

# Chordal Graphs<sup>1</sup>

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**Summary.** We are formalizing [9, pp. 81–84] where chordal graphs are defined and their basic characterization is given. This formalization is a part of the M.Sc. work of the first author under supervision of the second author.

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The terminology and notation used here are introduced in the following articles: [18], [21], [3], [16], [22], [5], [6], [4], [1], [8], [19], [2], [12], [11], [10], [7], [14], [17], [20], [15], and [13].

## 1. PRELIMINARIES

One can prove the following propositions:

- (1) For every non zero natural number  $n$  holds  $n - 1$  is a natural number and  $1 \leq n$ .
- (2) For every odd natural number  $n$  holds  $n - 1$  is a natural number and  $1 \leq n$ .
- (3) For all odd integers  $n, m$  such that  $n < m$  holds  $n \leq m - 2$ .
- (4) For all odd integers  $n, m$  such that  $m < n$  holds  $m + 2 \leq n$ .
- (5) For every odd natural number  $n$  such that  $1 \neq n$  there exists an odd natural number  $m$  such that  $m + 2 = n$ .
- (6) For every odd natural number  $n$  such that  $n \leq 2$  holds  $n = 1$ .
- (7) For every odd natural number  $n$  such that  $n \leq 4$  holds  $n = 1$  or  $n = 3$ .
- (8) For every odd natural number  $n$  such that  $n \leq 6$  holds  $n = 1$  or  $n = 3$  or  $n = 5$ .

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- (9) For every odd natural number  $n$  such that  $n \leq 8$  holds  $n = 1$  or  $n = 3$  or  $n = 5$  or  $n = 7$ .
- (10) For every even natural number  $n$  such that  $n \leq 1$  holds  $n = 0$ .
- (11) For every even natural number  $n$  such that  $n \leq 3$  holds  $n = 0$  or  $n = 2$ .
- (12) For every even natural number  $n$  such that  $n \leq 5$  holds  $n = 0$  or  $n = 2$  or  $n = 4$ .
- (13) For every even natural number  $n$  such that  $n \leq 7$  holds  $n = 0$  or  $n = 2$  or  $n = 4$  or  $n = 6$ .
- (14) For every finite sequence  $p$  and for every non zero natural number  $n$  such that  $p$  is one-to-one and  $n \leq \text{len } p$  holds  $p(n) \leftrightarrow p = n$ .
- (15) Let  $p$  be a non empty finite sequence and  $T$  be a non empty subset of  $\text{rng } p$ . Then there exists a set  $x$  such that  $x \in T$  and for every set  $y$  such that  $y \in T$  holds  $x \leftrightarrow p \leq y \leftrightarrow p$ .

Let  $p$  be a finite sequence and let  $n$  be a natural number. The functor  $p.\text{followSet}(n)$  yields a finite set and is defined as follows:

(Def. 1)  $p.\text{followSet}(n) = \text{rng}\langle p(n), \dots, p(\text{len } p) \rangle$ .

The following three propositions are true:

- (16) Let  $p$  be a finite sequence,  $x$  be a set, and  $n$  be a natural number. Suppose  $x \in \text{rng } p$  and  $n \in \text{dom } p$  and  $p$  is one-to-one. Then  $x \in p.\text{followSet}(n)$  if and only if  $x \leftrightarrow p \geq n$ .
- (17) Let  $p, q$  be finite sequences and  $x$  be a set. If  $p = \langle x \rangle \wedge q$ , then for every non zero natural number  $n$  holds  $p.\text{followSet}(n + 1) = q.\text{followSet}(n)$ .
- (18) Let  $X$  be a set,  $f$  be a finite sequence of elements of  $X$ , and  $g$  be a FinSubsequence of  $f$ . If  $\text{len Seq } g = \text{len } f$ , then  $\text{Seq } g = f$ .

## 2. MISCELLANY ON GRAPHS

Next we state a number of propositions:

- (19) Let  $G$  be a graph,  $S$  be a subset of the vertices of  $G$ ,  $H$  be a subgraph of  $G$  induced by  $S$ , and  $u, v$  be sets. Suppose  $u \in S$  and  $v \in S$ . Let  $e$  be a set. If  $e$  joins  $u$  and  $v$  in  $G$ , then  $e$  joins  $u$  and  $v$  in  $H$ .
- (20) For every graph  $G$  and for every walk  $W$  of  $G$  holds  $W$  is trail-like iff  $\text{len } W = 2 \cdot \text{card}(W.\text{edges}()) + 1$ .
- (21) Let  $G$  be a graph,  $S$  be a subset of the vertices of  $G$ ,  $H$  be a subgraph of  $G$  with vertices  $S$  removed, and  $W$  be a walk of  $G$ . Suppose that for every odd natural number  $n$  such that  $n \leq \text{len } W$  holds  $W(n) \notin S$ . Then  $W$  is a walk of  $H$ .

