RESEARCH PAPER

INFINITELY MANY SIGN-CHANGING SOLUTIONS FOR THE BRÉZIS-NIRENBERG PROBLEM INVOLVING THE FRACTIONAL LAPLACIAN

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Abstract

In this paper, we consider the following Brézis-Nirenberg problem involving the fractional Laplacian operator:

\[
\begin{cases}
(\mathcal{L}^\alpha)u = \lambda u + |u|^{2^*_s - 2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( s \in (0, 1) \), \( \Omega \) is a bounded smooth domain of \( \mathbb{R}^N \) (\( N > 6s \)) and \( 2^*_s = \frac{2N}{N-2s} \) is the critical fractional Sobolev exponent. We show that, for each \( \lambda > 0 \), this problem has infinitely many sign-changing solutions by using a compactness result obtained in [34] and a combination of invariant sets method and Ljusternik-Schnirelman type minimax method.

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1. Introduction and main results

In this paper, we consider the following nonlinear problem with the fractional Laplacian

\[
\begin{cases}
(\mathcal{L}^\alpha)u = \lambda u + |u|^{2^*_s - 2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.1)
where $\lambda > 0$, $0 < s < 1$, $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$ and $2^*_s = \frac{2N}{N-2s}$ is the critical exponent in fractional Sobolev inequalities. Here the fractional Laplacian $(-\Delta)^s$ is defined as follows.

Let $\{\varphi_k\}$ be an orthonormal basis of $L^2(\Omega)$ with $\|\varphi_k\|_{L^2(\Omega)} = 1$ forming a spectral decomposition of $-\Delta$ in $\Omega$ with zero Dirichlet boundary data and $\lambda_k$ be the corresponding eigenvalues, i.e. $-\Delta \varphi_k = \lambda_k \varphi_k$ in $\Omega$ with $\varphi_k = 0$ on $\partial \Omega$. Let $0 < s < 1$ and

$$H_{0}^{s}(\Omega) = \left\{ u = \sum_{k=1}^{\infty} u_k \varphi_k \in L^2(\Omega) : \|u\|_{H_{0}^{s}(\Omega)} = \left( \sum_{k=1}^{\infty} \lambda_k^s u_k^2 \right)^{1/2} < \infty \right\}$$

be the fractional Sobolev space (see [2, 37]) with inner product

$$(u, v)_{H_{0}^{s}(\Omega)} = \sum_{k=0}^{\infty} \lambda_k^s u_k v_k = \int_{\Omega} \left( (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} v \right) dx.$$  

It is not difficult to see that $H_{0}^{s}(\Omega)$ is a Hilbert space. For any $u \in H_{0}^{s}(\Omega)$, $u = \sum_{k=1}^{\infty} u_k \varphi_k$ with $u_k = \int_{\Omega} u \varphi_k dx$, the spectral fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s u = \sum_{k=1}^{\infty} \lambda_k^s u_k \varphi_k.$$  

We say that $\{(\varphi_k, \lambda_k^s)\}$ are the eigenfunctions and eigenvalues of $(-\Delta)^s$ in $\Omega$ with zero Dirichlet boundary data. In the pioneering work [6], Brézis and Nirenberg considered the existence solutions of equation (1.1) with $s = 1$. They show that for $\lambda > 0$ the problem

$$\begin{cases} 
-\Delta u = \lambda u + |u|^{2^*-2} u & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $2^* = \frac{2N}{N-2}$, may admit non-trivial solutions under a subtle dependence on the dimension $N \geq 3$. After that, there are many results on this problem. See e.g. [14] [15] [17] [28] and the references therein. In particular, in [15], Devillanova and Solimini showed that, when $N \geq 7$, problem (1.1) with $s = 1$ has infinitely many solutions for each $\lambda > 0$ by using the uniform bounded theorem (see Theorem 1.1 in [18]). Recently, under the same assumptions of [18], Schechter and Zou [23] proved that this problem has infinitely many sign-changing solutions by combining the estimates of Morse indices of nodal solutions with the uniform bounded theorem due to Devillanova and Solimini [18].

Nonlinear problems involving the fractional Laplacian have been extensively studied recently. Caffarelli et al. [8] [9] studied the free boundary problem for the fractional Laplacian. Silvestre [27] investigated the regularity of the obstacle problem for the fractional Laplace operator. In [10],
Caffarelli and Silvestre given a new local realization of the fractional Laplacian \((-\Delta)^s\) by introducing the so-called \(s\)-harmonic extension. After that, several authors, using the localization method, have extended some results of the classical elliptic problems to the fractional Laplacian, see for example \([2, 5, 7, 13, 16, 31, 34, 35, 36, 24, 25, 26]\) and the references therein. In particular, Chang and Wang \([16]\), using the method of invariant sets of descending flow, obtained the existence and multiplicity of nodal solutions for the elliptic equaitons involving the fractional Laplacian \((-\Delta)^s\) for all \(s \in (0, 1)\) with subcritical nonlinearities; for the Brézis-Nirenberg type problem involving the fractional Laplacian \((-\Delta)^s\), Tan \([31]\) proved the existence of positive solutions with the special case \(s = \frac{1}{2}\) and Barrios et al. \([2]\) studied the general case with \(0 < s < 1\). For any \(\lambda > 0\), Yan et al. \([34]\) proved that problem \((1.1)\) possesses infinitely many solutions by using a compactness result for the subcritical perturbed problem associated to \((1.1)\). In \([21]\) the authors study bifurcation and multiplicity of solutions for the fractional Laplacian with critical exponential nonlinearity using critical point theorem of Bartolo, Benci and Fortunato \([3]\). Multiplicity of solutions for fractional differential equations via variational method is studied in \([1, 32, 37]\).

A natural question is whether problem \((1.1)\) has infinitely many sign-changing solutions for each \(\lambda > 0\) and \(s \in (0, 1)\). To the best of our knowledge, there is no result in the literature concerning this question. In this paper, we give a positive answer to this open question. The main result of this paper is the following.

