

# On the Permanent of a Matrix

Ewa Romanowicz<sup>1</sup>  
Institute of Mathematics  
University of Białystok  
Akademicka 2, 15-267 Białystok, Poland

Adam Grabowski<sup>2</sup>  
Institute of Mathematics  
University of Białystok  
Akademicka 2, 15-267 Białystok, Poland

**Summary.** We introduce the notion of a permanent [13] of a square matrix. It is a notion somewhat related to a determinant, so we follow closely the approach and theorems already introduced in the Mizar Mathematical Library for the determinant. Unfortunately, the formalization of the latter notion is at its early stage, so we had to prove many very elementary auxiliary facts.

MML identifier: `MATRIX_9`, version: 7.6.01 4.53.937

The articles [18], [25], [14], [1], [16], [9], [26], [4], [6], [5], [2], [3], [15], [20], [21], [12], [23], [17], [24], [7], [19], [10], [22], [8], [11], and [27] provide the terminology and notation for this paper.

## 1. PRELIMINARIES

In this paper  $i, n$  are natural numbers and  $K$  is a field.

We now state the proposition

- (1) For all sets  $a, A$  such that  $a \in A$  holds  $\{a\} \in \text{Fin } A$ .

---

<sup>1</sup>This is a part of the author's MSc thesis.

<sup>2</sup>This work has been partially supported by the KBN grant 4 T11C 039 24 and the FP6 IST grant TYPES No. 510996.

Let  $n$  be a natural number. Observe that there exists an element of  $\text{Fin}$  (the permutations of  $n$ -element set) which is non empty.

The scheme *NonEmptyFiniteX* deals with a natural number  $\mathcal{A}$ , a non empty element  $\mathcal{B}$  of  $\text{Fin}$  the permutations of  $\mathcal{A}$ -element set, and a unary predicate  $\mathcal{P}$ , and states that:

$$\mathcal{P}[\mathcal{B}]$$

provided the following conditions are met:

- For every element  $x$  of the permutations of  $\mathcal{A}$ -element set such that  $x \in \mathcal{B}$  holds  $\mathcal{P}[\{x\}]$ , and
- Let  $x$  be an element of the permutations of  $\mathcal{A}$ -element set and  $B$  be a non empty element of  $\text{Fin}$  (the permutations of  $\mathcal{A}$ -element set). If  $x \in \mathcal{B}$  and  $B \subseteq \mathcal{B}$  and  $x \notin B$  and  $\mathcal{P}[B]$ , then  $\mathcal{P}[B \cup \{x\}]$ .

Let us consider  $n$ . Observe that there exists a function from  $\text{Seg } n$  into  $\text{Seg } n$  which is one-to-one and finite sequence-like.

Let us consider  $n$ . Observe that  $\text{id}_{\text{Seg } n}$  is finite sequence-like.

One can prove the following two propositions:

- (2)  $(\text{Rev}(\text{idseq}(2)))(1) = 2$  and  $(\text{Rev}(\text{idseq}(2)))(2) = 1$ .
- (3) For every one-to-one function  $f$  such that  $\text{dom } f = \text{Seg } 2$  and  $\text{rng } f = \text{Seg } 2$  holds  $f = \text{id}_{\text{Seg } 2}$  or  $f = \text{Rev}(\text{id}_{\text{Seg } 2})$ .

## 2. PERMUTATIONS

One can prove the following propositions:

- (4)  $\text{Rev}(\text{idseq}(n)) \in$  the permutations of  $n$ -element set.
- (5) Let  $f$  be a finite sequence. Suppose  $n \neq 0$  and  $f \in$  the permutations of  $n$ -element set. Then  $\text{Rev}(f) \in$  the permutations of  $n$ -element set.
- (6) The permutations of 2-element set =  $\{\text{idseq}(2), \text{Rev}(\text{idseq}(2))\}$ .

## 3. THE PERMANENT OF A MATRIX

Let us consider  $n, K$  and let  $M$  be a matrix over  $K$  of dimension  $n$ . The functor  $\text{PPath } M$  yielding a function from the permutations of  $n$ -element set into the carrier of  $K$  is defined by:

- (Def. 1) For every element  $p$  of the permutations of  $n$ -element set holds  $(\text{PPath } M)(p) = (\text{the multiplication of } K) \otimes (p\text{-Path } M)$ .