- (22) Let  $G$  be a graph and  $a, b$  be sets. Suppose  $a \neq b$ . Let  $W$  be a walk of  $G$ . If  $W.\text{vertices}() = \{a, b\}$ , then there exists a set  $e$  such that  $e$  joins  $a$  and  $b$  in  $G$ .
- (23) Let  $G$  be a graph,  $S$  be a non empty subset of the vertices of  $G$ ,  $H$  be a subgraph of  $G$  induced by  $S$ , and  $W$  be a walk of  $G$ . If  $W.\text{vertices}() \subseteq S$ , then  $W$  is a walk of  $H$ .
- (24) Let  $G_1, G_2$  be graphs. Suppose  $G_1 =_G G_2$ . Let  $W_1$  be a walk of  $G_1$  and  $W_2$  be a walk of  $G_2$ . If  $W_1 = W_2$ , then if  $W_1$  is cycle-like, then  $W_2$  is cycle-like.
- (25) Let  $G$  be a graph,  $P$  be a path of  $G$ , and  $m, n$  be odd natural numbers. Suppose  $m \leq \text{len } P$  and  $n \leq \text{len } P$  and  $P(m) = P(n)$ . Then  $m = n$  or  $m = 1$  and  $n = \text{len } P$  or  $m = \text{len } P$  and  $n = 1$ .
- (26) Let  $G$  be a graph and  $P$  be a path of  $G$ . Suppose  $P$  is open. Let  $a, e, b$  be sets. Suppose  $a \notin P.\text{vertices}()$  and  $b = P.\text{first}()$  and  $e$  joins  $a$  and  $b$  in  $G$ . Then  $(G.\text{walkOf}(a, e, b)).\text{append}(P)$  is path-like.
- (27) Let  $G$  be a graph and  $P, H$  be paths of  $G$ . Suppose  $P.\text{edges}()$  misses  $H.\text{edges}()$  and  $P$  is non trivial and open and  $H$  is non trivial and open and  $P.\text{vertices}() \cap H.\text{vertices}() = \{P.\text{first}(), P.\text{last}()\}$  and  $H.\text{first}() = P.\text{last}()$  and  $H.\text{last}() = P.\text{first}()$ . Then  $P.\text{append}(H)$  is cycle-like.
- (28) For every graph  $G$  and for all walks  $W_1, W_2$  of  $G$  such that  $W_1.\text{last}() = W_2.\text{first}()$  holds  $(W_1.\text{append}(W_2)).\text{length}() = W_1.\text{length}() + W_2.\text{length}()$ .
- (29) Let  $G$  be a graph and  $A, B$  be non empty subsets of the vertices of  $G$ . Suppose  $B \subseteq A$ . Let  $H_1$  be a subgraph of  $G$  induced by  $A$ . Then every subgraph of  $H_1$  induced by  $B$  is a subgraph of  $G$  induced by  $B$ .
- (30) Let  $G$  be a graph and  $A, B$  be non empty subsets of the vertices of  $G$ . Suppose  $B \subseteq A$ . Let  $H_1$  be a subgraph of  $G$  induced by  $A$ . Then every subgraph of  $G$  induced by  $B$  is a subgraph of  $H_1$  induced by  $B$ .
- (31) Let  $G$  be a graph and  $S, T$  be non empty subsets of the vertices of  $G$ . If  $T \subseteq S$ , then for every subgraph  $G_2$  of  $G$  induced by  $S$  holds  $G_2.\text{edgesBetween}(T) = G.\text{edgesBetween}(T)$ .

The scheme *FinGraphOrderCompInd* concerns a unary predicate  $\mathcal{P}$ , and states that:

For every finite graph  $G$  holds  $\mathcal{P}[G]$

provided the parameters meet the following condition:

- Let  $k$  be a non zero natural number. Suppose that for every finite graph  $G_3$  such that  $G_3.\text{order}() < k$  holds  $\mathcal{P}[G_3]$ . Let  $G_4$  be a finite graph. If  $G_4.\text{order}() = k$ , then  $\mathcal{P}[G_4]$ .

We now state two propositions:

- (32) For every graph  $G$  and for every walk  $W$  of  $G$  such that  $W$  is open and path-like holds  $W$  is vertex-distinct.

- (33) Let  $G$  be a graph and  $P$  be a path of  $G$ . Suppose  $P$  is open and  $\text{len } P > 3$ . Let  $e$  be a set. If  $e$  joins  $P.\text{last}()$  and  $P.\text{first}()$  in  $G$ , then  $P.\text{addEdge}(e)$  is cycle-like.

### 3. SHORTEST TOPOLOGICAL PATH

Let  $G$  be a graph and let  $W$  be a walk of  $G$ . We say that  $W$  is minimum length if and only if:

- (Def. 2) For every walk  $W_2$  of  $G$  such that  $W_2$  is walk from  $W.\text{first}()$  to  $W.\text{last}()$  holds  $\text{len } W_2 \geq \text{len } W$ .

