

# Determinant of Some Matrices of Field Elements

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**Summary.** Here, we present determinants of some square matrices of field elements. First, the determinant of  $2 \times 2$  matrix is shown. Secondly, the determinants of zero matrix and unit matrix are shown, which are equal to 0 in the field and 1 in the field respectively. Thirdly, the determinant of diagonal matrix is shown, which is a product of all diagonal elements of the matrix. At the end, we prove that the determinant of a matrix is the same as the determinant of its transpose.

MML identifier: MATRIX\_7, version: 7.6.01 4.53.937

The articles [19], [26], [2], [27], [5], [4], [8], [24], [18], [17], [14], [6], [23], [7], [25], [20], [21], [3], [12], [28], [10], [15], [16], [11], [13], [1], [9], and [22] provide the notation and terminology for this paper.

In this paper  $n, i, l$  are natural numbers.

The following propositions are true:

- (1) For every permutation  $f$  of Seg 2 holds  $f = \langle 1, 2 \rangle$  or  $f = \langle 2, 1 \rangle$ .
- (2) For every finite sequence  $f$  such that  $f = \langle 1, 2 \rangle$  or  $f = \langle 2, 1 \rangle$  holds  $f$  is a permutation of Seg 2.
- (3) The permutations of 2-element set =  $\{\langle 1, 2 \rangle, \langle 2, 1 \rangle\}$ .
- (4) For every permutation  $p$  of Seg 2 such that  $p$  is a transposition holds  $p = \langle 2, 1 \rangle$ .
- (5) Let  $D$  be a non empty set,  $f$  be a finite sequence of elements of  $D$ , and  $k_2$  be a natural number. If  $1 \leq k_2$  and  $k_2 < \text{len } f$ , then  $f = (\text{mid}(f, 1, k_2)) \hat{\ } \text{mid}(f, k_2 + 1, \text{len } f)$ .

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- (6) For every non empty set  $D$  and for every finite sequence  $f$  of elements of  $D$  such that  $2 \leq \text{len } f$  holds  $f = (f \upharpoonright (\text{len } f - 1)) \hat{\ } \text{mid}(f, \text{len } f - 1, \text{len } f)$ .
- (7) For every non empty set  $D$  and for every finite sequence  $f$  of elements of  $D$  such that  $1 \leq \text{len } f$  holds  $f = (f \upharpoonright (\text{len } f - 1)) \hat{\ } \text{mid}(f, \text{len } f, \text{len } f)$ .
- (8) Let  $a$  be an element of  $A_2$ . Given an element  $q$  of the permutations of 2-element set such that  $q = a$  and  $q$  is a transposition. Then  $a = \langle 2, 1 \rangle$ .
- (9) Let  $n$  be a natural number,  $a, b$  be elements of  $A_n$ , and  $p_2, p_1$  be elements of the permutations of  $n$ -element set. If  $a = p_2$  and  $b = p_1$ , then  $a \cdot b = p_1 \cdot p_2$ .
- (10) Let  $a, b$  be elements of  $A_2$ . Suppose that
- (i) there exists an element  $p$  of the permutations of 2-element set such that  $p = a$  and  $p$  is a transposition, and
  - (ii) there exists an element  $q$  of the permutations of 2-element set such that  $q = b$  and  $q$  is a transposition.
- Then  $a \cdot b = \langle 1, 2 \rangle$ .
- (11) Let  $l$  be a finite sequence of elements of  $A_2$ . Suppose that
- (i)  $\text{len } l \bmod 2 = 0$ , and
  - (ii) for every  $i$  such that  $i \in \text{dom } l$  there exists an element  $q$  of the permutations of 2-element set such that  $l(i) = q$  and  $q$  is a transposition.
- Then  $\prod l = \langle 1, 2 \rangle$ .
- (12) For every field  $K$  and for every matrix  $M$  over  $K$  of dimension 2 holds  $\text{Det } M = M_{1,1} \cdot M_{2,2} - M_{1,2} \cdot M_{2,1}$ .

