \end{aligned}$$

Definition **3.3**. [8, 9, 21] The Caputo fractional derivative of order $\alpha \in (0,1]$ of a function $h \in L^1([0,b])$ is defined by

$$\begin{aligned} {}^cD_0^\alpha h(t) &= I_0^{1-\alpha} \frac{d}{dt} h(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds} h(s) ds, \text{ for almost all } t \in [0,b]. \end{aligned}$$

Definition **3.4**. [16, 23] Let $\alpha \in (0, \infty)$ and $u \in L^1(J)$. The partial Riemann-Liouville integral of order α of $u(x,y)$ with respect to x is defined by the expression

$$I_{0,x}^\alpha u(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s,y) ds,$$

for almost all $x \in [0,a]$ and almost all $y \in [0,b]$.

Analogously, we define the integral

$$I_{0,y}^\alpha u(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^y (y-s)^{\alpha-1} u(x,s) ds,$$

for almost all $x \in [0,a]$ and almost all $y \in [0,b]$.

Definition **3.5**. [16, 23] Let $\alpha \in (0,1]$ and $u \in L^1(J)$. The Riemann-Liouville fractional derivative of order α of $u(x,y)$ with respect to x is defined by

$$(D^{\alpha}_{0,x}u)(x,y) = \frac{\partial}{\partial x} I^{1-\alpha}_{0,x} u(x,y),$$

for almost all $x \in [0,a]$ and almost all $y \in [0,b]$.

Analogously, we define the derivative

$$(D^{\alpha}_{0,y}u)(x,y) = \frac{\partial}{\partial y} I^{1-\alpha}_{0,y} u(x,y),$$

for almost all $x \in [0,a]$ and almost all $y \in [0,b]$.

Definition **3.6**. [16, 23] Let $\alpha \in (0,1]$ and $u \in L^1(J)$. The Caputo fractional derivative of order α of $u(x,y)$ with respect to x is defined by the expression

$${}^{c}D^{\alpha}_{0,x}u(x,y) = I^{1-\alpha}_{0,x} \frac{\partial}{\partial x} u(x,y),$$

for almost all $x \in [0,a]$ and almost all $y \in [0,b]$.

Analogously, we define the derivative

$${}^{c}D^{\alpha}_{0,y}u(x,y) = I^{1-\alpha}_{0,y} \frac{\partial}{\partial y} u(x,y),$$

for almost all $x \in [0,a]$ and almost all $y \in [0,b]$.

Definition **3.7**. [27] Let $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, $\theta = (0,0)$ and $u \in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I^{r}_{\theta}u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t)\,dt\,ds.$$

In particular,

$$(I^{\theta}_{\theta}u)(x,y) = u(x,y), \ (I^{\sigma}_{\theta}u)(x,y) = \int_0^x \int_0^y u(s,t)\,dt\,ds;$$

for almost all $(x,y) \in J$, where $\sigma = (1,1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_\theta^r u) \in C(J)$, moreover

$$(I_\theta^r u)(x,0) = (I_\theta^r u)(0,y) = 0;\ x \in [0,a],\ y \in [0,b].$$

EXAMPLE **3.1**. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty)\times(0,\infty)$, then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2}, \text{ for almost all } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in (0,1]\times(0,1]$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

DEFINITION **3.8**. [27] Let $r \in (0,1]\times(0,1]$ and $u \in L^1(J)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression $D_\theta^r u(x,y) = (D_{xy}^2 I_\theta^{1-r} u)(x,y)$ and the Caputo fractional-order derivative of order r of u is defined by the expression ${}^cD_\theta^r u(x,y) = (I_\theta^{1-r} D_{xy}^2 u)(x,y)$.