The following propositions are true:

- (34) For every graph  $G$  and for every walk  $W$  of  $G$  and for every subwalk  $S$  of  $W$  such that  $S.\text{first}() = W.\text{first}()$  and  $S.\text{edgeSeq}() = W.\text{edgeSeq}()$  holds  $S = W$ .
- (35) For every graph  $G$  and for every walk  $W$  of  $G$  and for every subwalk  $S$  of  $W$  such that  $\text{len } S = \text{len } W$  holds  $S = W$ .
- (36) For every graph  $G$  and for every walk  $W$  of  $G$  such that  $W$  is minimum length holds  $W$  is path-like.
- (37) For every graph  $G$  and for every walk  $W$  of  $G$  such that  $W$  is minimum length holds  $W$  is path-like.
- (38) Let  $G$  be a graph and  $W$  be a walk of  $G$ . Suppose that for every path  $P$  of  $G$  such that  $P$  is walk from  $W.\text{first}()$  to  $W.\text{last}()$  holds  $\text{len } P \geq \text{len } W$ . Then  $W$  is minimum length.
- (39) For every graph  $G$  and for every walk  $W$  of  $G$  holds there exists a path of  $G$  which is walk from  $W.\text{first}()$  to  $W.\text{last}()$  and minimum length.
- (40) Let  $G$  be a graph and  $W$  be a walk of  $G$ . Suppose  $W$  is minimum length. Let  $m, n$  be odd natural numbers. Suppose  $m + 2 < n$  and  $n \leq \text{len } W$ . Then it is not true that there exists a set  $e$  such that  $e$  joins  $W(m)$  and  $W(n)$  in  $G$ .
- (41) Let  $G$  be a graph,  $S$  be a non empty subset of the vertices of  $G$ ,  $H$  be a subgraph of  $G$  induced by  $S$ , and  $W$  be a walk of  $H$ . Suppose  $W$  is minimum length. Let  $m, n$  be odd natural numbers. Suppose  $m + 2 < n$  and  $n \leq \text{len } W$ . Then it is not true that there exists a set  $e$  such that  $e$  joins  $W(m)$  and  $W(n)$  in  $G$ .
- (42) Let  $G$  be a graph and  $W$  be a walk of  $G$ . Suppose  $W$  is minimum length. Let  $m, n$  be odd natural numbers. If  $m \leq n$  and  $n \leq \text{len } W$ , then  $W.\text{cut}(m, n)$  is minimum length.
- (43) Let  $G$  be a graph. Suppose  $G$  is connected. Let  $A, B$  be non empty subsets of the vertices of  $G$ . Suppose  $A$  misses  $B$ . Then there exists a path  $P$  of  $G$  such that

- (i)  $P$  is minimum length and non trivial,
- (ii)  $P.first() \in A$ ,
- (iii)  $P.last() \in B$ , and
- (iv) for every odd natural number  $n$  such that  $1 < n$  and  $n < \text{len } P$  holds  $P(n) \notin A$  and  $P(n) \notin B$ .

#### 4. ADJACENCY AND COMPLETE GRAPHS

Let  $G$  be a graph and let  $a, b$  be vertices of  $G$ . We say that  $a$  and  $b$  are adjacent if and only if:

(Def. 3) There exists a set  $e$  such that  $e$  joins  $a$  and  $b$  in  $G$ .

Let us note that the predicate  $a$  and  $b$  are adjacent is symmetric.

Next we state several propositions:

- (44) Let  $G_1, G_2$  be graphs. Suppose  $G_1 =_G G_2$ . Let  $u_1, v_1$  be vertices of  $G_1$ . Suppose  $u_1$  and  $v_1$  are adjacent. Let  $u_2, v_2$  be vertices of  $G_2$ . If  $u_1 = u_2$  and  $v_1 = v_2$ , then  $u_2$  and  $v_2$  are adjacent.
- (45) Let  $G$  be a graph,  $S$  be a non empty subset of the vertices of  $G$ ,  $H$  be a subgraph of  $G$  induced by  $S$ ,  $u, v$  be vertices of  $G$ , and  $t, w$  be vertices of  $H$ . Suppose  $u = t$  and  $v = w$ . Then  $u$  and  $v$  are adjacent if and only if  $t$  and  $w$  are adjacent.
- (46) For every graph  $G$  and for every walk  $W$  of  $G$  such that  $W.first() \neq W.last()$  and  $W.first()$  and  $W.last()$  are not adjacent holds  $W.length() \geq 2$ .
- (47) Let  $G$  be a graph and  $v_1, v_2, v_3$  be vertices of  $G$ . Suppose  $v_1 \neq v_2$  and  $v_1 \neq v_3$  and  $v_2 \neq v_3$  and  $v_1$  and  $v_2$  are adjacent and  $v_2$  and  $v_3$  are adjacent. Then there exists a path  $P$  of  $G$  and there exist sets  $e_1, e_2$  such that  $P$  is open and  $\text{len } P = 5$  and  $P.length() = 2$  and  $e_1$  joins  $v_1$  and  $v_2$  in  $G$  and  $e_2$  joins  $v_2$  and  $v_3$  in  $G$  and  $P.edges() = \{e_1, e_2\}$  and  $P.vertices() = \{v_1, v_2, v_3\}$  and  $P(1) = v_1$  and  $P(3) = v_2$  and  $P(5) = v_3$ .
- (48) Let  $G$  be a graph and  $v_1, v_2, v_3, v_4$  be vertices of  $G$ . Suppose that  $v_1 \neq v_2$  and  $v_1 \neq v_3$  and  $v_2 \neq v_3$  and  $v_2 \neq v_4$  and  $v_3 \neq v_4$  and  $v_1$  and  $v_2$  are adjacent and  $v_2$  and  $v_3$  are adjacent and  $v_3$  and  $v_4$  are adjacent. Then there exists a path  $P$  of  $G$  such that  $\text{len } P = 7$  and  $P.length() = 3$  and  $P.vertices() = \{v_1, v_2, v_3, v_4\}$  and  $P(1) = v_1$  and  $P(3) = v_2$  and  $P(5) = v_3$  and  $P(7) = v_4$ .