Let  $n$  be a natural number, let  $K$  be a field, let  $M$  be a matrix over  $K$  of dimension  $n$ , and let  $a$  be an element of  $K$ . Then  $a \cdot M$  is a matrix over  $K$  of dimension  $n$ .

The following three propositions are true:

- (13) For every field  $K$  and for all natural numbers  $n, m$  holds
- $$\text{len} \left( \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times m} \right) = n \text{ and } \text{dom} \left( \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times m} \right) = \text{Seg } n.$$
- (14) Let  $K$  be a field,  $n$  be a natural number,  $p$  be an element of the permutations of  $n$ -element set, and  $i$  be a natural number. If  $i \in \text{Seg } n$ , then  $p(i) \in \text{Seg } n$ .
- (15) For every field  $K$  and for every natural number  $n$  such that  $n \geq 1$  holds
- $$\text{Det} \left( \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n} \right) = 0_K.$$

Let  $x, y, a, b$  be sets. The functor  $\text{IFIN}(x, y, a, b)$  is defined by:

(Def. 1)  $\text{IFIN}(x, y, a, b) = \begin{cases} a, & \text{if } x \in y, \\ b, & \text{otherwise.} \end{cases}$

We now state the proposition

(16) For every field  $K$  and for every natural number  $n$  such that  $n \geq 1$  holds

$$\text{Det} \left( \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n \times n, K} \right) = 1_K.$$

Let  $K$  be a field, let  $n$  be a natural number, and let  $M$  be a matrix over  $K$  of dimension  $n$ . We say that  $M$  being diagonal if and only if:

(Def. 2) For all natural numbers  $i, j$  such that  $i \in \text{Seg } n$  and  $j \in \text{Seg } n$  and  $i \neq j$  holds  $M_{i,j} = 0_K$ .

One can prove the following propositions:

(17) Let  $K$  be a field,  $n$  be a natural number, and  $A$  be a matrix over  $K$  of dimension  $n$ . Suppose  $n \geq 1$  and  $A$  being diagonal. Then  $\text{Det } A = (\text{the multiplication of } K) \otimes (\text{the diagonal of } A)$ .

(18) Let  $n$  be a natural number and  $p$  be an element of the permutations of  $n$ -element set. Then  $p^{-1}$  is an element of the permutations of  $n$ -element set.

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Next we state the proposition

(19) Let  $n$  be a natural number,  $K$  be a field, and  $A$  be a matrix over  $K$  of dimension  $n$ . Then  $A^T$  is a matrix over  $K$  of dimension  $n$ .

Let  $n$  be a natural number, let  $K$  be a field, and let  $A$  be a matrix over  $K$  of dimension  $n$ . The functor  $A^T$  yields a matrix over  $K$  of dimension  $n$  and is defined as follows:

(Def. 3)  $A^T = (A \text{ qua matrix over } K)^T$ .

The following proposition is true

(20) For every group  $G$  and for all finite sequences  $f_1, f_2$  of elements of  $G$  holds  $(\prod(f_1 \wedge f_2))^{-1} = (\prod f_2)^{-1} \cdot (\prod f_1)^{-1}$ .

Let  $G$  be a group and let  $f$  be a finite sequence of elements of  $G$ . The functor  $f^{-1}$  yields a finite sequence of elements of  $G$  and is defined by:

(Def. 4)  $\text{len}(f^{-1}) = \text{len } f$  and for every natural number  $i$  such that  $i \in \text{Seg len } f$  holds  $(f^{-1})_i = (f_i)^{-1}$ .

One can prove the following propositions:

(21) For every group  $G$  holds  $(\varepsilon_{(\text{the carrier of } G)})^{-1} = \varepsilon_{(\text{the carrier of } G)}$ .

(22) For every group  $G$  and for all finite sequences  $f, g$  of elements of  $G$  holds  $(f \wedge g)^{-1} = (f^{-1}) \wedge g^{-1}$ .

(23) For every group  $G$  and for every element  $a$  of  $G$  holds  $\langle a \rangle^{-1} = \langle a^{-1} \rangle$ .

