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Chordal Graphs[1]

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Summary. We are formalizing [9, pp. 81–84] where chordal graphs are defined and their basic characterization is given. This formalization is a part of the M.Sc. work of the first author under supervision of the second author.

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The terminology and notation used here are introduced in the following articles: [18], [21], [3], [16], [22], [5], [6], [4], [1], [8], [19], [2], [12], [11], [10], [7], [14], [17], [20], [15], and [13].

1. Preliminaries

One can prove the following propositions:

(1) For every non zero natural number n holds $n-1$ is a natural number and $1 \leq n$.

(2) For every odd natural number n holds $n-1$ is a natural number and $1 \leq n$.

(3) For all odd integers n, m such that $n < m$ holds $n \leq m-2$.

(4) For all odd integers n, m such that $m < n$ holds $m+2 \leq n$.

(5) For every odd natural number n such that $1 \neq n$ there exists an odd natural number m such that $m+2=n$.

(6) For every odd natural number n such that $n \leq 2$ holds $n=1$.

(7) For every odd natural number n such that $n \leq 4$ holds $n=1$ or $n=3$.

(8) For every odd natural number n such that $n \leq 6$ holds $n=1$ or $n=3$ or $n=5$.

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(9) For every odd natural number n such that $n \leq 8$ holds $n = 1$ or $n = 3$ or $n = 5$ or $n = 7$.

(10) For every even natural number n such that $n \leq 1$ holds $n = 0$.

(11) For every even natural number n such that $n \leq 3$ holds $n = 0$ or $n = 2$.

(12) For every even natural number n such that $n \leq 5$ holds $n = 0$ or $n = 2$ or $n = 4$.

(13) For every even natural number n such that $n \leq 7$ holds $n = 0$ or $n = 2$ or $n = 4$ or $n = 6$.

(14) For every finite sequence p and for every non zero natural number n such that p is one-to-one and $n \leq \operatorname{len} p$ holds $p(n) \hookleftarrow p = n$.

(15) Let p be a non empty finite sequence and T be a non empty subset of $\operatorname{rng} p$. Then there exists a set x such that $x \in T$ and for every set y such that $y \in T$ holds $x \hookleftarrow p \leq y \hookleftarrow p$.

Let p be a finite sequence and let n be a natural number. The functor $p.\mathrm{followSet}(n)$ yields a finite set and is defined as follows:

(Def. 1) $p.\mathrm{followSet}(n) = \operatorname{rng}\langle p(n), \ldots, p(\operatorname{len} p)\rangle$.

The following three propositions are true:

(16) Let p be a finite sequence, x be a set, and n be a natural number. Suppose $x \in \operatorname{rng} p$ and $n \in \operatorname{dom} p$ and p is one-to-one. Then $x \in p.\mathrm{followSet}(n)$ if and only if $x \hookleftarrow p \geq n$.

(17) Let p, q be finite sequences and x be a set. If $p = \langle x \rangle \frown q$, then for every non zero natural number n holds $p.\mathrm{followSet}(n+1) = q.\mathrm{followSet}(n)$.

(18) Let X be a set, f be a finite sequence of elements of X, and g be a FinSubsequence of f. If $\operatorname{len} \operatorname{Seq} g = \operatorname{len} f$, then $\operatorname{Seq} g = f$.

2. Miscellany on Graphs

Next we state a number of propositions:

(19) Let G be a graph, S be a subset of the vertices of G, H be a subgraph of G induced by S, and u, v be sets. Suppose $u \in S$ and $v \in S$. Let e be a set. If e joins u and v in G, then e joins u and v in H.

(20) For every graph G and for every walk W of G holds W is trail-like iff $\operatorname{len} W = 2 \cdot \operatorname{card}(W.\mathrm{edges}()) + 1$.

(21) Let G be a graph, S be a subset of the vertices of G, H be a subgraph of G with vertices S removed, and W be a walk of G. Suppose that for every odd natural number n such that $n \leq \operatorname{len} W$ holds $W(n) \notin S$. Then W is a walk of H.

(22) Let G be a graph and a, b be sets. Suppose $a \neq b$. Let W be a walk of G. If W.vertices() $= \{a, b\}$, then there exists a set e such that e joins a and b in G.

(23) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, and W be a walk of G. If W.vertices() $\subseteq S$, then W is a walk of H.