Let  $G$  be a graph and let  $S$  be a set. The functor  $G.adjacentSet(S)$  yields a subset of the vertices of  $G$  and is defined as follows:

(Def. 4)  $G.adjacentSet(S) = \{u; u \text{ ranges over vertices of } G: u \notin S \wedge \bigvee_{v: \text{vertex of } G} (v \in S \wedge u \text{ and } v \text{ are adjacent})\}$ .

One can prove the following propositions:

- (49) For every graph  $G$  and for all sets  $S$ ,  $x$  such that  $x \in G.\text{adjacentSet}(S)$  holds  $x \notin S$ .
- (50) Let  $G$  be a graph,  $S$  be a set, and  $u$  be a vertex of  $G$ . Then  $u \in G.\text{adjacentSet}(S)$  if and only if the following conditions are satisfied:
- (i)  $u \notin S$ , and
  - (ii) there exists a vertex  $v$  of  $G$  such that  $v \in S$  and  $u$  and  $v$  are adjacent.
- (51) For all graphs  $G_1, G_2$  such that  $G_1 =_G G_2$  and for every set  $S$  holds  $G_1.\text{adjacentSet}(S) = G_2.\text{adjacentSet}(S)$ .
- (52) For every graph  $G$  and for all vertices  $u, v$  of  $G$  holds  $u \in G.\text{adjacentSet}(\{v\})$  iff  $u \neq v$  and  $v$  and  $u$  are adjacent.
- (53) For every graph  $G$  and for all sets  $x, y$  holds  $x \in G.\text{adjacentSet}(\{y\})$  iff  $y \in G.\text{adjacentSet}(\{x\})$ .
- (54) Let  $G$  be a graph and  $C$  be a path of  $G$ . Suppose  $C$  is cycle-like and  $C.\text{length}() > 3$ . Let  $x$  be a vertex of  $G$ . Suppose  $x \in C.\text{vertices}()$ . Then there exist odd natural numbers  $m, n$  such that  $m + 2 < n$  and  $n \leq \text{len } C$  and  $m = 1$  and  $n = \text{len } C$  and  $m = 1$  and  $n = \text{len } C - 2$  and  $m = 3$  and  $n = \text{len } C$  and  $C(m) \neq C(n)$  and  $C(m) \in G.\text{adjacentSet}(\{x\})$  and  $C(n) \in G.\text{adjacentSet}(\{x\})$ .
- (55) Let  $G$  be a graph and  $C$  be a path of  $G$ . Suppose  $C$  is cycle-like and  $C.\text{length}() > 3$ . Let  $x$  be a vertex of  $G$ . Suppose  $x \in C.\text{vertices}()$ . Then there exist odd natural numbers  $m, n$  such that
- (i)  $m + 2 < n$ ,
  - (ii)  $n \leq \text{len } C$ ,
  - (iii)  $C(m) \neq C(n)$ ,
  - (iv)  $C(m) \in G.\text{adjacentSet}(\{x\})$ ,
  - (v)  $C(n) \in G.\text{adjacentSet}(\{x\})$ , and
  - (vi) for every set  $e$  such that  $e \in C.\text{edges}()$  holds  $e$  does not join  $C(m)$  and  $C(n)$  in  $G$ .
- (56) For every loopless graph  $G$  and for every vertex  $u$  of  $G$  holds  $G.\text{adjacentSet}(\{u\}) = \emptyset$  iff  $u$  is isolated.
- (57) Let  $G$  be a graph,  $G_0$  be a subgraph of  $G$ ,  $S$  be a non empty subset of the vertices of  $G$ ,  $x$  be a vertex of  $G$ ,  $G_1$  be a subgraph of  $G$  induced by  $S$ , and  $G_2$  be a subgraph of  $G$  induced by  $S \cup \{x\}$ . If  $G_1$  is connected and  $x \in G.\text{adjacentSet}(\text{the vertices of } G_1)$ , then  $G_2$  is connected.
- (58) Let  $G$  be a graph,  $S$  be a non empty subset of the vertices of  $G$ ,  $H$  be a subgraph of  $G$  induced by  $S$ , and  $u$  be a vertex of  $G$ . Suppose  $u \in S$  and  $G.\text{adjacentSet}(\{u\}) \subseteq S$ . Let  $v$  be a vertex of  $H$ . If  $u = v$ , then  $G.\text{adjacentSet}(\{u\}) = H.\text{adjacentSet}(\{v\})$ .

Let  $G$  be a graph and let  $S$  be a set. A subgraph of  $G$  is called an adjacency graph of  $S$  in  $G$  if:

(Def. 5) It is a subgraph of  $G$  induced by  $G.\text{adjacentSet}(S)$  if  $S$  is a subset of the vertices of  $G$ .