**Theorem 1.1.** Suppose that \(N > 6s\) and \(\lambda > 0\), then problem \((1.1)\) has infinitely many sign-changing solutions.

**Remark 1.1.** Denote \(\lambda^*_1\) the first eigenvalue of \((-\Delta)^s\) in \(\Omega\) with zero Dirichlet boundary condition. Multiplying the first eigenfunction and integrating both sides, one can easily check that if \(\lambda \geq \lambda^*_1\), any nontrivial solution of \((1.1)\) is sign-changing. Therefore, by the results of \([34]\), to prove Theorem 1.1 it suffices to consider the case of \(\lambda \in (0, \lambda^*_1)\).

Theorem 1.1 extends the result in \([23]\) to the fractional Laplacian. Motivated by \([30]\) which used the more simple proof than \([23]\) to obtain the same result, we will prove Theorem 1.1 by applying the usual Ljusternik-Schnirelman type minimax method in conjunction with invariant set method. However, due to the fact that the operator \((-\Delta)^s\) is nonlocal, the techniques of constructing invariant sets of descending flow in \([4, 19, 20]\) cannot be directly applied to problem \((1.1)\). In order to construct
invariant sets, we adopt an idea from [13, 16] to introduce an auxiliary operator $A_\varepsilon$ (see Section 3) associated to the subcritical perturbed problem (2.3). Then we can follow the same way as in [30] with the help of the compactness result (Theorem 2.1, see Section 2) due to Yan et al. [34] to obtain Theorem 1.1.

This paper is organized as follows. In Section 2, we describe a variational setting of the problem and state a compactness result due to Yan et al. [34] for the solutions of the perturbed problem (2.4). In Section 3, we introduce an auxiliary operator $A_\varepsilon$ and then construct the invariant sets, the proof of Theorem 1.1 is given at the end of this section.

2. Preliminaries and functional setting

Denote $H^{-s}(\Omega)$ the dual space of $H_0^s(\Omega)$. Define the inner product in $H_0^s(\Omega)$ by

$$(u,v)_{H_0^s(\Omega)} := \int_\Omega (-\Delta)^{s/2}u(-\Delta)^{s/2}v dx.$$ 

**Definition 2.1.** We say that $u \in H_0^s(\Omega)$ is a weak solution of (1.1) if the identity

$$\int_\Omega (-\Delta)^{s/2}u(-\Delta)^{s/2}\phi dx = \int_\Omega (\lambda u\phi + |u|^{2^*_s-2}u\phi) dx$$

holds for every $\phi \in H_0^s(\Omega)$.

Note that the right hand side of the identity in the above definition is well defined, since $\phi \in H_0^s(\Omega) \hookrightarrow L^{2^*_s}(\Omega)$, and for $u \in H_0^s(\Omega)$, $\lambda u + |u|^{2^*_s-2}u \in L^{2^*_s}(\Omega)$. It is standard (see e.g. [22]) to show that the weak solutions of problem (1.1) correspond to the critical points of the energy functional $I : H_0^s(\Omega) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \int_\Omega |(-\Delta)^{s/2}u|^2 dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx - \frac{1}{2^*_s} \int_\Omega |u|^{2^*_s} dx, \quad \forall u \in H_0^s(\Omega).$$

Clearly, $I \in C^1(H_0^s(\Omega), \mathbb{R})$.

Define $\mathbb{R}^{N+1}_+ = \{(x,y) : x \in \mathbb{R}^N, y > 0\}$, the upper half space in $\mathbb{R}^{N+1}$. Associate to the bounded domain $\Omega$, we consider the cylinder $\mathcal{C} = \Omega \times (0, \infty) \subset \mathbb{R}^{N+1}_+$ and denote its lateral boundary by $\partial L \mathcal{C} = \partial \Omega \times [0, \infty)$.

Note that $(-\Delta)^s$ is a nonlocal operator, motivated by the work of Caffarelli and Silvestre [10]. Using the so-called $s$-harmonic extension, several authors have considered an equivalent definition of the operator $(-\Delta)^s$ defined through the spectral decomposition as above. Then the nonlocal problems can be transformed into a local problem see e.g. [2, 5, 7, 13, 16, 34, 35].
For a given $u \in H_0^s(\Omega)$, we define its $s$-harmonic extension $w = E_s(u)$ to $\mathcal{C}$ as the solution of the problem
\[
\begin{cases}
-\text{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathcal{C}, \\
w = 0 & \text{on } \partial L \mathcal{C}, \\
w(x,0) = u & \text{on } \Omega.
\end{cases}
\] (2.1)

Following [10], we can define the fractional Laplacian operator by the Dirichlet to Neumann map as follows.

**Definition 2.2.** For any $u \in H_0^s(\Omega)$, the fractional Laplacian $(-\Delta)^s$ acting on $u$ is defined by
\[
(-\Delta)^s u(x) := -\frac{1}{k_s} \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x,y), \quad \forall \ x \in \Omega,
\]
where $w = E_s(u)$ and $k_s = \frac{2^{1-2s} \Gamma(1-s)}{\Gamma(s)}$ is a normalization constant.

Define $H_{0,L}^s(\mathcal{C})$ as the closure of $C_0^\infty(\mathcal{C})$ under the norm
\[
\|w\|_{H_{0,L}^s(\mathcal{C})} = \left( k_s \int_{\mathcal{C}} y^{1-2s} |\nabla w|^2 dxdy \right)^{1/2}.
\]

Denote by $tr_\Omega$ the trace operator on $\Omega \times \{0\}$ for functions in $H_{0,L}^s(\mathcal{C})$:
\[
tr_\Omega w = w(\cdot,0), \quad \text{for } w \in H_{0,L}^s(\mathcal{C}).
\]
Then for any $w \in H_{0,L}^s(\mathcal{C})$, the following trace inequality holds
\[
\|tr_\Omega w\|_{H_0^s(\Omega)} \leq \|w\|_{H_{0,L}^s(\mathcal{C})}.
\]
Moreover, we have the following result (see [2, 16]).