(24) Let G_1, G_2 be graphs. Suppose $G_1 =_G G_2$. Let W_1 be a walk of G_1 and W_2 be a walk of G_2. If $W_1 = W_2$, then if W_1 is cycle-like, then W_2 is cycle-like.

(25) Let G be a graph, P be a path of G, and m, n be odd natural numbers. Suppose $m \leq \operatorname{len} P$ and $n \leq \operatorname{len} P$ and $P(m) = P(n)$. Then $m = n$ or $m = 1$ and $n = \operatorname{len} P$ or $m = \operatorname{len} P$ and $n = 1$.

(26) Let G be a graph and P be a path of G. Suppose P is open. Let a, e, b be sets. Suppose $a \notin P$.vertices() and $b = P$.first() and e joins a and b in G. Then (G.walkOf(a, e, b)).append(P) is path-like.

(27) Let G be a graph and P, H be paths of G. Suppose P.edges() misses H.edges() and P is non trivial and open and H is non trivial and open and P.vertices() $\cap$ H.vertices() $= \{P$.first()$, P$.last()$\}$ and H.first() $= P$.last() and H.last() $= P$.first(). Then P.append(H) is cycle-like.

(28) For every graph G and for all walks W_1, W_2 of G such that W_1.last() $=$ W_2.first() holds (W_1.append(W_2)).length() $= W_1$.length() $+ W_2$.length().

(29) Let G be a graph and A, B be non empty subsets of the vertices of G. Suppose $B \subseteq A$. Let H_1 be a subgraph of G induced by A. Then every subgraph of H_1 induced by B is a subgraph of G induced by B.

(30) Let G be a graph and A, B be non empty subsets of the vertices of G. Suppose $B \subseteq A$. Let H_1 be a subgraph of G induced by A. Then every subgraph of G induced by B is a subgraph of H_1 induced by B.

(31) Let G be a graph and S, T be non empty subsets of the vertices of G. If $T \subseteq S$, then for every subgraph G_2 of G induced by S holds G_2.edgesBetween(T) $= G$.edgesBetween(T).

The scheme *FinGraphOrderCompInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every finite graph G holds $\mathcal{P}[G]$

provided the parameters meet the following condition:

- Let k be a non zero natural number. Suppose that for every finite graph G_3 such that G_3.order() $< k$ holds $\mathcal{P}[G_3]$. Let G_4 be a finite graph. If G_4.order() $= k$, then $\mathcal{P}[G_4]$.

We now state two propositions:

(32) For every graph G and for every walk W of G such that W is open and path-like holds W is vertex-distinct.

(33) Let G be a graph and P be a path of G. Suppose P is open and $\operatorname{len} P > 3$. Let e be a set. If e joins P.last() and P.first() in G, then P.addEdge(e) is cycle-like.

3. Shortest Topological Path

Let G be a graph and let W be a walk of G. We say that W is minimum length if and only if:

(Def. 2) For every walk W_2 of G such that W_2 is walk from W.first() to W.last() holds $\operatorname{len} W_2 \geq \operatorname{len} W$.

The following propositions are true:

(34) For every graph G and for every walk W of G and for every subwalk S of W such that S.first() $=$ W.first() and S.edgeSeq() $=$ W.edgeSeq() holds $S = W$.

(35) For every graph G and for every walk W of G and for every subwalk S of W such that $\operatorname{len} S = \operatorname{len} W$ holds $S = W$.

(36) For every graph G and for every walk W of G such that W is minimum length holds W is path-like.

(37) For every graph G and for every walk W of G such that W is minimum length holds W is path-like.

(38) Let G be a graph and W be a walk of G. Suppose that for every path P of G such that P is walk from W.first() to W.last() holds $\operatorname{len} P \geq \operatorname{len} W$. Then W is minimum length.

(39) For every graph G and for every walk W of G holds there exists a path of G which is walk from W.first() to W.last() and minimum length.

(40) Let G be a graph and W be a walk of G. Suppose W is minimum length. Let m, n be odd natural numbers. Suppose $m + 2 < n$ and $n \leq \operatorname{len} W$. Then it is not true that there exists a set e such that e joins $W(m)$ and $W(n)$ in G.