Next we state two propositions:

- (59) Let  $G_1, G_2$  be graphs. Suppose  $G_1 =_G G_2$ . Let  $u_1$  be a vertex of  $G_1$  and  $u_2$  be a vertex of  $G_2$ . Suppose  $u_1 = u_2$ . Let  $H_1$  be an adjacency graph of  $\{u_1\}$  in  $G_1$  and  $H_2$  be an adjacency graph of  $\{u_2\}$  in  $G_2$ . Then  $H_1 =_G H_2$ .
- (60) Let  $G$  be a graph,  $S$  be a non empty subset of the vertices of  $G$ ,  $H$  be a subgraph of  $G$  induced by  $S$ , and  $u$  be a vertex of  $G$ . Suppose  $u \in S$  and  $G.\text{adjacentSet}(\{u\}) \subseteq S$  and  $G.\text{adjacentSet}(\{u\}) \neq \emptyset$ . Let  $v$  be a vertex of  $H$ . Suppose  $u = v$ . Let  $G_5$  be an adjacency graph of  $\{u\}$  in  $G$  and  $H_3$  be an adjacency graph of  $\{v\}$  in  $H$ . Then  $G_5 =_G H_3$ .

Let  $G$  be a graph. We say that  $G$  is complete if and only if:

(Def. 6) For all vertices  $u, v$  of  $G$  such that  $u \neq v$  holds  $u$  and  $v$  are adjacent.

We now state the proposition

- (61) For every graph  $G$  such that  $G$  is trivial holds  $G$  is complete.

One can check that every graph which is trivial is also complete.

Let us note that there exists a graph which is trivial, simple, and complete and there exists a graph which is non trivial, finite, simple, and complete.

The following propositions are true:

- (62) For all graphs  $G_1, G_2$  such that  $G_1 =_G G_2$  holds if  $G_1$  is complete, then  $G_2$  is complete.
- (63) For every complete graph  $G$  and for every subset  $S$  of the vertices of  $G$  holds every subgraph of  $G$  induced by  $S$  is complete.

## 5. SIMPLICIAL VERTEX

Let  $G$  be a graph and let  $v$  be a vertex of  $G$ . We say that  $v$  is simplicial if and only if:

(Def. 7) If  $G.\text{adjacentSet}(\{v\}) \neq \emptyset$ , then every adjacency graph of  $\{v\}$  in  $G$  is complete.

The following propositions are true:

- (64) For every complete graph  $G$  holds every vertex of  $G$  is simplicial.
- (65) For every trivial graph  $G$  holds every vertex of  $G$  is simplicial.
- (66) Let  $G_1, G_2$  be graphs. Suppose  $G_1 =_G G_2$ . Let  $u_1$  be a vertex of  $G_1$  and  $u_2$  be a vertex of  $G_2$ . If  $u_1 = u_2$  and  $u_1$  is simplicial, then  $u_2$  is simplicial.
- (67) Let  $G$  be a graph,  $S$  be a non empty subset of the vertices of  $G$ ,  $H$  be a subgraph of  $G$  induced by  $S$ , and  $u$  be a vertex of  $G$ . Suppose  $u \in S$  and  $G.\text{adjacentSet}(\{u\}) \subseteq S$ . Let  $v$  be a vertex of  $H$ . If  $u = v$ , then  $u$  is simplicial iff  $v$  is simplicial.

- (68) Let  $G$  be a graph and  $v$  be a vertex of  $G$ . Suppose  $v$  is simplicial. Let  $a, b$  be sets. Suppose  $a \neq b$  and  $a \in G.\text{adjacentSet}(\{v\})$  and  $b \in G.\text{adjacentSet}(\{v\})$ . Then there exists a set  $e$  such that  $e$  joins  $a$  and  $b$  in  $G$ .
- (69) Let  $G$  be a graph and  $v$  be a vertex of  $G$ . Suppose  $v$  is not simplicial. Then there exist vertices  $a, b$  of  $G$  such that  $a \neq b$  and  $v \neq a$  and  $v \neq b$  and  $v$  and  $a$  are adjacent and  $v$  and  $b$  are adjacent and  $a$  and  $b$  are not adjacent.

## 6. VERTEX SEPARATOR

Let  $G$  be a graph and let  $a, b$  be vertices of  $G$ . Let us assume that  $a \neq b$  and  $a$  and  $b$  are not adjacent. A subset of the vertices of  $G$  is said to be a vertex separator of  $a$  and  $b$  if:

- (Def. 8)  $a \notin S$  and  $b \notin S$  and for every subgraph  $G_2$  of  $G$  with vertices  $S$  removed holds there exists no walk of  $G_2$  which is walk from  $a$  to  $b$ .

