

Some Properties of Some Special Matrices. Part II

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Summary. This article provides definitions of idempotent, nilpotent, involutory, self-reversible, similar, and congruent matrices, the trace of a matrix and their main properties.

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The terminology and notation used here are introduced in the following articles: [7], [3], [1], [9], [8], [6], [4], [2], [5], [11], and [10].

We adopt the following convention: n is a natural number, K is a field, and $M_1, M_2, M_3, M_4, M_5, M_6$ are matrices over K of dimension n .

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n . We say that M_1 is idempotent if and only if:

$$\text{(Def. 1)} \quad M_1 \cdot M_1 = M_1.$$

We say that M_1 is 2-nilpotent if and only if:

$$\text{(Def. 2)} \quad M_1 \cdot M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{n \times n, K}.$$

We say that M_1 is involutory if and only if:

$$(Def. 3) \quad M_1 \cdot M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}.$$

We say that M_1 is self invertible if and only if:

$$(Def. 4) \quad M_1 \text{ is invertible and } M_1^\smile = M_1.$$

We now state a number of propositions:

- (1) $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}$ is idempotent and involutory.
- (2) If $n > 0$, then $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n}$ is idempotent and 2-nilpotent.
- (3) If $n > 0$ and $M_2 = M_1^T$, then M_1 is idempotent iff M_2 is idempotent.
- (4) If M_1 is involutory, then M_1 is invertible.
- (5) If M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2 , then $M_1 \cdot M_1$ is permutable with $M_2 \cdot M_2$.
- (6) If $n > 0$ and M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2 and $M_1 \cdot M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n}$, then $M_1 + M_2$ is idempotent.
- (7) If $n > 0$ and M_1 is idempotent and M_2 is idempotent and $M_1 \cdot M_2 = -M_2 \cdot M_1$, then $M_1 + M_2$ is idempotent.
- (8) If M_1 is idempotent and M_2 is invertible, then $M_2^\smile \cdot M_1 \cdot M_2$ is idempotent.
- (9) If $n > 0$ and M_1 is invertible and idempotent, then M_1^\smile is idempotent.
- (10) If M_1 is invertible and idempotent, then $M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}$.
- (11) If M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2 , then $M_1 \cdot M_2$ is idempotent.
- (12) If $n > 0$ and M_1 is idempotent and M_2 is idempotent and M_1 is permutable with M_2 and $M_3 = M_1^T \cdot M_2^T$, then M_3 is idempotent.
- (13) If M_1 is idempotent and M_2 is idempotent and M_1 is invertible, then $M_1 \cdot M_2$ is idempotent.
- (14) If $n > 0$ and M_1 is idempotent and orthogonal, then M_1 is symmetrical.

(15) If M_1 is idempotent and M_2 is idempotent and $M_2 \cdot M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}$, then $M_1 \cdot M_2$ is idempotent.

(16) If M_1 is idempotent and orthogonal, then $M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}$.

(17) If $n > 0$ and M_1 is symmetrical and $M_2 = M_1^T$, then $M_1 \cdot M_2$ is symmetrical.

(18) If $n > 0$ and M_1 is symmetrical and $M_2 = M_1^T$, then $M_2 \cdot M_1$ is symmetrical.

(19) If M_1 is invertible and $M_1 \cdot M_2 = M_1 \cdot M_3$, then $M_2 = M_3$.

(20) If M_1 is invertible and $M_2 \cdot M_1 = M_3 \cdot M_1$, then $M_2 = M_3$.

(21) If $n > 0$ and M_1 is invertible and $M_2 \cdot M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n}$, then

$$M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n}.$$

(22) If $n > 0$ and M_1 is invertible and $M_2 \cdot M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n}$, then

$$M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n}.$$

(23) If M_1 is 2-nilpotent and permutable with M_2 and $n > 0$, then $M_1 \cdot M_2$ is 2-nilpotent.

(24) If $n > 0$ and M_1 is 2-nilpotent and M_2 is 2-nilpotent and M_1 is permutable with M_2 and $M_1 \cdot M_2 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n}$, then $M_1 + M_2$ is 2-nilpotent.

(25) If M_1 is 2-nilpotent and M_2 is 2-nilpotent and $M_1 \cdot M_2 = -M_2 \cdot M_1$ and $n > 0$, then $M_1 + M_2$ is 2-nilpotent.

(26) If M_1 is 2-nilpotent and $M_2 = M_1^T$ and $n > 0$, then M_2 is 2-nilpotent.

$$(27) \quad \text{If } M_1 \text{ is 2-nilpotent and idempotent, then } M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n}.$$

$$(28) \quad \text{If } n > 0, \text{ then } \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{K}^{n \times n} \neq \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}.$$

(29) If $n > 0$ and M_1 is 2-nilpotent, then M_1 is not invertible.

(30) If M_1 is self invertible, then M_1 is involutory.

$$(31) \quad \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n} \text{ is self invertible.}$$

$$(32) \quad \text{If } M_1 \text{ is self invertible and idempotent, then } M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}.$$

(33) If M_1 is self invertible and symmetrical, then M_1 is orthogonal.