The case $\sigma = (1,1)$ is included and we have

$$(D_\theta^\sigma u)(x,y) = ({}^cD_\theta^\sigma u)(x,y) = (D_{xy}^2 u)(x,y), \text{ for almost all } (x,y) \in J.$$

EXAMPLE **3.2**. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,1]\times(0,1]$, then

$$D_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2}, \text{ for almost all } (x,y) \in J.$$

DEFINITION **3.9**. [29] For a function $u : J \to \mathbb{R}^n$, we set

$$q(x,y) = u(x,y) - u(x,0) - u(0,y) + u(0,0).$$

By the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1]\times(0,1]$ of a function $u(x,y)$, we name the function

$$\overline{D}_\theta^r u(x,y) = D_\theta^r q(x,y).$$

The function

$$\overline{D}_{0,x}^{r_1} u(x,y) = D_{0,x}^{r_1}[u(x,y) - u(0,y)],$$

is called the partial r_1-order regularized derivative of the function $u(x,y) : J \to \mathbb{R}^n$ with respect to the variable x. Analogously, we define the derivative

$$\overline{D}^{r_2}_{0,y}u(x,y) = D^{r_2}_{0,y}[u(x,y) - u(x,0)].$$

4. Riemann-Liouville and Caputo partial fractional derivatives

For a function $h \in L^1([0,b])$; $b > 0$ and $\alpha \in (0,1]$. The connection between D^α_0 and ${}^cD^\alpha_0$ is given by

$$ {}^cD^\alpha_0 h(t) = D^\alpha_0 h(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} h(0^+), \text{ for almost all } t \in [0,b]. \tag{4.1}$$

For more details, see [16].

Corollary **4.1**. *For a function $u \in L^1(J)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$. The connection between $D^{r_1}_{0,x}u(x,y)$ and ${}^cD^{r_1}_{0,x}u(x,y)$ with respect to x is given by*

$$({}^cD^{r_1}_{0,x}u)(x,y) = (D^{r_1}_{0,x}u)(x,y) - \frac{x^{-r_1}}{\Gamma(1-r_1)}u(0^+,y). \tag{4.2}$$

Analogously, the connection between $D^{r_2}_{0,y}u(x,y)$ and ${}^cD^{r_2}_{0,y}u(x,y)$ with respect to y is given by

$$({}^cD^{r_2}_{0,y}u)(x,y) = (D^{r_2}_{0,y}u)(x,y) - \frac{y^{-r_2}}{\Gamma(1-r_2)}u(x,0^+). \tag{4.3}$$

Now, let us give the relation between D^r_θ and ${}^cD^r_\theta$, where $r = (r_1, r_2) \in (0,1] \times (0,1]$.

Theorem **4.1**. *For $u \in AC(J)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$ we have*

$$\begin{aligned}({}^cD^r_\theta u)(x,y) &= \overline{D}^r_\theta u(x,y) = (D^r_\theta u)(x,y) - \frac{x^{-r_1}}{\Gamma(1-r_1)}(D^{r_2}_{0,y}u)(0,y) \\ &\quad - \frac{y^{-r_2}}{\Gamma(1-r_2)}(D^{r_1}_{0,x}u)(x,0) + \frac{x^{-r_1}y^{-r_2}}{\Gamma(1-r_1)\Gamma(1-r_2)}u(0,0).\end{aligned}$$

P r o o f. According to [[28], Lemma 1] $(\overline{D}^r_\theta u)(x,y) = ({}^cD^r_\theta u)(x,y)$. Then

$$(\overline{D}^r_\theta u)(x,y) = (D^r_\theta q)(x,y) = D_{xy}I^{1-r}_\theta q(x,y),$$
$$q(x,y) = u(x,y) - \gamma(x,y),\ \gamma(x,y) = u(x,0) + u(0,y) - u(0,0),$$
$$I^{1-r}_\theta q(x,y) = I^{1-r}_\theta u(x,y) - I^{1-r}_\theta \gamma(x,y).$$

As $u(x,y) = \gamma(x,y) + I_\theta^\sigma v(x,y)$, then $q(x,y) = I_\theta^\sigma v(x,y)$, where $v(x,y) = D_{xy}u(x,y)$. Hence

$$\begin{aligned} I_\theta^{1-r}q(x,y) &= I_\theta^{1-r}(I_\theta^\sigma v(x,y)) \\ &= \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)}\int_0^x\int_0^y\Big(\int_0^s\int_0^t(s-z)^{-r_1}(t-\tau)^{-r_2} \\ &\times \quad v(\tau,z)dzd\tau\Big)dtds \in AC(J), \end{aligned}$$

$$\begin{aligned} I_\theta^{1-r}\gamma(x,y) &= \frac{y^{1-r_2}}{(1-r_2)\Gamma(1-r_2)}I_{0,x}^{1-r_1}u(x,0) \\ &+\frac{x^{1-r_1}}{(1-r_1)\Gamma(1-r_1)}I_{0,y}^{1-r_2}u(0,y) \\ &-\frac{x^{1-r_1}y^{1-r_2}}{(1-r_1)(1-r_2)\Gamma(1-r_1)\Gamma(1-r_2)}u(0,0), \end{aligned}$$

besides [[23], Lemma 2.1] $I_\theta^{1-r}\gamma(x,y) \in AC(J)$.