(41) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, and W be a walk of H. Suppose W is minimum length. Let m, n be odd natural numbers. Suppose $m + 2 < n$ and $n \leq \operatorname{len} W$. Then it is not true that there exists a set e such that e joins $W(m)$ and $W(n)$ in G.

(42) Let G be a graph and W be a walk of G. Suppose W is minimum length. Let m, n be odd natural numbers. If $m \leq n$ and $n \leq \operatorname{len} W$, then W.cut(m, n) is minimum length.

(43) Let G be a graph. Suppose G is connected. Let A, B be non empty subsets of the vertices of G. Suppose A misses B. Then there exists a path P of G such that

(i) P is minimum length and non trivial,
(ii) P.first() $\in A$,
(iii) P.last() $\in B$, and
(iv) for every odd natural number n such that $1 < n$ and $n < \operatorname{len} P$ holds $P(n) \notin A$ and $P(n) \notin B$.

4. Adjacency and Complete Graphs

Let G be a graph and let a, b be vertices of G. We say that a and b are adjacent if and only if:

(Def. 3) There exists a set e such that e joins a and b in G.

Let us note that the predicate a and b are adjacent is symmetric.

Next we state several propositions:

(44) Let G_1, G_2 be graphs. Suppose $G_1 =_G G_2$. Let u_1, v_1 be vertices of G_1. Suppose u_1 and v_1 are adjacent. Let u_2, v_2 be vertices of G_2. If $u_1 = u_2$ and $v_1 = v_2$, then u_2 and v_2 are adjacent.

(45) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, u, v be vertices of G, and t, w be vertices of H. Suppose $u = t$ and $v = w$. Then u and v are adjacent if and only if t and w are adjacent.

(46) For every graph G and for every walk W of G such that W.first() $\neq$ W.last() and W.first() and W.last() are not adjacent holds W.length() $\geq$ 2.

(47) Let G be a graph and v_1, v_2, v_3 be vertices of G. Suppose $v_1 \neq v_2$ and $v_1 \neq v_3$ and $v_2 \neq v_3$ and v_1 and v_2 are adjacent and v_2 and v_3 are adjacent. Then there exists a path P of G and there exist sets e_1, e_2 such that P is open and $\operatorname{len} P = 5$ and P.length() $= 2$ and e_1 joins v_1 and v_2 in G and e_2 joins v_2 and v_3 in G and P.edges() $= \{e_1, e_2\}$ and P.vertices() $= \{v_1, v_2, v_3\}$ and $P(1) = v_1$ and $P(3) = v_2$ and $P(5) = v_3$.

(48) Let G be a graph and v_1, v_2, v_3, v_4 be vertices of G. Suppose that $v_1 \neq v_2$ and $v_1 \neq v_3$ and $v_2 \neq v_3$ and $v_2 \neq v_4$ and $v_3 \neq v_4$ and v_1 and v_2 are adjacent and v_2 and v_3 are adjacent and v_3 and v_4 are adjacent. Then there exists a path P of G such that $\operatorname{len} P = 7$ and P.length() $= 3$ and P.vertices() $= \{v_1, v_2, v_3, v_4\}$ and $P(1) = v_1$ and $P(3) = v_2$ and $P(5) = v_3$ and $P(7) = v_4$.

Let G be a graph and let S be a set. The functor G.adjacentSet(S) yields a subset of the vertices of G and is defined as follows:

(Def. 4) G.adjacentSet(S) $= \{u; u$ ranges over vertices of G: $u \notin S \wedge \bigvee_{v\,:\,\text{vertex of } G} (v \in S \wedge u$ and v are adjacent$)\}$.

One can prove the following propositions:

(49) For every graph G and for all sets S, x such that $x \in G$.adjacentSet(S) holds $x \notin S$.

(50) Let G be a graph, S be a set, and u be a vertex of G. Then $u \in G$.adjacentSet(S) if and only if the following conditions are satisfied:
  (i) $u \notin S$, and
  (ii) there exists a vertex v of G such that $v \in S$ and u and v are adjacent.

(51) For all graphs G_1, G_2 such that $G_1 =_G G_2$ and for every set S holds G_1.adjacentSet(S) $= G_2$.adjacentSet(S).

(52) For every graph G and for all vertices u, v of G holds $u \in G$.adjacentSet($\{v\}$) iff $u \neq v$ and v and u are adjacent.

