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Determinant of Some Matrices of Field Elements

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Summary. Here, we present determinants of some square matrices of field elements. First, the determinat of $2 * 2$ matrix is shown. Secondly, the determinants of zero matrix and unit matrix are shown, which are equal to 0 in the field and 1 in the field respectively. Thirdly, the determinant of diagonal matrix is shown, which is a product of all diagonal elements of the matrix. At the end, we prove that the determinant of a matrix is the same as the determinant of its transpose.

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The articles [19], [26], [2], [27], [5], [4], [8], [24], [18], [17], [14], [6], [23], [7], [25], [20], [21], [3], [12], [28], [10], [15], [16], [11], [13], [1], [9], and [22] provide the notation and terminology for this paper.

In this paper n, i, l are natural numbers.

The following propositions are true:

(1) For every permutation f of $\mathrm{Seg}\,2$ holds $f = \langle 1, 2\rangle$ or $f = \langle 2, 1\rangle$.

(2) For every finite sequence f such that $f = \langle 1, 2\rangle$ or $f = \langle 2, 1\rangle$ holds f is a permutation of $\mathrm{Seg}\,2$.

(3) The permutations of 2-element set $= \{\langle 1, 2\rangle, \langle 2, 1\rangle\}$.

(4) For every permutation p of $\mathrm{Seg}\,2$ such that p is a transposition holds $p = \langle 2, 1\rangle$.

(5) Let D be a non empty set, f be a finite sequence of elements of D, and k_2 be a natural number. If $1 \leq k_2$ and $k_2 < \mathrm{len}\, f$, then $f = (\mathrm{mid}(f, 1, k_2)) \frown \mathrm{mid}(f, k_2 + 1, \mathrm{len}\, f)$.

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(6) For every non empty set D and for every finite sequence f of elements of D such that $2 \leq \operatorname{len} f$ holds $f = (f \restriction (\operatorname{len} f -' 2)) \mathbin{^\frown} \operatorname{mid}(f, \operatorname{len} f -' 1, \operatorname{len} f)$.

(7) For every non empty set D and for every finite sequence f of elements of D such that $1 \leq \operatorname{len} f$ holds $f = (f \restriction (\operatorname{len} f -' 1)) \mathbin{^\frown} \operatorname{mid}(f, \operatorname{len} f, \operatorname{len} f)$.

(8) Let a be an element of A_2. Given an element q of the permutations of 2-element set such that $q = a$ and q is a transposition. Then $a = \langle 2, 1 \rangle$.

(9) Let n be a natural number, a, b be elements of A_n, and p_2, p_1 be elements of the permutations of n-element set. If $a = p_2$ and $b = p_1$, then $a \cdot b = p_1 \cdot p_2$.

(10) Let a, b be elements of A_2. Suppose that

(i) there exists an element p of the permutations of 2-element set such that $p = a$ and p is a transposition, and

(ii) there exists an element q of the permutations of 2-element set such that $q = b$ and q is a transposition.

Then $a \cdot b = \langle 1, 2 \rangle$.

(11) Let l be a finite sequence of elements of A_2. Suppose that

(i) $\operatorname{len} l \bmod 2 = 0$, and

(ii) for every i such that $i \in \operatorname{dom} l$ there exists an element q of the permutations of 2-element set such that $l(i) = q$ and q is a transposition.

Then $\prod l = \langle 1, 2 \rangle$.

(12) For every field K and for every matrix M over K of dimension 2 holds $\operatorname{Det} M = M_{1,1} \cdot M_{2,2} - M_{1,2} \cdot M_{2,1}$.

Let n be a natural number, let K be a field, let M be a matrix over K of dimension n, and let a be an element of K. Then $a \cdot M$ is a matrix over K of dimension n.

The following three propositions are true:

(13) For every field K and for all natural numbers n, m holds $\operatorname{len}(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times m}) = n$ and $\operatorname{dom}(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times m}) = \operatorname{Seg} n$.

(14) Let K be a field, n be a natural number, p be an element of the permutations of n-element set, and i be a natural number. If $i \in \operatorname{Seg} n$, then $p(i) \in \operatorname{Seg} n$.

(15) For every field K and for every natural number n such that $n \geq 1$ holds $\operatorname{Det}(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n}) = 0_K$.

Let x, y, a, b be sets. The functor $\operatorname{IFIN}(x, y, a, b)$ is defined by:

(Def. 1) $\mathrm{IFIN}(x, y, a, b) = \begin{cases} a, & \text{if } x \in y, \\ b, & \text{otherwise.} \end{cases}$

We now state the proposition

(16) For every field K and for every natural number n such that $n \geq 1$ holds $\mathrm{Det}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}) = 1_K$.

Let K be a field, let n be a natural number, and let M be a matrix over K of dimension n. We say that M being diagonal if and only if:

(Def. 2) For all natural numbers i, j such that $i \in \mathrm{Seg}\, n$ and $j \in \mathrm{Seg}\, n$ and $i \neq j$ holds $M_{i,j} = 0_K$.