Let us consider  $n, K$  and let  $M$  be a matrix over  $K$  of dimension  $n$ . The functor  $\text{Per } M$  yielding an element of  $K$  is defined as follows:

- (Def. 2)  $\text{Per } M = (\text{the addition of } K) - \sum_{\Omega_{\text{the permutations of } n\text{-element set}}^f} \text{PPath } M$ .

In the sequel  $a, b, c, d$  denote elements of  $K$ .

The following propositions are true:

- (7)  $\text{Per}\langle\langle a \rangle\rangle = a.$
- (8) For every field  $K$  and for every natural number  $n$  such that  $n \geq 1$  holds
 
$$\text{Per}\left(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{n \times n}^K\right) = 0_K.$$
- (9) For every element  $p$  of the permutations of 2-element set such that  $p = \text{idseq}(2)$  holds  $p\text{-Path}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \langle a, d \rangle.$
- (10) For every element  $p$  of the permutations of 2-element set such that  $p = \text{Rev}(\text{idseq}(2))$  holds  $p\text{-Path}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \langle b, c \rangle.$
- (11) (The multiplication of  $K$ )  $\otimes \langle a, b \rangle = a \cdot b.$

4. MATRICES WITH THE DIMENSION 2 AND 3

One can check that there exists a permutation of Seg 2 which is odd.

Let  $n$  be a natural number. Observe that there exists a permutation of Seg  $n$  which is even.

One can prove the following four propositions:

- (12)  $\langle 2, 1 \rangle$  is an odd permutation of Seg 2.
- (13)  $\text{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot d - b \cdot c.$
- (14)  $\text{Per}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot d + b \cdot c.$
- (15)  $\text{Rev}(\text{idseq}(3)) = \langle 3, 2, 1 \rangle.$

In the sequel  $D$  is a non empty set.

One can prove the following propositions:

- (16) For all elements  $x, y, z$  of  $D$  and for every finite sequence  $f$  of elements of  $D$  such that  $f = \langle x, y, z \rangle$  holds  $\text{Rev}(f) = \langle z, y, x \rangle.$
- (17) Let  $f, g$  be finite sequences. Suppose  $f \wedge g \in$  the permutations of  $n$ -element set. Then  $f \wedge \text{Rev}(g) \in$  the permutations of  $n$ -element set.
- (18) Let  $f, g$  be finite sequences. Suppose  $f \wedge g \in$  the permutations of  $n$ -element set. Then  $g \wedge f \in$  the permutations of  $n$ -element set.
- (19) The permutations of 3-element set =  $\{\langle 1, 2, 3 \rangle, \langle 3, 2, 1 \rangle, \langle 1, 3, 2 \rangle, \langle 2, 3, 1 \rangle, \langle 2, 1, 3 \rangle, \langle 3, 1, 2 \rangle\}.$
- (20) Let  $a, b, c, d, e, f, g, h, i$  be elements of  $K$  and  $M$  be a matrix over  $K$  of dimension 3. Suppose  $M = \langle\langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle\rangle.$  Let  $p$  be an element of the permutations of 3-element set. If  $p = \langle 1, 2, 3 \rangle,$  then  $p\text{-Path } M = \langle a, e, i \rangle.$