Next we state several propositions:

- (70) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Then every vertex separator of  $a$  and  $b$  is a vertex separator of  $b$  and  $a$ .
- (71) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a subset of the vertices of  $G$ . Then  $S$  is a vertex separator of  $a$  and  $b$  if and only if  $a \notin S$  and  $b \notin S$  and for every walk  $W$  of  $G$  such that  $W$  is walk from  $a$  to  $b$  there exists a vertex  $x$  of  $G$  such that  $x \in S$  and  $x \in W.\text{vertices}()$ .
- (72) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$  and  $W$  be a walk of  $G$ . Suppose  $W$  is walk from  $a$  to  $b$ . Then there exists an odd natural number  $k$  such that  $1 < k$  and  $k < \text{len } W$  and  $W(k) \in S$ .
- (73) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$ . If  $S = \emptyset$ , then there exists no walk of  $G$  which is walk from  $a$  to  $b$ .
- (74) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent and there exists no walk of  $G$  which is walk from  $a$  to  $b$ . Then  $\emptyset$  is a vertex separator of  $a$  and  $b$ .
- (75) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$ ,  $G_2$  be a subgraph of  $G$  with vertices  $S$  removed, and  $a_2$  be a vertex of  $G_2$ . If  $a_2 = a$ , then  $(G_2.\text{reachableFrom}(a_2)) \cap S = \emptyset$ .



- (76) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$ ,  $G_2$  be a subgraph of  $G$  with vertices  $S$  removed, and  $a_2, b_2$  be vertices of  $G_2$ . If  $a_2 = a$  and  $b_2 = b$ , then  $(G_2.\text{reachableFrom}(a_2)) \cap (G_2.\text{reachableFrom}(b_2)) = \emptyset$ .
- (77) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$  and  $G_2$  be a subgraph of  $G$  with vertices  $S$  removed. Then  $a$  is a vertex of  $G_2$  and  $b$  is a vertex of  $G_2$ .

Let  $G$  be a graph, let  $a, b$  be vertices of  $G$ , and let  $S$  be a vertex separator of  $a$  and  $b$ . We say that  $S$  is minimal if and only if:

- (Def. 9) For every subset  $T$  of  $S$  such that  $T \neq S$  holds  $T$  is not a vertex separator of  $a$  and  $b$ .

Next we state several propositions:

- (78) Let  $G$  be a graph,  $a, b$  be vertices of  $G$ , and  $S$  be a vertex separator of  $a$  and  $b$ . If  $S = \emptyset$ , then  $S$  is minimal.
- (79) For every finite graph  $G$  and for all vertices  $a, b$  of  $G$  holds there exists a vertex separator of  $a$  and  $b$  which is minimal.
- (80) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$ . Suppose  $S$  is minimal. Let  $T$  be a vertex separator of  $b$  and  $a$ . If  $S = T$ , then  $T$  is minimal.
- (81) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$ . Suppose  $S$  is minimal. Let  $x$  be a vertex of  $G$ . If  $x \in S$ , then there exists a walk  $W$  of  $G$  such that  $W$  is walk from  $a$  to  $b$  and  $x \in W.\text{vertices}()$ .
- (82) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$ . Suppose  $S$  is minimal. Let  $H$  be a subgraph of  $G$  with vertices  $S$  removed and  $a_1$  be a vertex of  $H$ . Suppose  $a_1 = a$ . Let  $x$  be a vertex of  $G$ . Suppose  $x \in S$ . Then there exists a vertex  $y$  of  $G$  such that  $y \in H.\text{reachableFrom}(a_1)$  and  $x$  and  $y$  are adjacent.
- (83) Let  $G$  be a graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$ . Suppose  $S$  is minimal. Let  $H$  be a subgraph of  $G$  with vertices  $S$  removed and  $a_1$  be a vertex of  $H$ . Suppose  $a_1 = b$ . Let  $x$  be a vertex of  $G$ . Suppose  $x \in S$ . Then there exists a vertex  $y$  of  $G$  such that  $y \in H.\text{reachableFrom}(a_1)$  and  $x$  and  $y$  are adjacent.

## 7. CHORDAL GRAPHS

Let  $G$  be a graph and let  $W$  be a walk of  $G$ . We say that  $W$  is chordal if and only if the condition (Def. 10) is satisfied.

- (Def. 10) There exist odd natural numbers  $m, n$  such that
- (i)  $m + 2 < n$ ,
  - (ii)  $n \leq \text{len } W$ ,
  - (iii)  $W(m) \neq W(n)$ ,
  - (iv) there exists a set  $e$  such that  $e$  joins  $W(m)$  and  $W(n)$  in  $G$ , and
  - (v) for every set  $f$  such that  $f \in W.\text{edges}()$  holds  $f$  does not join  $W(m)$  and  $W(n)$  in  $G$ .

Let  $G$  be a graph and let  $W$  be a walk of  $G$ . We introduce  $W$  is chordless as an antonym of  $W$  is chordal.