Then

$$I_\theta^{1-r}u(x,y) = I_\theta^{1-r}q(x,y) + I_\theta^{1-r}\gamma(x,y) \in AC(J).$$

Finally

$$\begin{aligned} \overline{D}_\theta^r u(x,y) &= D_{xy}I_\theta^{1-r}q(x,y) = (D_\theta^r u)(x,y) - \frac{y^{-r_2}}{\Gamma(1-r_2)}(D_{0,x}^{r_1}u)(x,0) \\ &-\frac{x^{-r_1}}{\Gamma(1-r_1)}(D_{0,y}^{r_2}u)(0,y) + \frac{x^{-r_1}y^{-r_2}}{\Gamma(1-r_1)\Gamma(1-r_2)}u(0,0). \end{aligned}$$

□

5. Existence of solutions

Let us start by defining what we mean by a solution of the problem (1.1)-(1.2).

DEFINITION **5.1**. A function $u \in C(J)$ such that $u(x,y)$, $\overline{D}_{0,x}^{r_1}u(x,y)$, $\overline{D}_{0,y}^{r_2}u(x,y), \overline{D}_\theta^r u(x,y)$ are continuous for $(x,y) \in J$ and $I_\theta^{1-r}u(x,y) \in AC(J)$ is said to be a solution of (1.1)-(1.2) if u satisfies equation (1.1) and conditions (1.2) on J.

For the existence of solutions for the problem (1.1)-(1.2) we need the following lemma.

LEMMA **5.1**. [29] *Let a function* $f(x,y,u,z) : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous. Then problem (1.1)-(1.2) is equivalent to the problem of the solution of the equation*

$$g(x,y) = f(x,y,\mu(x,y) + I_\theta^r g(x,y), g(x,y)),$$

and if $g \in C(J)$ *is the solution of this equation, then* $u(x,y) = \mu(x,y) + I_\theta^r g(x,y)$, *where*

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Further, we present conditions for the existence and uniqueness of a solution of problem (1.1)-(1.2).

THEOREM **5.1**. *Assume:*

(H_1) *The function* $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous,*

(H_2) *For any* $u, v, w, z \in \mathbb{R}^n$ *and* $(x,y) \in J$, *there exist constants* $k > 0$ *and* $0 < l < 1$ *such that*

$$\|f(x,y,u,z) - f(x,y,v,w)\| \le k\|u - v\| + l\|z - w\|.$$

If

$$\frac{k a^{r_1} b^{r_2}}{(1-l)\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{5.1}$$

then there exists a unique solution for IVP *(1.1)-(1.2) on J.*

P r o o f. Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $N : C(J) \to C(J)$ defined by,

$$N(u)(x,y) = \mu(x,y) + I_\theta^r g(x,y), \tag{5.2}$$

where $g \in C(J)$ such that

$$g(x,y) = f(x,y,u(x,y), g(x,y)).$$

By Lemma 5.1, the problem of finding the solutions of the IVP (1.1)-(1.2) is reduced to finding the solutions of the operator equation $N(u) = u$.