- (21) Let  $a, b, c, d, e, f, g, h, i$  be elements of  $K$  and  $M$  be a matrix over  $K$  of dimension 3. Suppose  $M = \langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle$ . Let  $p$  be an element of the permutations of 3-element set. If  $p = \langle 3, 2, 1 \rangle$ , then  $p$ -Path  $M = \langle c, e, g \rangle$ .
- (22) Let  $a, b, c, d, e, f, g, h, i$  be elements of  $K$  and  $M$  be a matrix over  $K$  of dimension 3. Suppose  $M = \langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle$ . Let  $p$  be an element of the permutations of 3-element set. If  $p = \langle 1, 3, 2 \rangle$ , then  $p$ -Path  $M = \langle a, f, h \rangle$ .
- (23) Let  $a, b, c, d, e, f, g, h, i$  be elements of  $K$  and  $M$  be a matrix over  $K$  of dimension 3. Suppose  $M = \langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle$ . Let  $p$  be an element of the permutations of 3-element set. If  $p = \langle 2, 3, 1 \rangle$ , then  $p$ -Path  $M = \langle b, f, g \rangle$ .
- (24) Let  $a, b, c, d, e, f, g, h, i$  be elements of  $K$  and  $M$  be a matrix over  $K$  of dimension 3. Suppose  $M = \langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle$ . Let  $p$  be an element of the permutations of 3-element set. If  $p = \langle 2, 1, 3 \rangle$ , then  $p$ -Path  $M = \langle b, d, i \rangle$ .
- (25) Let  $a, b, c, d, e, f, g, h, i$  be elements of  $K$  and  $M$  be a matrix over  $K$  of dimension 3. Suppose  $M = \langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle$ . Let  $p$  be an element of the permutations of 3-element set. If  $p = \langle 3, 1, 2 \rangle$ , then  $p$ -Path  $M = \langle c, d, h \rangle$ .
- (26) (The multiplication of  $K$ )  $\otimes \langle a, b, c \rangle = a \cdot b \cdot c$ .
- (27)(i)  $\langle 1, 3, 2 \rangle \in$  the permutations of 3-element set,  
(ii)  $\langle 2, 3, 1 \rangle \in$  the permutations of 3-element set,  
(iii)  $\langle 2, 1, 3 \rangle \in$  the permutations of 3-element set,  
(iv)  $\langle 3, 1, 2 \rangle \in$  the permutations of 3-element set,  
(v)  $\langle 1, 2, 3 \rangle \in$  the permutations of 3-element set, and  
(vi)  $\langle 3, 2, 1 \rangle \in$  the permutations of 3-element set.
- (28)  $\langle 2, 3, 1 \rangle^{-1} = \langle 3, 1, 2 \rangle$ .
- (29) For every element  $a$  of  $A_3$  such that  $a = \langle 2, 3, 1 \rangle$  holds  $a^{-1} = \langle 3, 1, 2 \rangle$ .

## 5. TRANSPOSITIONS

The following propositions are true:

- (30) For every permutation  $p$  of Seg 3 such that  $p = \langle 1, 3, 2 \rangle$  holds  $p$  is a transposition.
- (31) For every permutation  $p$  of Seg 3 such that  $p = \langle 2, 1, 3 \rangle$  holds  $p$  is a transposition.
- (32) For every permutation  $p$  of Seg 3 such that  $p = \langle 3, 2, 1 \rangle$  holds  $p$  is a transposition.

- (33) For every permutation  $p$  of  $\text{Seg } n$  such that  $p = \text{id}_{\text{Seg } n}$  holds  $p$  is not a transposition.
- (34) For every permutation  $p$  of  $\text{Seg } 3$  such that  $p = \langle 3, 1, 2 \rangle$  holds  $p$  is not a transposition.
- (35) For every permutation  $p$  of  $\text{Seg } 3$  such that  $p = \langle 2, 3, 1 \rangle$  holds  $p$  is not a transposition.

## 6. EVEN AND ODD PERMUTATIONS

One can prove the following propositions:

- (36) Every permutation of  $\text{Seg } n$  is a finite sequence of elements of  $\text{Seg } n$ .
- (37)  $\langle 2, 1, 3 \rangle \cdot \langle 1, 3, 2 \rangle = \langle 2, 3, 1 \rangle$  and  $\langle 1, 3, 2 \rangle \cdot \langle 2, 1, 3 \rangle = \langle 3, 1, 2 \rangle$  and  $\langle 2, 1, 3 \rangle \cdot \langle 3, 2, 1 \rangle = \langle 3, 1, 2 \rangle$  and  $\langle 3, 2, 1 \rangle \cdot \langle 2, 1, 3 \rangle = \langle 2, 3, 1 \rangle$  and  $\langle 3, 2, 1 \rangle \cdot \langle 3, 2, 1 \rangle = \langle 1, 2, 3 \rangle$  and  $\langle 2, 1, 3 \rangle \cdot \langle 2, 1, 3 \rangle = \langle 1, 2, 3 \rangle$  and  $\langle 1, 3, 2 \rangle \cdot \langle 1, 3, 2 \rangle = \langle 1, 2, 3 \rangle$  and  $\langle 1, 3, 2 \rangle \cdot \langle 2, 3, 1 \rangle = \langle 3, 2, 1 \rangle$  and  $\langle 2, 3, 1 \rangle \cdot \langle 2, 3, 1 \rangle = \langle 3, 1, 2 \rangle$  and  $\langle 2, 3, 1 \rangle \cdot \langle 3, 1, 2 \rangle = \langle 1, 2, 3 \rangle$  and  $\langle 3, 1, 2 \rangle \cdot \langle 2, 3, 1 \rangle = \langle 1, 2, 3 \rangle$  and  $\langle 3, 1, 2 \rangle \cdot \langle 3, 1, 2 \rangle = \langle 2, 3, 1 \rangle$  and  $\langle 1, 3, 2 \rangle \cdot \langle 3, 2, 1 \rangle = \langle 2, 3, 1 \rangle$  and  $\langle 3, 2, 1 \rangle \cdot \langle 1, 3, 2 \rangle = \langle 3, 1, 2 \rangle$ .
- (38) For every permutation  $p$  of  $\text{Seg } 3$  such that  $p$  is a transposition holds  $p = \langle 2, 1, 3 \rangle$  or  $p = \langle 1, 3, 2 \rangle$  or  $p = \langle 3, 2, 1 \rangle$ .
- (39) For all elements  $f, g$  of the permutations of  $n$ -element set holds  $f \cdot g \in$  the permutations of  $n$ -element set.
- (40) Let  $l$  be a finite sequence of elements of  $A_n$ . Suppose that
- $\text{len } l \bmod 2 = 0$ , and
  - for every natural number  $i$  such that  $i \in \text{dom } l$  there exists an element  $q$  of the permutations of  $n$ -element set such that  $l(i) = q$  and  $q$  is a transposition.
- Then  $\prod l$  is an even permutation of  $\text{Seg } n$ .
- (41) Let  $l$  be a finite sequence of elements of  $A_3$ . Suppose that
- $\text{len } l \bmod 2 = 0$ , and
  - for every natural number  $i$  such that  $i \in \text{dom } l$  there exists an element  $q$  of the permutations of 3-element set such that  $l(i) = q$  and  $q$  is a transposition.
- Then  $\prod l = \langle 1, 2, 3 \rangle$  or  $\prod l = \langle 2, 3, 1 \rangle$  or  $\prod l = \langle 3, 1, 2 \rangle$ .

Let us mention that there exists a permutation of  $\text{Seg } 3$  which is odd.

We now state four propositions:

- (42)  $\langle 3, 2, 1 \rangle$  is an odd permutation of  $\text{Seg } 3$ .
- (43)  $\langle 2, 1, 3 \rangle$  is an odd permutation of  $\text{Seg } 3$ .
- (44)  $\langle 1, 3, 2 \rangle$  is an odd permutation of  $\text{Seg } 3$ .

- (45) For every odd permutation  $p$  of  $\text{Seg } 3$  holds  $p = \langle 3, 2, 1 \rangle$  or  $p = \langle 1, 3, 2 \rangle$  or  $p = \langle 2, 1, 3 \rangle$ .

## 7. DETERMINANT AND PERMANENT

One can prove the following propositions:

- (46) Let  $a, b, c, d, e, f, g, h, i$  be elements of  $K$  and  $M$  be a matrix over  $K$  of dimension 3. If  $M = \langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle$ , then  $\text{Det } M = ((a \cdot e \cdot i - c \cdot e \cdot g - a \cdot f \cdot h) + b \cdot f \cdot g) - b \cdot d \cdot i + c \cdot d \cdot h$ .
- (47) Let  $a, b, c, d, e, f, g, h, i$  be elements of  $K$  and  $M$  be a matrix over  $K$  of dimension 3. If  $M = \langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle$ , then  $\text{Per } M = a \cdot e \cdot i + c \cdot e \cdot g + a \cdot f \cdot h + b \cdot f \cdot g + b \cdot d \cdot i + c \cdot d \cdot h$ .
- (48) Let  $i, n$  be natural numbers and  $p$  be an element of the permutations of  $n$ -element set. If  $i \in \text{Seg } n$ , then there exists a natural number  $k$  such that  $k \in \text{Seg } n$  and  $i = p(k)$ .
- (49) Let  $M$  be a matrix over  $K$  of dimension  $n$ . Given a natural number  $i$  such that  $i \in \text{Seg } n$  and for every natural number  $k$  such that  $k \in \text{Seg } n$  holds  $M_{\square, i}(k) = 0_K$ . Let  $p$  be an element of the permutations of  $n$ -element set. Then there exists a natural number  $l$  such that  $l \in \text{Seg } n$  and  $(p\text{-Path } M)(l) = 0_K$ .
- (50) Let  $p$  be an element of the permutations of  $n$ -element set and  $M$  be a matrix over  $K$  of dimension  $n$ . Given a natural number  $i$  such that  $i \in \text{Seg } n$  and for every natural number  $k$  such that  $k \in \text{Seg } n$  holds  $M_{\square, i}(k) = 0_K$ . Then  $(\text{the product on paths of } M)(p) = 0_K$ .
- (51) Let  $M$  be a matrix over  $K$  of dimension  $n$ . Given a natural number  $i$  such that  $i \in \text{Seg } n$  and for every natural number  $k$  such that  $k \in \text{Seg } n$  holds  $M_{\square, i}(k) = 0_K$ . Then  $(\text{the addition of } K)\text{-}\sum_{\text{the permutations of } n\text{-element set}}^{\Omega^f} (\text{the product on paths of } M) = 0_K$ .
- (52) Let  $p$  be an element of the permutations of  $n$ -element set and  $M$  be a matrix over  $K$  of dimension  $n$ . Given a natural number  $i$  such that  $i \in \text{Seg } n$  and for every natural number  $k$  such that  $k \in \text{Seg } n$  holds  $M_{\square, i}(k) = 0_K$ . Then  $(\text{PPath } M)(p) = 0_K$ .
- (53) Let  $M$  be a matrix over  $K$  of dimension  $n$ . Given a natural number  $i$  such that  $i \in \text{Seg } n$  and for every natural number  $k$  such that  $k \in \text{Seg } n$  holds  $M_{\square, i}(k) = 0_K$ . Then  $\text{Det } M = 0_K$ .
- (54) Let  $M$  be a matrix over  $K$  of dimension  $n$ . Given a natural number  $i$  such that  $i \in \text{Seg } n$  and for every natural number  $k$  such that  $k \in \text{Seg } n$  holds  $M_{\square, i}(k) = 0_K$ . Then  $\text{Per } M = 0_K$ .

## 8. ON THE PATHS OF MATRICES

One can prove the following two propositions:

- (55) Let  $M, N$  be matrices over  $K$  of dimension  $n$ . Suppose  $i \in \text{Seg } n$ . Let  $p$  be an element of the permutations of  $n$ -element set. Then there exists a natural number  $k$  such that  $k \in \text{Seg } n$  and  $i = p(k)$  and  $(N_{\square, i})_k = (p\text{-Path } N)_k$ .
- (56) Let  $a$  be an element of  $K$  and  $M, N$  be matrices over  $K$  of dimension  $n$ . Given a natural number  $i$  such that
- (i)  $i \in \text{Seg } n$ ,
  - (ii) for every natural number  $k$  such that  $k \in \text{Seg } n$  holds  $M_{\square, i}(k) = a \cdot (N_{\square, i})_k$ , and
  - (iii) for every natural number  $l$  such that  $l \neq i$  and  $l \in \text{Seg } n$  holds  $M_{\square, l} = N_{\square, l}$ .
- Let  $p$  be an element of the permutations of  $n$ -element set. Then there exists a natural number  $l$  such that  $l \in \text{Seg } n$  and  $(p\text{-Path } M)_l = a \cdot (p\text{-Path } N)_l$ .

## REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(3):529–536, 1990.
- [4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [7] Czesław Byliński. Semigroup operations on finite subsets. *Formalized Mathematics*, 1(4):651–656, 1990.
- [8] Czesław Byliński. Some properties of restrictions of finite sequences. *Formalized Mathematics*, 5(2):241–245, 1996.
- [9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [10] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(4):475–480, 1991.
- [11] Katarzyna Jankowska. Transpose matrices and groups of permutations. *Formalized Mathematics*, 2(5):711–717, 1991.
- [12] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(2):335–342, 1990.
- [13] H. Minc. *Permanents*. Addison-Wesley, 1978.
- [14] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
- [15] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [16] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(1):25–34, 1990.
- [17] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(2):369–376, 1990.
- [18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [19] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(1):187–190, 1990.

- [20] Wojciech A. Trybulec. Binary operations on finite sequences. *Formalized Mathematics*, 1(5):979–981, 1990.
- [21] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(5):821–827, 1990.
- [22] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(3):575–579, 1990.
- [23] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [24] Wojciech A. Trybulec. Lattice of subgroups of a group. Frattini subgroup. *Formalized Mathematics*, 2(1):41–47, 1991.
- [25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [26] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [27] Katarzyna Zawadzka. The product and the determinant of matrices with entries in a field. *Formalized Mathematics*, 4(1):1–8, 1993.

*Received January 4, 2006*

---