Next we state a number of propositions:

- (84) Let  $G$  be a graph and  $W$  be a walk of  $G$ . Suppose  $W$  is chordal. Then there exist odd natural numbers  $m, n$  such that
- (i)  $m + 2 < n$ ,
  - (ii)  $n \leq \text{len } W$ ,
  - (iii)  $W(m) \neq W(n)$ ,
  - (iv) there exists a set  $e$  such that  $e$  joins  $W(m)$  and  $W(n)$  in  $G$ , and
  - (v) if  $W$  is cycle-like, then  $m = 1$  and  $n = \text{len } W$  and  $m = 1$  and  $n = \text{len } W - 2$  and  $m = 3$  and  $n = \text{len } W$ .
- (85) Let  $G$  be a graph and  $P$  be a path of  $G$ . Given odd natural numbers  $m, n$  such that
- (i)  $m + 2 < n$ ,
  - (ii)  $n \leq \text{len } P$ ,
  - (iii) there exists a set  $e$  such that  $e$  joins  $P(m)$  and  $P(n)$  in  $G$ , and
  - (iv) if  $P$  is cycle-like, then  $m = 1$  and  $n = \text{len } P$  and  $m = 1$  and  $n = \text{len } P - 2$  and  $m = 3$  and  $n = \text{len } P$ .
- Then  $P$  is chordal.
- (86) Let  $G_1, G_2$  be graphs. Suppose  $G_1 =_G G_2$ . Let  $W_1$  be a walk of  $G_1$  and  $W_2$  be a walk of  $G_2$ . If  $W_1 = W_2$ , then if  $W_1$  is chordal, then  $W_2$  is chordal.
- (87) Let  $G$  be a graph,  $S$  be a non empty subset of the vertices of  $G$ ,  $H$  be a subgraph of  $G$  induced by  $S$ ,  $W_1$  be a walk of  $G$ , and  $W_2$  be a walk of  $H$ . If  $W_1 = W_2$ , then  $W_2$  is chordal iff  $W_1$  is chordal.
- (88) Let  $G$  be a graph and  $W$  be a walk of  $G$ . Suppose  $W$  is cycle-like and chordal and  $W.\text{length}() = 4$ . Then there exists a set  $e$  such that  $e$  joins  $W(1)$  and  $W(5)$  in  $G$  or  $e$  joins  $W(3)$  and  $W(7)$  in  $G$ .
- (89) For every graph  $G$  and for every walk  $W$  of  $G$  such that  $W$  is minimum length holds  $W$  is chordless.

- (90) Let  $G$  be a graph and  $W$  be a walk of  $G$ . Suppose  $W$  is open and  $\text{len } W = 5$  and  $W.\text{first}()$  and  $W.\text{last}()$  are not adjacent. Then  $W$  is chordless.
- (91) For every graph  $G$  and for every walk  $W$  of  $G$  holds  $W$  is chordal iff  $W.\text{reverse}()$  is chordal.
- (92) Let  $G$  be a graph and  $P$  be a path of  $G$ . Suppose  $P$  is open and chordless. Let  $m, n$  be odd natural numbers. Suppose  $m < n$  and  $n \leq \text{len } P$ . Then there exists a set  $e$  such that  $e$  joins  $P(m)$  and  $P(n)$  in  $G$  if and only if  $m + 2 = n$ .
- (93) Let  $G$  be a graph and  $P$  be a path of  $G$ . Suppose  $P$  is open and chordless. Let  $m, n$  be odd natural numbers. If  $m < n$  and  $n \leq \text{len } P$ , then  $P.\text{cut}(m, n)$  is chordless and  $P.\text{cut}(m, n)$  is open.
- (94) Let  $G$  be a graph,  $S$  be a non empty subset of the vertices of  $G$ ,  $H$  be a subgraph of  $G$  induced by  $S$ ,  $W$  be a walk of  $G$ , and  $V$  be a walk of  $H$ . If  $W = V$ , then  $W$  is chordless iff  $V$  is chordless.

Let  $G$  be a graph. We say that  $G$  is chordal if and only if:

- (Def. 11) For every walk  $P$  of  $G$  such that  $P.\text{length}() > 3$  and  $P$  is cycle-like holds  $P$  is chordal.

Next we state two propositions:

- (95) For all graphs  $G_1, G_2$  such that  $G_1 =_G G_2$  holds if  $G_1$  is chordal, then  $G_2$  is chordal.
- (96) For every finite graph  $G$  such that  $\text{card}(\text{the vertices of } G) \leq 3$  holds  $G$  is chordal.

One can verify the following observations:

- \* there exists a graph which is trivial, finite, and chordal,
- \* there exists a graph which is non trivial, finite, simple, and chordal, and
- \* every graph which is complete is also chordal.

Let  $G$  be a chordal graph and let  $V$  be a set. One can check that every subgraph of  $G$  induced by  $V$  is chordal.

Next we state several propositions:

- (97) Let  $G$  be a chordal graph and  $P$  be a path of  $G$ . Suppose  $P$  is open and chordless. Let  $x, e$  be sets. Suppose  $x \notin P.\text{vertices}()$  and  $e$  joins  $P.\text{last}()$  and  $x$  in  $G$  and it is not true that there exists a set  $f$  such that  $f$  joins  $P(\text{len } P - 2)$  and  $x$  in  $G$ . Then  $P.\text{addEdge}(e)$  is path-like and  $P.\text{addEdge}(e)$  is open and  $P.\text{addEdge}(e)$  is chordless.
- (98) Let  $G$  be a chordal graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$ . If  $S$  is minimal and non empty, then every subgraph of  $G$  induced by  $S$  is complete.
- (99) Let  $G$  be a finite graph. Suppose that for all vertices  $a, b$  of  $G$  such that

$a \neq b$  and  $a$  and  $b$  are not adjacent and for every vertex separator  $S$  of  $a$  and  $b$  such that  $S$  is minimal and non empty holds every subgraph of  $G$  induced by  $S$  is complete. Then  $G$  is chordal.