(53) For every graph G and for all sets x, y holds $x \in G$.adjacentSet($\{y\}$) iff $y \in G$.adjacentSet($\{x\}$).

(54) Let G be a graph and C be a path of G. Suppose C is cycle-like and C.length() > 3. Let x be a vertex of G. Suppose $x \in C$.vertices(). Then there exist odd natural numbers m, n such that $m + 2 < n$ and $n \leq \operatorname{len} C$ and $m = 1$ and $n = \operatorname{len} C$ and $m = 1$ and $n = \operatorname{len} C - 2$ and $m = 3$ and $n = \operatorname{len} C$ and $C(m) \neq C(n)$ and $C(m) \in G$.adjacentSet($\{x\}$) and $C(n) \in G$.adjacentSet($\{x\}$).

(55) Let G be a graph and C be a path of G. Suppose C is cycle-like and C.length() > 3. Let x be a vertex of G. Suppose $x \in C$.vertices(). Then there exist odd natural numbers m, n such that
  (i) $m + 2 < n$,
  (ii) $n \leq \operatorname{len} C$,
  (iii) $C(m) \neq C(n)$,
  (iv) $C(m) \in G$.adjacentSet($\{x\}$),
  (v) $C(n) \in G$.adjacentSet($\{x\}$), and
  (vi) for every set e such that $e \in C$.edges() holds e does not join $C(m)$ and $C(n)$ in G.

(56) For every loopless graph G and for every vertex u of G holds G.adjacentSet($\{u\}$) $= \emptyset$ iff u is isolated.

(57) Let G be a graph, G_0 be a subgraph of G, S be a non empty subset of the vertices of G, x be a vertex of G, G_1 be a subgraph of G induced by S, and G_2 be a subgraph of G induced by $S \cup \{x\}$. If G_1 is connected and $x \in G$.adjacentSet(the vertices of G_1), then G_2 is connected.

(58) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, and u be a vertex of G. Suppose $u \in S$ and G.adjacentSet($\{u\}$) $\subseteq S$. Let v be a vertex of H. If $u = v$, then G.adjacentSet($\{u\}$) $= H$.adjacentSet($\{v\}$).

Let G be a graph and let S be a set. A subgraph of G is called an adjacency graph of S in G if:

(Def. 5) It is a subgraph of G induced by G.adjacentSet(S) if S is a subset of the vertices of G.

Next we state two propositions:

(59) Let G_1, G_2 be graphs. Suppose $G_1 =_G G_2$. Let u_1 be a vertex of G_1 and u_2 be a vertex of G_2. Suppose $u_1 = u_2$. Let H_1 be an adjacency graph of $\{u_1\}$ in G_1 and H_2 be an adjacency graph of $\{u_2\}$ in G_2. Then $H_1 =_G H_2$.

(60) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, and u be a vertex of G. Suppose $u \in S$ and G.adjacentSet$(\{u\}) \subseteq S$ and G.adjacentSet$(\{u\}) \neq \emptyset$. Let v be a vertex of H. Suppose $u = v$. Let G_5 be an adjacency graph of $\{u\}$ in G and H_3 be an adjacency graph of $\{v\}$ in H. Then $G_5 =_G H_3$.

Let G be a graph. We say that G is complete if and only if:

(Def. 6) For all vertices u, v of G such that $u \neq v$ holds u and v are adjacent.

We now state the proposition

(61) For every graph G such that G is trivial holds G is complete.

One can check that every graph which is trivial is also complete.

Let us note that there exists a graph which is trivial, simple, and complete and there exists a graph which is non trivial, finite, simple, and complete.

The following propositions are true:

(62) For all graphs G_1, G_2 such that $G_1 =_G G_2$ holds if G_1 is complete, then G_2 is complete.

(63) For every complete graph G and for every subset S of the vertices of G holds every subgraph of G induced by S is complete.

5. Simplicial Vertex

Let G be a graph and let v be a vertex of G. We say that v is simplicial if and only if:

(Def. 7) If G.adjacentSet$(\{v\}) \neq \emptyset$, then every adjacency graph of $\{v\}$ in G is complete.

The following propositions are true:

(64) For every complete graph G holds every vertex of G is simplicial.