Let n be a natural number, let K be a field, and let M_1, M_2 be matrices over K of dimension n . We say that M_1 is similar to M_2 if and only if:

(Def. 5) There exists a matrix M over K of dimension n such that M is invertible and $M_1 = M^{-1} \cdot M_2 \cdot M$.

Let us notice that the predicate M_1 is similar to M_2 is reflexive and symmetric.

The following propositions are true:

(34) If M_1 is similar to M_2 and M_2 is similar to M_3 and $n > 0$, then M_1 is similar to M_3 .

(35) If M_1 is similar to M_2 and M_2 is idempotent, then M_1 is idempotent.

(36) If M_1 is similar to M_2 and M_2 is 2-nilpotent and $n > 0$, then M_1 is 2-nilpotent.

(37) If M_1 is similar to M_2 and M_2 is involutory, then M_1 is involutory.

$$(38) \quad \text{If } M_1 \text{ is similar to } M_2 \text{ and } n > 0, \text{ then } M_1 + \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n} \text{ is similar}$$

$$\text{to } M_2 + \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}.$$

(39) If M_1 is similar to M_2 and $n > 0$, then $M_1 + M_1$ is similar to $M_2 + M_2$.

(40) If M_1 is similar to M_2 and $n > 0$, then $M_1 + M_1 + M_1$ is similar to $M_2 + M_2 + M_2$.

- (41) If M_1 is invertible, then $M_2 \cdot M_1$ is similar to $M_1 \cdot M_2$.
- (42) If M_2 is invertible and M_1 is similar to M_2 and $n > 0$, then M_1 is invertible.
- (43) If M_2 is invertible and M_1 is similar to M_2 and $n > 0$, then M_1^\sim is similar to M_2^\sim .

Let n be a natural number, let K be a field, and let M_1, M_2 be matrices over K of dimension n . We say that M_1 is congruent to M_2 if and only if:

(Def. 6) There exists a matrix M over K of dimension n such that M is invertible and $M_1 = M^T \cdot M_2 \cdot M$.

Next we state several propositions:

- (44) If $n > 0$, then M_1 is congruent to M_1 .
- (45) If M_1 is congruent to M_2 and $n > 0$, then M_2 is congruent to M_1 .
- (46) If M_1 is congruent to M_2 and M_2 is congruent to M_3 and $n > 0$, then M_1 is congruent to M_3 .
- (47) If M_1 is congruent to M_2 and $n > 0$, then $M_1 + M_1$ is congruent to $M_2 + M_2$.
- (48) If M_1 is congruent to M_2 and $n > 0$, then $M_1 + M_1 + M_1$ is congruent to $M_2 + M_2 + M_2$.
- (49) If M_1 is orthogonal, then $M_2 \cdot M_1$ is congruent to $M_1 \cdot M_2$.
- (50) If M_2 is invertible and M_1 is congruent to M_2 and $n > 0$, then M_1 is invertible.
- (51) If M_2 is invertible and M_1 is congruent to M_2 and $n > 0$ and $M_5 = M_1^T$ and $M_6 = M_2^T$, then M_5 is congruent to M_6 .
- (52) If M_4 is orthogonal and $M_1 = M_4^T \cdot M_2 \cdot M_4$, then M_1 is similar to M_2 .

Let n be a natural number, let K be a field, and let M be a matrix over K of dimension n . The functor $\text{Trace}(M)$ yields an element of K and is defined by:

(Def. 7) $\text{Trace}(M) = \sum$ (the diagonal of M).

The following propositions are true:

- (53) If $M_2 = M_1^T$, then $\text{Trace}(M_1) = \text{Trace}(M_2)$.
- (54) $\text{Trace}(M_1 + M_2) = \text{Trace}(M_1) + \text{Trace}(M_2)$.
- (55) $\text{Trace}(M_1 + M_2 + M_3) = \text{Trace}(M_1) + \text{Trace}(M_2) + \text{Trace}(M_3)$.
- (56) $\text{Trace}\left(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{n \times n}^K\right) = 0_K$.
- (57) If $n > 0$, then $\text{Trace}(-M_1) = -\text{Trace}(M_1)$.
- (58) If $n > 0$, then $-\text{Trace}(-M_1) = \text{Trace}(M_1)$.
- (59) If $n > 0$, then $\text{Trace}(M_1 + -M_1) = 0_K$.

- (60) If $n > 0$, then $\text{Trace}(M_1 - M_2) = \text{Trace}(M_1) - \text{Trace}(M_2)$.
- (61) If $n > 0$, then $\text{Trace}((M_1 - M_2) + M_3) = (\text{Trace}(M_1) - \text{Trace}(M_2)) + \text{Trace}(M_3)$.
- (62) If $n > 0$, then $\text{Trace}((M_1 + M_2) - M_3) = (\text{Trace}(M_1) + \text{Trace}(M_2)) - \text{Trace}(M_3)$.
- (63) If $n > 0$, then $\text{Trace}(M_1 - M_2 - M_3) = \text{Trace}(M_1) - \text{Trace}(M_2) - \text{Trace}(M_3)$.

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