- (100) Let  $G$  be a finite chordal graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$ . Suppose  $S$  is minimal. Let  $H$  be a subgraph of  $G$  with vertices  $S$  removed and  $a_3$  be a vertex of  $H$ . Suppose  $a = a_3$ . Then there exists a vertex  $c$  of  $G$  such that  $c \in H.\text{reachableFrom}(a_3)$  and for every vertex  $x$  of  $G$  such that  $x \in S$  holds  $c$  and  $x$  are adjacent.
- (101) Let  $G$  be a finite chordal graph and  $a, b$  be vertices of  $G$ . Suppose  $a \neq b$  and  $a$  and  $b$  are not adjacent. Let  $S$  be a vertex separator of  $a$  and  $b$ . Suppose  $S$  is minimal. Let  $H$  be a subgraph of  $G$  with vertices  $S$  removed and  $a_3$  be a vertex of  $H$ . Suppose  $a = a_3$ . Let  $x, y$  be vertices of  $G$ . Suppose  $x \in S$  and  $y \in S$ . Then there exists a vertex  $c$  of  $G$  such that  $c \in H.\text{reachableFrom}(a_3)$  and  $c$  and  $x$  are adjacent and  $c$  and  $y$  are adjacent.
- (102) Let  $G$  be a non trivial finite chordal graph. Suppose  $G$  is not complete. Then there exist vertices  $a, b$  of  $G$  such that  $a \neq b$  and  $a$  and  $b$  are not adjacent and  $a$  is simplicial and  $b$  is simplicial.
- (103) For every finite chordal graph  $G$  holds there exists a vertex of  $G$  which is simplicial.

## 8. VERTEX ELIMINATION SCHEME

Let  $G$  be a finite graph. A finite sequence of elements of the vertices of  $G$  is said to be a vertex scheme of  $G$  if:

(Def. 12) It is one-to-one and  $\text{rng it} = \text{the vertices of } G$ .

Let  $G$  be a finite graph. Note that every vertex scheme of  $G$  is non empty.

The following three propositions are true:

- (104) For every finite graph  $G$  and for every vertex scheme  $S$  of  $G$  holds  $\text{len } S = \text{card}(\text{the vertices of } G)$ .
- (105) For every finite graph  $G$  and for every vertex scheme  $S$  of  $G$  holds  $1 \leq \text{len } S$ .
- (106) For all finite graphs  $G, H$  and for every vertex scheme  $g$  of  $G$  such that  $G =_G H$  holds  $g$  is a vertex scheme of  $H$ .

Let  $G$  be a finite graph, let  $S$  be a vertex scheme of  $G$ , and let  $x$  be a vertex of  $G$ . Then  $x \leftarrow S$  is a non zero element of  $\mathbb{N}$ .

Let  $G$  be a finite graph, let  $S$  be a vertex scheme of  $G$ , and let  $n$  be a natural number. Then  $S.\text{followSet}(n)$  is a subset of the vertices of  $G$ .

Next we state the proposition

(107) Let  $G$  be a finite graph,  $S$  be a vertex scheme of  $G$ , and  $n$  be a non zero natural number. If  $n \leq \text{len } S$ , then  $S.\text{followSet}(n)$  is non empty.

Let  $G$  be a finite graph and let  $S$  be a vertex scheme of  $G$ . We say that  $S$  is perfect if and only if the condition (Def. 13) is satisfied.

(Def. 13) Let  $n$  be a non zero natural number. Suppose  $n \leq \text{len } S$ . Let  $G_6$  be a subgraph of  $G$  induced by  $S.\text{followSet}(n)$  and  $v$  be a vertex of  $G_6$ . If  $v = S(n)$ , then  $v$  is simplicial.

One can prove the following propositions:

(108) Let  $G$  be a finite trivial graph and  $v$  be a vertex of  $G$ . Then there exists a vertex scheme  $S$  of  $G$  such that  $S = \langle v \rangle$  and  $S$  is perfect.

(109) Let  $G$  be a finite graph and  $V$  be a vertex scheme of  $G$ . Then  $V$  is perfect if and only if for all vertices  $a, b, c$  of  $G$  such that  $b \neq c$  and  $a$  and  $b$  are adjacent and  $a$  and  $c$  are adjacent and for all natural numbers  $v_5, v_6, v_7$  such that  $v_5 \in \text{dom } V$  and  $v_6 \in \text{dom } V$  and  $v_7 \in \text{dom } V$  and  $V(v_5) = a$  and  $V(v_6) = b$  and  $V(v_7) = c$  and  $v_5 < v_6$  and  $v_5 < v_7$  holds  $b$  and  $c$  are adjacent.

Let  $G$  be a finite chordal graph. One can check that there exists a vertex scheme of  $G$  which is perfect.

The following propositions are true:

(110) Let  $G, H$  be finite chordal graphs and  $g$  be a perfect vertex scheme of  $G$ . If  $G =_G H$ , then  $g$  is a perfect vertex scheme of  $H$ .

(111) For every finite graph  $G$  such that there exists a vertex scheme of  $G$  which is perfect holds  $G$  is chordal.

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