(65) For every trivial graph G holds every vertex of G is simplicial.

(66) Let G_1, G_2 be graphs. Suppose $G_1 =_G G_2$. Let u_1 be a vertex of G_1 and u_2 be a vertex of G_2. If $u_1 = u_2$ and u_1 is simplicial, then u_2 is simplicial.

(67) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, and u be a vertex of G. Suppose $u \in S$ and G.adjacentSet$(\{u\}) \subseteq S$. Let v be a vertex of H. If $u = v$, then u is simplicial iff v is simplicial.

(68) Let G be a graph and v be a vertex of G. Suppose v is simplicial. Let a, b be sets. Suppose $a \neq b$ and $a \in G$.adjacentSet($\{v\}$) and $b \in G$.adjacentSet($\{v\}$). Then there exists a set e such that e joins a and b in G.

(69) Let G be a graph and v be a vertex of G. Suppose v is not simplicial. Then there exist vertices a, b of G such that $a \neq b$ and $v \neq a$ and $v \neq b$ and v and a are adjacent and v and b are adjacent and a and b are not adjacent.

6. Vertex Separator

Let G be a graph and let a, b be vertices of G. Let us assume that $a \neq b$ and a and b are not adjacent. A subset of the vertices of G is said to be a vertex separator of a and b if:

(Def. 8) $a \notin$ it and $b \notin$ it and for every subgraph G_2 of G with vertices it removed holds there exists no walk of G_2 which is walk from a to b.

Next we state several propositions:

(70) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Then every vertex separator of a and b is a vertex separator of b and a.

(71) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a subset of the vertices of G. Then S is a vertex separator of a and b if and only if $a \notin S$ and $b \notin S$ and for every walk W of G such that W is walk from a to b there exists a vertex x of G such that $x \in S$ and $x \in W$.vertices().

(72) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b and W be a walk of G. Suppose W is walk from a to b. Then there exists an odd natural number k such that $1 < k$ and $k < \operatorname{len} W$ and $W(k) \in S$.

(73) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. If $S = \emptyset$, then there exists no walk of G which is walk from a to b.

(74) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent and there exists no walk of G which is walk from a to b. Then $\emptyset$ is a vertex separator of a and b.

(75) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b, G_2 be a subgraph of G with vertices S removed, and a_2 be a vertex of G_2. If $a_2 = a$, then $(G_2.\text{reachableFrom}(a_2)) \cap S = \emptyset$.

(76) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b, G_2 be a subgraph of G with vertices S removed, and a_2, b_2 be vertices of G_2. If $a_2 = a$ and $b_2 = b$, then $(G_2.\mathrm{reachableFrom}(a_2)) \cap (G_2.\mathrm{reachableFrom}(b_2)) = \emptyset$.

(77) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b and G_2 be a subgraph of G with vertices S removed. Then a is a vertex of G_2 and b is a vertex of G_2.

Let G be a graph, let a, b be vertices of G, and let S be a vertex separator of a and b. We say that S is minimal if and only if:

(Def. 9) For every subset T of S such that $T \neq S$ holds T is not a vertex separator of a and b.

Next we state several propositions:

(78) Let G be a graph, a, b be vertices of G, and S be a vertex separator of a and b. If $S = \emptyset$, then S is minimal.

(79) For every finite graph G and for all vertices a, b of G holds there exists a vertex separator of a and b which is minimal.

(80) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let T be a vertex separator of b and a. If $S = T$, then T is minimal.

(81) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let x be a vertex of G. If $x \in S$, then there exists a walk W of G such that W is walk from a to b and $x \in W.\mathrm{vertices}()$.

(82) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let H be a subgraph of G with vertices S removed and a_1 be a vertex of H. Suppose $a_1 = a$. Let x be a vertex of G. Suppose $x \in S$. Then there exists a vertex y of G such that $y \in H.\mathrm{reachableFrom}(a_1)$ and x and y are adjacent.

(83) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let H be a subgraph of G with vertices S removed and a_1 be a vertex of H. Suppose $a_1 = b$. Let x be a vertex of G. Suppose $x \in S$. Then there exists a vertex y of G such that $y \in H.\mathrm{reachableFrom}(a_1)$ and x and y are adjacent.

7. Chordal Graphs

Let G be a graph and let W be a walk of G. We say that W is chordal if and only if the condition (Def. 10) is satisfied.