Let $v, w \in C(J)$. Then, for $(x,y) \in J$, we have

$$\|N(v)(x,y) - N(w)(x,y)\| \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \|g(s,t) - h(s,t)\| dt ds, \tag{5.3}$$

where $g, h \in C(J)$ such that

$$g(x,y) = f(x,y,v(x,y), g(x,y))$$

and

$$h(x,y)=f(x,y,w(x,y),h(x,y)).$$

By (H_2), we get

$$\|g(x,y)-h(x,y)\|\leq k\|v(x,y)-w(x,y)\|+l\|g(x,y)-h(x,y)\|.$$

Then

$$\begin{aligned}\|g(x,y)-h(x,y)\| &\leq \frac{k}{1-l}\|v(x,y)-w(x,y)\|\\ &\leq \frac{k}{1-l}\|v-w\|_\infty.\end{aligned}$$

Thus, (5.3) implies that

$$\begin{aligned}\|N(v)-N(w)\|_\infty &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\\ &\times \frac{k}{1-l}\|v-w\|_\infty dtds\\ &\leq \frac{ka^{r_1}b^{r_2}}{(1-l)\Gamma(1+r_1)\Gamma(1+r_2)}\|v-w\|_\infty.\end{aligned}$$

Hence

$$\|N(v)-N(w)\|_\infty\leq\frac{ka^{r_1}b^{r_2}}{(1-l)\Gamma(1+r_1)\Gamma(1+r_2)}\|v-w\|_\infty.$$

By (5.1), N is a contraction, and hence N has a unique fixed point by Banach's contraction principle. □

THEOREM **5.2**. [12] (Nonlinear alternative of Leray-Schauder type) *Let X be a Banach space and C a nonempty convex subset of X. Let U a nonempty open subset of C with $0\in U$ and $T:\overline{U}\to C$ continuous and compact operator.*

Then,

(a) either T has fixed points,

(b) or there exist $u\in\partial U$ and $\lambda\in[0,1]$ with $u=\lambda T(u)$.

THEOREM **5.3**. *Assume (H_1) and the following hypothesis holds:*

(H_3) There exist $p,q,d\in C(J,\mathbb{R}_+)$ such that

$$\|f(x,y,u,z)\|\leq p(x,y)+q(x,y)\|u\|+d(x,y)\|z\|$$

for $(x,y)\in J$ and each $u,z\in\mathbb{R}^n$.

If

$$d^*+\frac{q^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}<1, \tag{5.4}$$

where $d^* = \sup_{(x,y)\in J} d(x,y)$ *and* $q^* = \sup_{(x,y)\in J} q(x,y)$, *then the IVP (1.1)-(1.2) has at least one solution on J.*

P r o o f. Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator N defined in (5.2). We shall show that the operator N is continuous and compact.

Step 1: N is continuous.
Let $\{u_n\}_{n\in\{\mathcal{N}}$ be a sequence such that $u_n \to u$ in $C(J)$. Let $\eta > 0$ be such that $\|u_n\| \leq \eta$. Then

$$\|N(u_n)(x,y) - N(u)(x,y)\| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \|g_n(s,t) - g(s,t)\| dtds, \tag{5.5}$$

where $g_n, g \in C(J)$ such that

$$g_n(x,y) = f(x,y,u_n(x,y),g_n(x,y))$$

and

$$g(x,y) = f(x,y,u(x,y),g(x,y)).$$

Since $u_n \to u$ as $n \to \infty$ and f is a continuous function, we get

$$g_n(x,y) \to g(x,y) \text{ as } n \to \infty, \text{ for each } (x,y) \in J.$$

Hence, (5.5) gives

$$\|N(u_n) - N(u)\|_\infty \leq \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|g_n - g\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2: N maps bounded sets into bounded sets in $C(J)$.
Indeed, it is enough show that for any $\eta^* > 0$, there exists a positive constant M^* such that, for each

$$u \in B_{\eta^*} = \{u \in C(J) : \|u\|_\infty \leq \eta^*\},$$

we have $\|N(u)\|_\infty \leq M^*$. For $(x,y) \in J$, we have

$$\|N(u)(x,y)\| \leq \|\mu(x,y)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \times \|g(s,t)\| dtds, \tag{5.6}$$

where $g \in C(J)$ such that

$$g(x,y) = f(x,y,u(x,y),g(x,y)).$$

By (H_3) we have for each $(x,y) \in J$,

$$\begin{aligned} \|g(x,y)\| &\leq p(x,y) + q(x,y)\|\mu(x,y) + I_\theta^r g(x,y)\| + d(x,y)\|g(x,y)\| \\ &\leq p^* + q^*\Big(\|\mu\|_\infty + \frac{a^{r_1}b^{r_2}\|g(x,y)\|}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big) + d^*\|g(x,y)\|, \end{aligned}$$

where $p^* = \sup_{(x,y)\in J} p(x,y)$. Then, by (5.4) we have

$$\|g\|_\infty \quad \leq \quad \frac{p^* + q^*\|\mu\|_\infty}{1 - d^* - \frac{q^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}} := M.$$