**Lemma 2.1 (Lemma 2.3, [16]).** (i) $E_s(\cdot)$ is an isometry between $H_0^s(\Omega)$ and $H_{0,L}^s(\mathcal{C})$, that is
\[
\|u\|_{H_0^s(\Omega)} = \|E_s(u)\|_{H_{0,L}^s(\mathcal{C})};
\]
(ii) For any $w \in H_{0,L}^s(\mathcal{C})$, there exists a constant $C$ independent of $w$ such that
\[
\|tr_\Omega w\|_{L^r(\Omega)} \leq C \|w\|_{H_{0,L}^s(\mathcal{C})}
\]
holds for all $r \in [2, 2^*_s]$. Moreover, $H_{0,L}^s(\mathcal{C})$ is compactly embedded into $L^r(\Omega)$ for every $r \in [2, 2^*_s]$.

Set
\[
\partial_s^\nu w(x) := -\frac{1}{k_s} \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x,y).
\]
With the above extension, from [10], we can transform the nonlocal problem (1.1) into the following local problem

\[
\begin{aligned}
& -\text{div}(y^{1-2s}\nabla w) = 0 & \quad & \text{in } C, \\
& w = 0 & \quad & \text{on } \partial_L C, \\
& \partial_C^s w = \lambda w(x,0) + |w(x,0)|^{2^*_s-2}w(x,0) & \quad & \text{on } \Omega.
\end{aligned}
\]

(2.2)

A weak solution to this problem is a function \( w \in H^s_{0,L}(C) \) such that

\[
 k_s \int_C y^{1-2s}(\nabla w, \nabla \psi) \, dx \, dy = \int_{\Omega} (\lambda w(x,0) + |w(x,0)|^{2^*_s}w(x,0)) \text{tr}_\Omega \psi \, dx,
\]

for all \( \psi \in H^s_{0,L}(C) \). Then, critical points of the functional

\[
 J(w) = \frac{k_s}{2} \int_C y^{1-2s}|\nabla w|^2 \, dx \, dy - \frac{\lambda}{2} \int_{\Omega} |w(x,0)|^2 \, dx - \frac{1}{2s^*_2} \int_{\Omega} |w(x,0)|^{2^*_s} \, dx,
\]

defined on \( H^s_{0,L}(C) \) correspond to the solutions of (2.2). For any weak solution \( w \in H^s_{0,L}(C) \) to (2.2), the function \( u = \text{tr}_\Omega w \in H^s_0(\Omega) \) is a weak solution of problem (1.1) and is a critical point of \( I \). The converse is also true. Therefore, these two formulations are equivalent, and we will use both formulations in the sequel.

Given \( \varepsilon > 0 \) small enough, associated to problems (1.1) and (2.2), we consider the following subcritical perturbed nonlocal problem:

\[
\begin{aligned}
& (-\Delta)^s u = \lambda u + |u|^{2^*_s-2-\varepsilon} u & \quad & \text{in } \Omega, \\
& u = 0 & \quad & \text{on } \partial \Omega,
\end{aligned}
\]

(2.3)

and the local problem

\[
\begin{aligned}
& -\text{div}(y^{1-2s}\nabla w) = 0 & \quad & \text{in } C, \\
& w = 0 & \quad & \text{on } \partial_L C, \\
& \partial_C^s w = \lambda w(x,0) + |w(x,0)|^{2^*_s-2-\varepsilon}w(x,0) & \quad & \text{on } \Omega.
\end{aligned}
\]

(2.4)

The functional \( I_\varepsilon : H^s_0(\Omega) \to \mathbb{R} \) corresponding to (2.3) is defined as follows

\[
 I_\varepsilon(u) = \frac{1}{2} \int_{\Omega} |(-\Delta)^{s/2} u|^2 \, dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 \, dx - \frac{1}{2s^*_2} \int_{\Omega} |u|^{2^*_s-\varepsilon} \, dx, \forall u \in H^s_0(\Omega).
\]

It is easy to check that \( I_\varepsilon \in C^1(\mathcal{H}^s_0(\Omega), \mathbb{R}) \).

Now we state the following compactness result due to Yan et al. [31], which plays a very important role in our proof.

\textbf{Theorem 2.1} (\textit{[31], Theorem 1.1}). \textit{Suppose } \( N > 6s \) \textit{and } \( \lambda > 0 \). \textit{Assume that } \( w_n(n = 1, 2, \cdots) \) \textit{is a nontrivial solution of (2.4) with } \( \varepsilon = \varepsilon_n > 0 \), \textit{and } \{w_n\}_{n \in \mathbb{N}} \textit{satisfies } \|w_n\| \leq C \textit{for some positive constant independent of } n. \textit{Then } \{w_n\}_{n \in \mathbb{N}} \textit{possesses a subsequence which converges strongly in } H^s_{0,L}(\Omega) \textit{as } n \to \infty. \)
3. Proof of the main result

In this section, we will prove Theorem 1.1.

3.1. Some technical lemmas. Let

\[ 0 < \lambda_1^s < \lambda_2^s \leq \ldots \leq \lambda_m^s \leq \ldots \]

be the eigenvalues of \((-\Delta)^s, H_0^s(\Omega))\) introduced in Section 1 and \(\varphi_m\) be the eigenfunction corresponding to \(\lambda_m^s\). Denote

\[ E_m := \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_m\}. \]

Fix \(\zeta \in (2, 2^*_s)\). In the following, we will always assume that

\[ \lambda \in (0, \lambda_1^s) \quad \text{and} \quad \varepsilon \in (0, 2^*_s - \zeta). \]

In order to construct the minimax values for the perturbed functional \(I_\varepsilon\), the following three technical lemmas are needed.