(Def. 10) There exist odd natural numbers m, n such that

(i) $m + 2 < n$,

(ii) $n \leq \operatorname{len} W$,

(iii) $W(m) \neq W(n)$,

(iv) there exists a set e such that e joins $W(m)$ and $W(n)$ in G, and

(v) for every set f such that $f \in W$.edges() holds f does not join $W(m)$ and $W(n)$ in G.

Let G be a graph and let W be a walk of G. We introduce W is chordless as an antonym of W is chordal.

Next we state a number of propositions:

(84) Let G be a graph and W be a walk of G. Suppose W is chordal. Then there exist odd natural numbers m, n such that

(i) $m + 2 < n$,

(ii) $n \leq \operatorname{len} W$,

(iii) $W(m) \neq W(n)$,

(iv) there exists a set e such that e joins $W(m)$ and $W(n)$ in G, and

(v) if W is cycle-like, then $m = 1$ and $n = \operatorname{len} W$ and $m = 1$ and $n = \operatorname{len} W - 2$ and $m = 3$ and $n = \operatorname{len} W$.

(85) Let G be a graph and P be a path of G. Given odd natural numbers m, n such that

(i) $m + 2 < n$,

(ii) $n \leq \operatorname{len} P$,

(iii) there exists a set e such that e joins $P(m)$ and $P(n)$ in G, and

(iv) if P is cycle-like, then $m = 1$ and $n = \operatorname{len} P$ and $m = 1$ and $n = \operatorname{len} P - 2$ and $m = 3$ and $n = \operatorname{len} P$.

Then P is chordal.

(86) Let G_1, G_2 be graphs. Suppose $G_1 =_G G_2$. Let W_1 be a walk of G_1 and W_2 be a walk of G_2. If $W_1 = W_2$, then if W_1 is chordal, then W_2 is chordal.

(87) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, W_1 be a walk of G, and W_2 be a walk of H. If $W_1 = W_2$, then W_2 is chordal iff W_1 is chordal.

(88) Let G be a graph and W be a walk of G. Suppose W is cycle-like and chordal and W.length() $= 4$. Then there exists a set e such that e joins $W(1)$ and $W(5)$ in G or e joins $W(3)$ and $W(7)$ in G.

(89) For every graph G and for every walk W of G such that W is minimum length holds W is chordless.

(90) Let G be a graph and W be a walk of G. Suppose W is open and len $W = 5$ and W.first() and W.last() are not adjacent. Then W is chordless.

(91) For every graph G and for every walk W of G holds W is chordal iff W.reverse() is chordal.

(92) Let G be a graph and P be a path of G. Suppose P is open and chordless. Let m, n be odd natural numbers. Suppose $m < n$ and $n \leq \operatorname{len} P$. Then there exists a set e such that e joins $P(m)$ and $P(n)$ in G if and only if $m + 2 = n$.

(93) Let G be a graph and P be a path of G. Suppose P is open and chordless. Let m, n be odd natural numbers. If $m < n$ and $n \leq \operatorname{len} P$, then $P.\mathrm{cut}(m, n)$ is chordless and $P.\mathrm{cut}(m, n)$ is open.

(94) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, W be a walk of G, and V be a walk of H. If $W = V$, then W is chordless iff V is chordless.

Let G be a graph. We say that G is chordal if and only if:

(Def. 11) For every walk P of G such that P.length() > 3 and P is cycle-like holds P is chordal.

Next we state two propositions:

(95) For all graphs G_1, G_2 such that $G_1 =_G G_2$ holds if G_1 is chordal, then G_2 is chordal.

(96) For every finite graph G such that card (the vertices of G) ≤ 3 holds G is chordal.

One can verify the following observations:

* there exists a graph which is trivial, finite, and chordal,
* there exists a graph which is non trivial, finite, simple, and chordal, and
* every graph which is complete is also chordal.

Let G be a chordal graph and let V be a set. One can check that every subgraph of G induced by V is chordal.

Next we state several propositions:

(97) Let G be a chordal graph and P be a path of G. Suppose P is open and chordless. Let x, e be sets. Suppose $x \notin P$.vertices() and e joins P.last() and x in G and it is not true that there exists a set f such that f joins $P(\operatorname{len} P - 2)$ and x in G. Then P.addEdge(e) is path-like and P.addEdge(e) is open and P.addEdge(e) is chordless.