- (24) For every group  $G$  and for every finite sequence  $f$  of elements of  $G$  holds  $\prod(f \wedge (\text{Rev}(f))^{-1}) = 1_G$ .
- (25) For every group  $G$  and for every finite sequence  $f$  of elements of  $G$  holds  $\prod(((\text{Rev}(f))^{-1}) \wedge f) = 1_G$ .
- (26) For every group  $G$  and for every finite sequence  $f$  of elements of  $G$  holds  $(\prod f)^{-1} = \prod((\text{Rev}(f))^{-1})$ .
- (27) Let  $I_1$  be an element of the permutations of  $n$ -element set and  $I_2$  be an element of  $A_n$ . If  $I_2 = I_1$  and  $n \geq 1$ , then  $I_1^{-1} = I_2^{-1}$ .
- (28) Let  $n$  be a natural number and  $I_3$  be an element of the permutations of  $n$ -element set. If  $n \geq 1$ , then  $I_3$  is even iff  $I_3^{-1}$  is even.
- (29) Let  $n$  be a natural number,  $K$  be a field,  $p$  be an element of the permutations of  $n$ -element set, and  $x$  be an element of  $K$ . If  $n \geq 1$ , then  $(-1)^{\text{sgn}(p)}x = (-1)^{\text{sgn}(p^{-1})}x$ .
- (30) Let  $K$  be a field and  $f_1, f_2$  be finite sequences of elements of  $K$ . Then  $(\text{the multiplication of } K) \otimes (f_1 \wedge f_2) = ((\text{the multiplication of } K) \otimes (f_1)) \cdot ((\text{the multiplication of } K) \otimes (f_2))$ .
- (31) Let  $K$  be a field and  $R_1, R_2$  be finite sequences of elements of  $K$ . Suppose  $R_1$  and  $R_2$  are fiberwise equipotent. Then  $(\text{the multiplication of } K) \otimes (R_1) = (\text{the multiplication of } K) \otimes (R_2)$ .
- (32) Let  $n$  be a natural number,  $K$  be a field,  $p$  be an element of the permutations of  $n$ -element set, and  $f, g$  be finite sequences of elements of  $K$ . If  $n \geq 1$  and  $\text{len } f = n$  and  $g = f \cdot p$ , then  $f$  and  $g$  are fiberwise equipotent.
- (33) Let  $n$  be a natural number,  $K$  be a field,  $p$  be an element of the permutations of  $n$ -element set, and  $f, g$  be finite sequences of elements of  $K$ . Suppose  $n \geq 1$  and  $\text{len } f = n$  and  $g = f \cdot p$ . Then  $(\text{the multiplication of } K) \otimes f = (\text{the multiplication of } K) \otimes g$ .
- (34) Let  $n$  be a natural number,  $K$  be a field,  $p$  be an element of the permutations of  $n$ -element set, and  $f$  be a finite sequence of elements of  $K$ . If  $n \geq 1$  and  $\text{len } f = n$ , then  $f \cdot p$  is a finite sequence of elements of  $K$ .
- (35) Let  $n$  be a natural number,  $K$  be a field,  $p$  be an element of the permutations of  $n$ -element set, and  $A$  be a matrix over  $K$  of dimension  $n$ . If  $n \geq 1$ , then  $p^{-1}\text{-Path } A^T = (p\text{-Path } A) \cdot p^{-1}$ .
- (36) Let  $n$  be a natural number,  $K$  be a field,  $p$  be an element of the permutations of  $n$ -element set, and  $A$  be a matrix over  $K$  of dimension  $n$ . Suppose  $n \geq 1$ . Then  $(\text{the product on paths of } A^T)(p^{-1}) = (\text{the product on paths of } A)(p)$ .
- (37) Let  $n$  be a natural number,  $K$  be a field, and  $A$  be a matrix over  $K$  of dimension  $n$ . If  $n \geq 1$ , then  $\text{Det } A = \text{Det}(A^T)$ .

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Received January 4, 2006

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