**Lemma 3.1.** For any \(\varepsilon \in (0, 2^*_s - \zeta)\), the functional \(I_\varepsilon\) satisfies the Palais-Smale (PS) condition.

**Proof.** Suppose \(\{u_n\} \subset H_0^s(\Omega)\) is a (PS) sequence for \(I_\varepsilon\), i.e.,

\[ I_\varepsilon(u_n) \to c \in \mathbb{R} \quad \text{and} \quad I_\varepsilon'(u_n) \to 0 \quad \text{as} \quad n \to \infty. \]

We have

\[ c + o(1)(1 + \|u_n\|_{H_0^s(\Omega)}) = I_\varepsilon(u_n) - \frac{1}{2s}(I_\varepsilon'(u_n), u_n)_{H^{-s}(\Omega), H_0^s(\Omega)} \]

\[ = \left( \frac{1}{2} - \frac{1}{2s} \right) \left( \|u_n\|_{H_0^s(\Omega)}^2 - \lambda \|u_n\|_{H_0^s(\Omega)}^2 \right) \]

\[ \geq \left( \frac{1}{2} - \frac{1}{2s} \right) \frac{\lambda^s - \lambda}{\lambda_1^s} \|u_n\|_{H_0^s(\Omega)}^2, \]

which implies that \(\{u_n\}\) is bounded sequence in \(H_0^s(\Omega)\). Hence, there exists \(u_0 \in H_0^s(\Omega)\) such that

\[ u_n \rightharpoonup u_0 \quad \text{in} \quad H_0^s(\Omega). \]

By Lemma 2.1, we have

\[ u_n \to u_0 \quad \text{in} \quad L^r(\Omega) \quad \text{with} \quad r \in [2, 2^*_s) \]

and

\[ u_n(x) \to u_0(x) \quad \text{for a.e.} \quad x \in \Omega. \]
It follows that
\[ o(1) = \langle I'(u_n), u_n - u_0 \rangle_{H^{-s} (\Omega), H^s_0 (\Omega)} \]
\[ = \int_{\Omega} ( - \Delta )^{s/2} u_n ( - \Delta )^{s/2} (u_n - u_0) dx + \int_{\Omega} (\lambda + |u_n|^{2^*_s - 2 - \varepsilon}) u_n (u_n - u_0) dx \]
\[ = \| u_n - u_0 \|^2_{H^s_0 (\Omega)} + o(1), \text{ as } n \to \infty, \]
which implies that
\[ u_n \to u_0 \text{ in } H^s_0 (\Omega) \]
and the proof is completed. \(\square\)

**Lemma 3.2.** Suppose \( m \geq 1 \). Then there exists \( R = R(E_m) > 0 \), such that for all \( \varepsilon \in (0, 2^*_s - \zeta) \),
\[ \sup_{B_R \cap E_m} I_{\varepsilon} < 0, \]
where \( B_R := H^s_0 (\Omega) \setminus B_R \) and \( B_R = \{ u \in H^s_0 (\Omega) : \| u \|_{H^s_0 (\Omega)} \leq R \} \).

**Proof.** Define an auxiliary functional \( I_\ast : H^s_0 (\Omega) \to \mathbb{R} \) given by
\[ I_\ast(u) = \frac{1}{2} \int_{\Omega} (|(-\Delta)^{s/2} u|^2 - \lambda u^2) dx - \frac{1}{2^*_s} \int_{\Omega} |u|^\varepsilon dx. \tag{3.1} \]
Noting that
\[ \frac{1}{2^*_s} (|u|^\varepsilon - 1) \leq \frac{1}{2^*_s - \varepsilon} |u|^{2^*_s - \varepsilon}, \]
it is easy to check that
\[ I_{\varepsilon}(u) \leq I_\ast(u) + \frac{|\Omega|}{2^*_s}, \]
holds for any \( \varepsilon \in (0, 2^*_s - \zeta) \). Since any norm in finite dimensional space is equivalent,
\[ \lim_{\|u\|_{H^s_0 (\Omega)} \to \infty, u \in E_m} I_\ast(u) = -\infty \]
for any fixed \( m \geq 1 \). Thus the result follows. \(\square\)

**Lemma 3.3.** For any \( \varepsilon \in (0, 2^*_s - \zeta) \), there exist \( \rho_\varepsilon > 0 \) and \( \alpha_\varepsilon > 0 \) such that
\[ \inf_{\partial B_{\rho_\varepsilon}} I_{\varepsilon} \geq \alpha_\varepsilon, \]
where \( B_{\rho_\varepsilon} = \{ u \in H^s_0 (\Omega) : \| u \|_{H^s_0 (\Omega)} \leq \rho_\varepsilon \} \).

**Proof.** For \( u \in H^s_0 (\Omega) \), by Lemma 2.1, there exists \( C(\varepsilon) > 0 \) such that
Clearly, such that $H$ implies and positive compact operator.

Moreover, by Proposition 4.2 in [16], the operator $(\epsilon I - \Delta)^{s/2}$ solves

\[ \begin{cases} \text{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathcal{C}, \\ w = 0 & \text{on } \partial \mathcal{C}, \\ \partial_w w = g(x) & \text{on } \Omega. \end{cases} \tag{3.2} \]

As in [16] (see also [7]), the above definition is well defined and

\[ T_s \circ (\Delta)^{s} = \text{id} |_{H^s_0(\Omega)}, \quad (\Delta)^{s} \circ T_s = \text{id} |_{H^{-s}(\Omega)}, \]

which implies $T_s$ is the inverse of the operator $(\Delta)^{s}$. Denote $T_s$ by $(\Delta)^{-s}$.

Clearly,

\[ (\Delta)^{-s} = ((\Delta)^{s})^{-1}. \]

Moreover, by Proposition 4.2 in [16], the operator $(\Delta)^{-s}$ is a self-adjoint and positive compact operator.