(98) Let G be a chordal graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. If S is minimal and non empty, then every subgraph of G induced by S is complete.

(99) Let G be a finite graph. Suppose that for all vertices a, b of G such that

$a \neq b$ and a and b are not adjacent and for every vertex separator S of a and b such that S is minimal and non empty holds every subgraph of G induced by S is complete. Then G is chordal.

(100) Let G be a finite chordal graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let H be a subgraph of G with vertices S removed and a_3 be a vertex of H. Suppose $a = a_3$. Then there exists a vertex c of G such that $c \in H.\mathrm{reachableFrom}(a_3)$ and for every vertex x of G such that $x \in S$ holds c and x are adjacent.

(101) Let G be a finite chordal graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let H be a subgraph of G with vertices S removed and a_3 be a vertex of H. Suppose $a = a_3$. Let x, y be vertices of G. Suppose $x \in S$ and $y \in S$. Then there exists a vertex c of G such that $c \in H.\mathrm{reachableFrom}(a_3)$ and c and x are adjacent and c and y are adjacent.

(102) Let G be a non trivial finite chordal graph. Suppose G is not complete. Then there exist vertices a, b of G such that $a \neq b$ and a and b are not adjacent and a is simplicial and b is simplicial.

(103) For every finite chordal graph G holds there exists a vertex of G which is simplicial.

8. Vertex Elimination Scheme

Let G be a finite graph. A finite sequence of elements of the vertices of G is said to be a vertex scheme of G if:

(Def. 12) It is one-to-one and $\mathrm{rng}\,\mathrm{it} =$ the vertices of G.

Let G be a finite graph. Note that every vertex scheme of G is non empty.

The following three propositions are true:

(104) For every finite graph G and for every vertex scheme S of G holds $\mathrm{len}\, S = \mathrm{card}\,(\text{the vertices of } G)$.

(105) For every finite graph G and for every vertex scheme S of G holds $1 \leq \mathrm{len}\, S$.

(106) For all finite graphs G, H and for every vertex scheme g of G such that $G =_G H$ holds g is a vertex scheme of H.

Let G be a finite graph, let S be a vertex scheme of G, and let x be a vertex of G. Then $x \leftarrow\!\!\!\!\shortmid\; S$ is a non zero element of $\mathbb{N}$.

Let G be a finite graph, let S be a vertex scheme of G, and let n be a natural number. Then $S.\mathrm{followSet}(n)$ is a subset of the vertices of G.

Next we state the proposition

(107) Let G be a finite graph, S be a vertex scheme of G, and n be a non zero natural number. If $n \leq \operatorname{len} S$, then S.followSet(n) is non empty.

Let G be a finite graph and let S be a vertex scheme of G. We say that S is perfect if and only if the condition (Def. 13) is satisfied.

(Def. 13) Let n be a non zero natural number. Suppose $n \leq \operatorname{len} S$. Let G_6 be a subgraph of G induced by S.followSet(n) and v be a vertex of G_6. If $v = S(n)$, then v is simplicial.

One can prove the following propositions:

(108) Let G be a finite trivial graph and v be a vertex of G. Then there exists a vertex scheme S of G such that $S = \langle v \rangle$ and S is perfect.

(109) Let G be a finite graph and V be a vertex scheme of G. Then V is perfect if and only if for all vertices a, b, c of G such that $b \neq c$ and a and b are adjacent and a and c are adjacent and for all natural numbers v_5, v_6, v_7 such that $v_5 \in \operatorname{dom} V$ and $v_6 \in \operatorname{dom} V$ and $v_7 \in \operatorname{dom} V$ and $V(v_5) = a$ and $V(v_6) = b$ and $V(v_7) = c$ and $v_5 < v_6$ and $v_5 < v_7$ holds b and c are adjacent.

Let G be a finite chordal graph. One can check that there exists a vertex scheme of G which is perfect.

The following propositions are true:

(110) Let G, H be finite chordal graphs and g be a perfect vertex scheme of G. If $G =_G H$, then g is a perfect vertex scheme of H.

(111) For every finite graph G such that there exists a vertex scheme of G which is perfect holds G is chordal.

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