Thus, (5.6) implies that

$$\|N(u)\|_\infty \leq \|\mu\|_\infty + \frac{M a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} := M^*.$$

Step 3: N maps bounded sets into equicontinuous sets in $C(J)$.
Let $(x_1, y_1), (x_2, y_2) \in J$, $x_1 < x_2$, $y_1 < y_2$, B_{η^*} be a bounded set of $C(J)$ as in Step 2, and let $u \in B_{\eta^*}$. Then

$$\begin{aligned}
&\|N(u)(x_2,y_2) - N(u)(x_1,y_1)\| \\
&\leq \|\mu(x_2,y_2) - \mu(x_1,y_1)\| \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} \\
&-(x_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\|g(s,t)\|dtds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds,
\end{aligned}$$

where $g \in C(J)$ such that

$$g(x,y) = f(x,y,u(x,y),g(x,y)).$$

But $\|g\|_\infty \leq M$. Thus

$$\begin{aligned}
&\|N(u)(x_2,y_2) - N(u)(x_1,y_1)\| \leq \|\mu(x_2,y_2) - \mu(x_1,y_1)\| \\
+ \quad &\frac{M}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big[2y_2^{r_2}(x_2-x_1)^{r_1} + 2x_2^{r_1}(y_2-y_1)^{r_2} \\
+ \quad &x_1^{r_1}y_1^{r_2} - x_2^{r_1}y_2^{r_2} - 2(x_2-x_1)^{r_1}(y_2-y_1)^{r_2}\Big].
\end{aligned}$$

As $x_1 \to x_2$, $y_1 \to y_2$ the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous.

Step 4: *A priori bounds.*
We now show there exists an open set $U \subseteq C(J)$ with $u \neq \lambda N(u)$, for

$\lambda \in (0,1)$ and $u \in \partial U$. Let $u \in C(J)$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(x,y) \in J$, we have

$$u(x,y) = \lambda\mu(x,y) + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)dtds.$$

This implies by (H_3) and as in step 2 that, for each $(x,y) \in J$, we get $\|u\| \leq M^*$.
Set

$$U = \{u \in C(J) : \|u\|_\infty < M^* + 1\}.$$

By our choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of Theorem **5.2**, we deduce that N has a fixed point u in $\overline{U}$ which is a solution to problem (1.1)-(1.2). □

6. An example

As an application of our results we consider the following partial hyperbolic functional differential equations of the form

$$\overline{D}_\theta^r u(x,y) = \frac{1}{5e^{x+y+2}(1 + |u(x,y)| + |\overline{D}_\theta^r u(x,y)|)};$$

$$\text{if } (x,y) \in [0,1] \times [0,1], \tag{6.1}$$

$$u(x,0) = x,\ u(0,y) = y^2;\ x, y \in [0,1]. \tag{6.2}$$

Set

$$f(x,y,u(x,y),\overline{D}_\theta^r u(x,y)) = \frac{1}{5e^{x+y+2}(1 + |u(x,y)| + |\overline{D}_\theta^r u(x,y)|)};$$

for $(x,y) \in [0,1] \times [0,1]$. Clearly, the function f is continuous. For each $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ and $(x,y) \in [0,1] \times [0,1]$ we have

$$|f(x,y,u,v) - f(x,y,\overline{u},\overline{v})| \leq \frac{1}{5e^2}(\|u - \overline{u}\| + \|v - \overline{v}\|).$$

Hence condition (H_2) is satisfied with $k = l = \frac{1}{5e^2}$. We shall show that condition (5.1) holds with $a = b = 1$. Indeed

$$\frac{k a^{r_1} b^{r_2}}{(1-l)\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{(5e^2 - 1)\Gamma(1+r_1)\Gamma(1+r_2)} < 1,$$

which is satisfied for each $(r_1, r_2) \in (0,1] \times (0,1]$. Consequently Theorem **5.1** implies that problem (6.1)-(6.2) has a unique solution defined on $[0,1] \times [0,1]$.

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