Now we define the operator $A_{\epsilon} : H^s_0(\Omega) \to H^s_0(\Omega)$ by

\[ A_{\epsilon}(u) = (\Delta)^{-s}[\lambda u + |u|^{2s - 2} - \epsilon u] \]

for $u \in H^s_0(\Omega)$. Then the gradient of $I_{\epsilon}$ has the form

\[ I'_{\epsilon}(u) = u - A_{\epsilon}(u). \]

Indeed, we have

\[ \langle I'_{\epsilon}(u), \varphi \rangle_{H^{-s}(\Omega), H^s_0(\Omega)} = \int_{\Omega} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi \, dx - \int_{\Omega} (\lambda u + |u|^{2s - 2} - \epsilon u) \varphi \, dx \]

\[ = \int_{\Omega} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi \, dx - \int_{\Omega} (-\Delta)^s A_{\epsilon}(u) \varphi \, dx \]

\[ = \int_{\Omega} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi \, dx - \int_{\Omega} (-\Delta)^s A_{\epsilon}(u)(-\Delta)^{s/2} \varphi \, dx \]

\[ = \langle u - A_{\epsilon}(u), \varphi \rangle_{H^s_0(\Omega)}, \quad \forall u, \varphi \in H^s_0(\Omega). \]
Note that the set of fixed points of $A_\varepsilon$ is the same as the set of critical points of $I_\varepsilon$, which is $K_\varepsilon := \{ u \in H^s_0(\Omega) : I'_\varepsilon(u) = 0 \}$.

3.3. **Invariant subsets of descending flow.** It is easy to check that $I'_\varepsilon$ is locally Lipschitz continuous. We consider the negative gradient flow $\varphi_\varepsilon$ of $I_\varepsilon$ defined by
\[
\left\{ \begin{array}{l}
\frac{d}{dt}\varphi_\varepsilon(t, u) = -I'_\varepsilon(\varphi_\varepsilon(t, u)) \quad \text{for} \quad t \geq 0, \\
\varphi_\varepsilon(0, u) = u.
\end{array} \right.
\]
Here and in the sequel, define the convex cones
\[ P^+ = \{ u \in H^s_0(\Omega) : u \geq 0 \} \quad \text{and} \quad P^- = \{ u \in H^s_0(\Omega) : u \leq 0 \}. \]
For $\vartheta > 0$, we denote
\[ P^+_\vartheta = \{ u \in H^s_0(\Omega) : \text{dist}(u, P^+) < \vartheta \} \]
and
\[ P^-_{\vartheta} = \{ u \in H^s_0(\Omega) : \text{dist}(u, P^-) < \vartheta \}, \]
where $\text{dist}(u, P^\pm) = \inf_{v \in P^\mp} \|u - v\|_{H^s_0(\Omega)}$. Obviously, $P^-_{\vartheta} = -P^+_\vartheta$. Let
\[ W = P^+_\vartheta \cup P^-_{\vartheta}. \]
Then, $W$ is an open and symmetric subset of $H^s_0(\Omega)$ and $Q := H^s_0(\Omega) \setminus W$ contains only sign-changing functions. By similar arguments as in [16] (see also [13] and [19], we have the following result which shows that for $\vartheta$ small, $P^\pm_{\vartheta}$ is an invariant set and all sign-changing solutions to (2.3) are contained in $Q$.

**Lemma 3.4.** There exists $\vartheta_0 > 0$ such that for any $\vartheta \in (0, \vartheta_0]$, there holds
\[ A_\varepsilon(\partial P^\pm_{\vartheta}) \subset P^\pm_{\vartheta}, \]
and
\[ \varphi_\varepsilon(t, u) \in P^\pm_{\vartheta} \quad \text{for all} \quad t > 0 \quad \text{and} \quad u \in P^\pm_{\vartheta}. \]
Moreover, every nontrivial solutions $u \in P^+_{\vartheta}$ and $u \in P^-_{\vartheta}$ of (2.3) are positive and negative, respectively.

**Remark 3.1.** Note that there exists a constant $C > 0$ independent of $p \in [2, 2^*_s]$ such that $\|u\|_p \leq C\|u\|_{2^*_s}$ for all $p \in [2, 2^*_s]$, as in the proof of Lemma 5.2 in [16], one can show that there exists $\vartheta_0 > 0$ such that for any $\vartheta \in (0, \vartheta_0]$, there holds $A_\varepsilon(\partial P^\pm_{\vartheta}) \subset P^\pm_{\vartheta}$ for all $\varepsilon > 0$ small enough.
In the following, we may choose an \( \vartheta > 0 \) small enough such that \( P^\pm_\vartheta \) is an invariant set. In order to construct nodal solution by using the combination of invariant sets method and minimax method, we need a deformation lemma in the presence of invariant sets. Since \( I_\varepsilon \) satisfies the (PS) condition, using similar arguments to Lemma 5.1 in [20], we have the following result.

**Lemma 3.5.** Define \( K^{1}_{\varepsilon,c} := K_{\varepsilon,c} \cap W, K^{2}_{\varepsilon,c} := K_{\varepsilon,c} \cap Q \), where \( K_{\varepsilon,c} := \{ u \in H^\varepsilon(\Omega) : I_\varepsilon(u) = c, I_\varepsilon'(u) = 0 \} \). Let \( \varrho > 0 \) be such that \( (K^{1}_{\varepsilon,c})_\varrho \subset W \) where \( (K^{1}_{\varepsilon,c})_\varrho := \{ u \in H^\varepsilon(\Omega) : \text{dist}(u, K^{1}_{\varepsilon,c}) < \varrho \} \). Then there exists an \( \delta_0 > 0 \) such that for any \( 0 < \delta < \delta_0 \), there exists \( \eta \in C([0,1] \times H^\varepsilon(\Omega), H^\varepsilon(\Omega)) \) satisfying:

- (i) \( \eta(t,u) = u \) for \( t = 0 \) or \( u \notin I^{-1}_\varepsilon([c - \delta_0, c + \delta_0]) \setminus (K^{2}_{\varepsilon,c})_\varrho \).
- (ii) \( \eta(1, I^{\varepsilon-\delta}_\varepsilon \cup W \setminus (K^{2}_{\varepsilon,c})_\varrho) \subset I^{\varepsilon-\delta}_\varepsilon \cup W \) and \( \eta(1, I^{\varepsilon+\delta}_\varepsilon \cup W) \subset I^{\varepsilon-\delta}_\varepsilon \cup W \) if \( K^{2}_{\varepsilon,c} = \emptyset \). Here \( I^d_\varepsilon = \{ u \in H^\varepsilon(\Omega) : I_\varepsilon(u) \leq d \} \) for any \( d \in \mathbb{R} \).
- (iii) \( \eta(t, \cdot) \) is odd and a homeomorphism of \( H^\varepsilon(\Omega) \) for \( t \in [0,1] \).
- (iv) \( I_\varepsilon(\eta(\cdot, u)) \) is non-increasing.
- (v) \( \eta(t,W) \subset W \) for any \( t \in [0,1] \).

**3.4. Existence of infinitely many sign-changing solutions.** Now we prove the existence of infinitely many sign-changing solutions to problem (1.1).

**Proof of Theorem 1.1.** Here and in the sequel, we fix \( \lambda \in (0,\lambda^*_R) \). As in [30], we divide the proof into three steps.

**Step 1.** For any \( \varepsilon \in (0,2^* - \zeta) \) small, we define the minimax value \( c_{\varepsilon,k} \) for the perturbed functional \( I_\varepsilon(u) \) with \( k = 2,3,\ldots \). Set

\[
G_m := \{ h \in C(B_R \cap E_m, H^\varepsilon(\Omega)) : h \text{ is odd and } h = id \text{ on } \partial B_R \cap E_m \},
\]

where \( R > 0 \) is given by Lemma 3.2. Note that \( id \in G_m \), thus \( G_m \neq \emptyset \).

For \( k \geq 2 \), we define

\[
\Gamma_k := \{ h(B_R \cap E_m \setminus Y) : h \in G_m, m \geq k, Y = -Y \text{ is open and } \gamma(Y) \leq m - k \},
\]

where \( \gamma(K) \) is the Krasnoselskii genus of the symmetric closed set \( K \), i.e. the smallest integer \( n \) such that there exists and odd continuous map \( \sigma : K \rightarrow S^{n-1} \). From [22], \( \Gamma_k \) possess the following properties:

- (1') \( \Gamma_k \neq \emptyset \) and \( \Gamma_{k+1} \subset \Gamma_k \) for all \( k \geq 2 \).
- (2') If \( \phi \in C(H^\varepsilon(\Omega), H^\varepsilon(\Omega)) \) is odd and \( \phi = id \) on \( \partial B_R \cap E_m \), then \( \phi(A) \in \Gamma_k \) if \( A \in \Gamma_k \) for all \( k \geq 2 \).
- (3') If \( A \in \Gamma_k, Z = -Z \) is open and \( \gamma(Z) \leq s < k \) and \( k - s \geq 2 \), then \( A \setminus Z \in \Gamma_{k-s} \).
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Now, for $k = 2, 3, \cdots$, we can define the minimax value $c_{\varepsilon,k}$ given by

$$c_{\varepsilon,k} := \inf_{A \in \Gamma_k} \sup_{u \in A \cap Q} I_{\varepsilon}(u).$$

We need to show that $c_{\varepsilon,k}$ ($k \geq 2$) are well defined (that is for any $A \in \Gamma_k$, $A \cap Q \neq \emptyset$) and $c_{\varepsilon,k} \geq \alpha_{\varepsilon} > 0$, where $\alpha_{\varepsilon}$ is given by Lemma 3.3.

Consider the attracting domain of 0 in $H^s_0(\Omega)$:

$$D := \{ u \in H^s_0(\Omega) : \varphi_{\varepsilon}(t,u) \to 0, \text{ as } t \to \infty \}.$$ 

Note that $D$ is open, since 0 is a local minimum of $I_{\varepsilon}$ and by the continuous dependence of ODE on initial data. Moreover, $\partial D$ is an invariant set and $\overline{P_{\gamma}^+ \cap P_{\gamma}^-} \subset D$.

In particular, there holds

$$I_{\varepsilon}(u) > 0 \text{ for every } u \in \overline{P_{\gamma}^+ \cap P_{\gamma}^-} \setminus \{0\}$$

by the similar arguments to Lemma 3.4 in [4]. Now we claim that for any $A \in \Gamma_k$ with $k \geq 2$, it holds

$$A \cap Q \cap \partial D \neq \emptyset. \quad (3.3)$$

Set

$$A = h(B_R \cap E_m \setminus Y)$$

with $\gamma(Y) \leq m - k$ and $k \geq 2$. Define

$$\mathcal{O} := \{ u \in B_R \cap E_m : h(u) \in D \}.$$ 

Obviously, $\mathcal{O}$ is a bounded open symmetric set with $0 \in \mathcal{O}$ and $\overline{\mathcal{O}} \subset B_R \cap E_m$. Thus, by the Borsuk-Ulam theorem that $\gamma(\partial \mathcal{O}) = m$ and by the continuity of $h$, $h(\partial \mathcal{O}) \subset \partial D$. As a consequence,

$$h(\partial \mathcal{O} \setminus Y) \subset A \cap \partial D,$$

and therefore

$$\gamma(A \cap \partial D) \geq \gamma(h(\partial \mathcal{O} \setminus Y)) \geq \gamma(\partial \mathcal{O} \setminus Y) \geq \gamma(\partial \mathcal{O}) - \gamma(Y) \geq k,$$

by the “monotone, sub-additive and supervariant” property of the genus (cf. Proposition 5.4 in [29]). Since $P_{\gamma}^+ \cap P_{\gamma}^- \cap \partial D = \emptyset$, one has