One can prove the following propositions:

(17) Let K be a field, n be a natural number, and A be a matrix over K of dimension n. Suppose $n \geq 1$ and A being diagonal. Then $\mathrm{Det}\, A =$ (the multiplication of K) $\circledast$ (the diagonal of A).

(18) Let n be a natural number and p be an element of the permutations of n-element set. Then p^{-1} is an element of the permutations of n-element set.

Let us consider n and let p be an element of the permutations of n-element set. Then p^{-1} is an element of the permutations of n-element set.

Next we state the proposition

(19) Let n be a natural number, K be a field, and A be a matrix over K of dimension n. Then A^{T} is a matrix over K of dimension n.

Let n be a natural number, let K be a field, and let A be a matrix over K of dimension n. The functor A^{T} yields a matrix over K of dimension n and is defined as follows:

(Def. 3) $A^{\mathrm{T}} = (A$ **qua** matrix over $K)^{\mathrm{T}}$.

The following proposition is true

(20) For every group G and for all finite sequences f_1, f_2 of elements of G holds $(\prod(f_1 \frown f_2))^{-1} = (\prod f_2)^{-1} \cdot (\prod f_1)^{-1}$.

Let G be a group and let f be a finite sequence of elements of G. The functor f^{-1} yields a finite sequence of elements of G and is defined by:

(Def. 4) $\mathrm{len}(f^{-1}) = \mathrm{len}\, f$ and for every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\, f$ holds $(f^{-1})_i = (f_i)^{-1}$.

One can prove the following propositions:

(21) For every group G holds $(\varepsilon_{(\text{the carrier of } G)})^{-1} = \varepsilon_{(\text{the carrier of } G)}$.

(22) For every group G and for all finite sequences f, g of elements of G holds $(f \frown g)^{-1} = (f^{-1}) \frown g^{-1}$.

(23) For every group G and for every element a of G holds $\langle a \rangle^{-1} = \langle a^{-1} \rangle$.

(24) For every group G and for every finite sequence f of elements of G holds $\prod(f \frown (\mathrm{Rev}(f))^{-1}) = 1_G$.

(25) For every group G and for every finite sequence f of elements of G holds $\prod(((\mathrm{Rev}(f))^{-1}) \frown f) = 1_G$.

(26) For every group G and for every finite sequence f of elements of G holds $(\prod f)^{-1} = \prod((\mathrm{Rev}(f))^{-1})$.

(27) Let I_1 be an element of the permutations of n-element set and I_2 be an element of A_n. If $I_2 = I_1$ and $n \geq 1$, then $I_1{}^{-1} = I_2{}^{-1}$.

(28) Let n be a natural number and I_3 be an element of the permutations of n-element set. If $n \geq 1$, then I_3 is even iff $I_3{}^{-1}$ is even.

(29) Let n be a natural number, K be a field, p be an element of the permutations of n-element set, and x be an element of K. If $n \geq 1$, then $(-1)^{\mathrm{sgn}(p)}x = (-1)^{\mathrm{sgn}(p^{-1})}x$.

(30) Let K be a field and f_1, f_2 be finite sequences of elements of K. Then (the multiplication of K) $\circledast$ $(f_1 \frown f_2)$ = ((the multiplication of K) $\circledast$ $(f_1)) \cdot$ ((the multiplication of K) $\circledast$ $(f_2))$.

(31) Let K be a field and R_1, R_2 be finite sequences of elements of K. Suppose R_1 and R_2 are fiberwise equipotent. Then (the multiplication of K) $\circledast$ (R_1) = (the multiplication of K) $\circledast$ (R_2).

(32) Let n be a natural number, K be a field, p be an element of the permutations of n-element set, and f, g be finite sequences of elements of K. If $n \geq 1$ and $\mathrm{len}\, f = n$ and $g = f \cdot p$, then f and g are fiberwise equipotent.

(33) Let n be a natural number, K be a field, p be an element of the permutations of n-element set, and f, g be finite sequences of elements of K. Suppose $n \geq 1$ and $\mathrm{len}\, f = n$ and $g = f \cdot p$. Then (the multiplication of K) $\circledast$ f = (the multiplication of K) $\circledast$ g.

(34) Let n be a natural number, K be a field, p be an element of the permutations of n-element set, and f be a finite sequence of elements of K. If $n \geq 1$ and $\mathrm{len}\, f = n$, then $f \cdot p$ is a finite sequence of elements of K.

(35) Let n be a natural number, K be a field, p be an element of the permutations of n-element set, and A be a matrix over K of dimension n. If $n \geq 1$, then p^{-1}-Path $A^{\mathrm{T}} = (p\text{-Path } A) \cdot p^{-1}$.

(36) Let n be a natural number, K be a field, p be an element of the permutations of n-element set, and A be a matrix over K of dimension n. Suppose $n \geq 1$. Then (the product on paths of A^{T})(p^{-1}) = (the product on paths of A)(p).

(37) Let n be a natural number, K be a field, and A be a matrix over K of dimension n. If $n \geq 1$, then $\mathrm{Det}\, A = \mathrm{Det}(A^{\mathrm{T}})$.

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