$$\gamma(W \cap \partial D) \leq 1.$$ 

Thus for $k \geq 2$, we conclude that

$$\gamma(A \cap Q \cap \partial D) \geq \gamma(A \cap \partial D) - \gamma(W \cap \partial D) \geq k - 1 \geq 1,$$

which proves (3.3). Therefore, it follows from (3.3) that $A \cap Q \neq \emptyset$. Moreover, we have

$$c_{\varepsilon,2} \geq \alpha_{\varepsilon} > 0,$$

because $\partial B_{\rho_{\varepsilon}} \subset D$ and $\sup_{A \cap Q} I_{\varepsilon} \geq \inf_{\partial D} I_{\varepsilon} \geq \inf_{\partial B_{\rho_{\varepsilon}}} I_{\varepsilon} \geq \alpha_{\varepsilon} > 0$ by Lemma 3.3.

Hence, $c_{\varepsilon,k}$ are well defined for all $k \geq 2$ and $0 < \alpha_{\varepsilon} \leq c_{\varepsilon,2} \leq c_{\varepsilon,3} \leq \cdots$. 
Now, we claim
\[ K_{\varepsilon,k} \cap Q \neq \emptyset, \] (3.4)
which implies that there exists a sign-changing critical point \( u_{\varepsilon,k} \) such that
\[ I_{\varepsilon}(u_{\varepsilon,k}) = c_{\varepsilon,k}, \]
and \( c_{\varepsilon,k} \to \infty \) as \( k \to \infty \). It can be done, using deformation Lemma 3.5 following the same arguments as in the proof of Step 1 in [30].

**Step 2.** We show that for any fixed \( k \geq 2 \), \( \|u_{\varepsilon,k}\|_{H^s_0(\Omega)} \) is uniformly bounded with respect to \( \varepsilon \), and then \( u_{\varepsilon,k} \) converges strongly to \( u_k \) in \( H^s_0(\Omega) \) as \( \varepsilon \to 0 \).

In fact, using the same \( \Gamma_k \) above, we can also define the minimax value for the auxiliary functional \( I_* \) (see (3.1)) by
\[ \beta_k := \inf_{\Lambda \in \Gamma_k} \sup_{u \in \Lambda} I_*(u), \quad k = 2, 3, \cdots. \]

Here, choosing \( R > 0 \) sufficiently large if necessary, we point out that Lemma 3.2 also holds for \( I_* \). Then from a \( \mathbb{Z}_2 \) version of the Mountain Pass Theorem (see Theorem 9.12 in [22]), for each \( k \geq 2 \), \( \beta_k > 0 \) is well defined and
\[ \beta_k \to \infty, \quad \text{as} \quad k \to \infty. \]

Since
\[ I_{\varepsilon}(u) \leq I_*(u) + \frac{\|\Omega\|}{2^s} \]
holds for any \( \varepsilon \in (0, 2^s - \zeta) \), by the definition of \( c_{\varepsilon,k} \) and \( \beta_k \), we have
\[ c_{\varepsilon,k} \leq \beta_k + \frac{\|\Omega\|}{2^s}. \]

Therefore, for fixed \( k \geq 2 \), \( c_{\varepsilon,k} \) is uniformly bounded for \( \varepsilon \in (0, 2^s - \zeta) \), i.e., there exists \( C = C(\beta_k, \Omega) > 0 \) independent on \( \varepsilon \), such that \( c_{\varepsilon,k} \leq C \) uniformly for \( \varepsilon \). Since \( u_{\varepsilon,k} \) is a nodal solution of (2.3) and \( I_{\varepsilon}(u_{\varepsilon,k}) = c_{\varepsilon,k} \), one concludes that
\[ \frac{\lambda^s_1 - \lambda}{\lambda^s_1} \int_{\Omega} |(-\Delta)^{s/2} u_{\varepsilon,k}|^2 \, dx \leq \int_{\Omega} |u_{\varepsilon,k}|^{2^*_s - \varepsilon} \, dx = \frac{2 (2^*_s - \varepsilon)}{2^*_s - \varepsilon - 2} c_{\varepsilon,k} \leq C, \] (3.5)
which implies that \( \|u_{\varepsilon,k}\|_{H^s_0(\Omega)} \leq C \) uniformly with respect to \( \varepsilon \). Denote
\[ w_{\varepsilon,k} = E_s(u_{\varepsilon,k}). \]

Then, \( w_{\varepsilon,k} \) is a solution of (2.4) satisfying \( \|w_{\varepsilon,k}\|_{H^s_0(\Omega)} \leq C \) uniformly with respect to \( \varepsilon \). So we can apply Theorem 2.1 and obtain a subsequence \( \{w_{\varepsilon,n,k}\}_{n \in \mathbb{N}} \), such that
\[ w_{\varepsilon,n,k} \to w_k \text{ strongly in } H^s_0(\Omega) \]
for some \( w_k \in H^s_0(\Omega) \). We set
\[ u_k = tr_{\Omega} w_k. \]

Then,
\[ u_{\varepsilon_n,k} \to u_k \text{ strongly in } H_0^s(\Omega) \]

by the trace inequality and also \( c_{\varepsilon_n,k} \to c_k \). Thus \( u_k \) is a solution of (1.1) and \( I(u_k) = c_k \). Moreover, since \( u_{\varepsilon_n,k} \) is sign-changing, similar to Step 1, by Lemma 2.1 we can show that \( u_k \) is still sign-changing.

**Step 3.** We are in a position to prove that the functional has infinitely many sign-changing critical points. Recalling that \( c_k \) is non-decreasing with respect to \( k \), we distinguish two cases:

**Case I:** There exist \( 2 \leq k_1 < \cdots < k_i < \cdots \), satisfying \( c_{k_1} < \cdots < c_{k_i} < \cdots \).

**Case II:** There is a positive integer \( l \) such that \( c_k = c \) for all \( k \geq l \).

Obviously, in Case I, problem (1.1) has infinitely many sign-changing solutions such that \( I(u_i) = c_{k_i} \), thus we are done. So we assume in the sequel that Case II holds. From now on, we suppose that there exists a \( \delta > 0 \), such that \( I(u) \) has no sign-changing critical point \( u \) with \( I(u) \in [c - \delta, c) \cup (c, c + \delta] \).

Otherwise, the result follows.

We claim that

\[ \gamma(K_c^2) \geq 2, \]

where \( K_c := \{ u \in H_0^s(\Omega) : I(u) = c, I'(u) = 0 \} \) and \( K_c^2 = K_c \cap Q \). If it is true, \( I(u) \) has infinitely many sign-changing critical points and thus we are done. Here we borrow some ideas used in [11]. Arguing by contradiction, suppose that

\[ \gamma(K_c^2) = 1 \]

(note that \( K_c^2 \neq \emptyset \)). Moreover, we assume \( K_c^2 \) contains only finitely many critical points, otherwise the proof is completed. As a consequence, \( K_c^2 \) is compact. Clearly, \( 0 \notin K_c^2 \). Thus there exists an open neighborhood \( N \) in \( H_0^s(\Omega) \) with \( K_c^2 \subset N \) such that \( \gamma(N) = \gamma(K_c^2) \).

Define

\[ U_\varepsilon := I_\varepsilon^{-1}([c - \delta, c + \delta]) \setminus N. \]

Now we claim that for \( \varepsilon > 0 \) small, \( I_\varepsilon \) has no sign-changing critical point in \( U_\varepsilon \). Indeed, if not, we suppose that there exist \( \varepsilon_n \to 0 \) and \( u_n \in U_{\varepsilon_n} \) satisfying \( I'_{\varepsilon_n}(u_n) = 0 \), with \( u_n^+ \neq 0 \), and \( u_n \notin N \). Obviously,

\[ I_{\varepsilon_n}(u_n) \in [c - \delta, c + \delta]. \]

Then, similar to (3.5), one can obtain that \( \|u_n\|_{H_0^s(\Omega)} \leq C \) uniformly with respect to \( n \). Set

\[ w_n = E_s(u_n). \]
By Lemma 2.1, \( w_n \) is a solution of (2.4) satisfying \( \| w_n \|_{H_{0,L}^s(C)} \leq C \) uniformly with respect to \( n \). Therefore, by Theorem 2.1, we obtain, up to a subsequence, that
\[ w_n \to w \quad \text{strongly in} \quad H_{0,L}^s(C) \]
for some \( w \in H_{0,L}^s(C) \). We set
\[ u = tr_\Omega w. \]
Clearly, \( u_n \to u \) strongly in \( H_{0}^s(\Omega) \). Thus
\[ I'(u) = 0, \quad I(u) \in [c - \delta, c + \delta] \quad \text{and} \quad u \notin K_c^2. \]
But \( u \) is still sign-changing, a contradiction.

From the above observation, one can easily show that for any \( \varepsilon > 0 \) small, there exists a constant \( \alpha_\varepsilon > 0 \) such that
\[ \| I'_\varepsilon(u) \| \geq \alpha_\varepsilon, \quad \text{for} \quad u \in I^{-1}_\varepsilon((c - \delta, c + \delta)) \setminus (N \cup W). \]
Then, as in [11], standard techniques show that for \( \varepsilon > 0 \) small enough, there exists an odd homeomorphism \( \eta \in C(H_0^s(\Omega), H_0^s(\Omega)) \) such that
\[ \eta(I^{-1}_\varepsilon \cup W \setminus N) \subset I^{-\delta}_\varepsilon \cup W. \quad (3.6) \]
See for example the proof of Theorem A.4 in [22] and also Lemma 5.1 in [20].

Now fix \( k > l \). Since \( c_{\varepsilon,k}, c_{\varepsilon,k+1} \to c \) as \( \varepsilon \to 0 \), we can find an \( \varepsilon > 0 \) small, such that
\[ c_{\varepsilon,k}, c_{\varepsilon,k+1} \in (c - \frac{\delta}{4}, c + \frac{\delta}{4}). \]
By the definition of \( c_{\varepsilon,k+1} \), we can find a set \( A \in \Gamma_{k+1} \), \( A = h(B_R \cap E_m \setminus \bar{Y}) \), where \( h \in G_m, \ m \geq k + 1 \), \( \gamma(\bar{Y}) \leq m - (k + 1) \), such that
\[ I_\varepsilon(u) \leq c_{\varepsilon,k+1} + \frac{\delta}{4} < c + \frac{\delta}{2}, \]
for any \( u \in A \cap Q \), which implies, \( A \subset I^{c_{\varepsilon}+\frac{\delta}{2}}_\varepsilon \cup W \). Then by (3.6), we have
\[ \eta(A \setminus N) \subset I^{-\delta}_\varepsilon \cup W. \quad (3.7) \]
Let \( \bar{Y} = Y \cup h^{-1}(N) \). Then \( \bar{Y} \) is symmetric and open, and
\[ \gamma(\bar{Y}) \leq \gamma(Y) + \gamma(h^{-1}(N)) \leq m - (k + 1) + 1 = m - k. \]
Therefore one can obtain that \( \bar{A} := \eta(h(B_R \cap E_m \setminus \bar{Y})) \in \Gamma_k \) by (2°) and (3°) above. Consequently, by (3.7),
\[ c_{\varepsilon,k} \leq \sup_{\bar{A} \cap Q} I_\varepsilon \leq \sup_{\eta(A \setminus N) \cap Q} I_\varepsilon \leq c - \frac{\delta}{2}, \]
which contradicts to
\[ c_{\varepsilon,k} > c - \frac{\delta}{4}. \]
Hence the proof is completed and the functional $I$ has infinitely many sign-changing critical points.

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References


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