

Connectedness and Continuous Sequences in Finite Topological Spaces

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Summary. First, equivalence conditions for connectedness are examined for a finite topological space (originated in [9]). Secondly, definitions of subspace, and components of the subspace of a finite topological space are given. Lastly, concepts of continuous finite sequence and minimum path of finite topological space are proposed.

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The articles [16], [5], [18], [13], [1], [19], [14], [3], [4], [2], [6], [12], [10], [15], [7], [11], [8], and [17] provide the terminology and notation for this paper.

1. CONNECTEDNESS AND SUBSPACES

In this paper F_1 denotes a non empty finite topology space and A, B, C denote subsets of F_1 .

Let us consider F_1 . One can check that $\emptyset_{(F_1)}$ is connected.

We now state two propositions:

- (1) For all subsets A, B of F_1 holds $(A \cup B)^b = A^b \cup B^b$.
- (2) $(\emptyset_{(F_1)})^b = \emptyset$.

Let us consider F_1 . Observe that $(\emptyset_{(F_1)})^b$ is empty.

Next we state the proposition

- (3) Let A be a subset of F_1 . Suppose that for all subsets B, C of F_1 such that $A = B \cup C$ and $B \neq \emptyset$ and $C \neq \emptyset$ and B misses C holds B^b meets C and B meets C^b . Then A is connected.

Let F_1 be a non empty finite topology space. We say that F_1 is connected if and only if:

(Def. 1) $\Omega_{(F_1)}$ is connected.

We now state four propositions:

- (4) Let A be a subset of F_1 . Suppose A is connected. Let A_2, B_2 be subsets of F_1 . Suppose $A = A_2 \cup B_2$ and A_2 misses B_2 and A_2 and B_2 are separated. Then $A_2 = \emptyset_{(F_1)}$ or $B_2 = \emptyset_{(F_1)}$.
- (5) Suppose F_1 is connected. Let A, B be subsets of F_1 . Suppose $\Omega_{(F_1)} = A \cup B$ and A misses B and A and B are separated. Then $A = \emptyset_{(F_1)}$ or $B = \emptyset_{(F_1)}$.
- (6) For all subsets A, B of F_1 such that F_1 is symmetric and A^b misses B holds A misses B^b .
- (7) Let A be a subset of F_1 . Suppose that
 - (i) F_1 is symmetric, and
 - (ii) for all subsets A_2, B_2 of F_1 such that $A = A_2 \cup B_2$ and A_2 misses B_2 and A_2 and B_2 are separated holds $A_2 = \emptyset_{(F_1)}$ or $B_2 = \emptyset_{(F_1)}$.
 Then A is connected.

Let T be a finite topology space. A finite topology space is said to be a subspace of T if it satisfies the conditions (Def. 2).

- (Def. 2)(i) The carrier of it \subseteq the carrier of T ,
- (ii) $\text{dom}(\text{the neighbour-map of it}) = \text{the carrier of it}$, and
 - (iii) for every element x of it such that $x \in \text{the carrier of it}$ holds (the neighbour-map of it)(x) = (the neighbour-map of T)(x) \cap the carrier of it.

Let T be a finite topology space. Note that there exists a subspace of T which is strict.

Let T be a non empty finite topology space. Note that there exists a subspace of T which is strict and non empty.

Let T be a non empty finite topology space and let P be a non empty subset of T . The functor $T \upharpoonright P$ yields a strict non empty subspace of T and is defined as follows:

(Def. 3) $\Omega_{T \upharpoonright P} = P$.

We now state the proposition

- (8) For every non empty subspace X of F_1 such that F_1 is filled holds X is filled.

Let F_1 be a filled non empty finite topology space. Note that every non empty subspace of F_1 is filled.

Next we state a number of propositions:

- (9) For every non empty subspace X of F_1 such that F_1 is symmetric holds X is symmetric.
- (10) For every subspace X' of F_1 holds every subset of X' is a subset of F_1 .

- (11) For every subset P of F_1 holds P is closed iff P^c is open.
- (12) Let A be a subset of F_1 . Then A is open if and only if the following conditions are satisfied:
- (i) for every element z of F_1 such that $U(z) \subseteq A$ holds $z \in A$, and
 - (ii) for every element x of F_1 such that $x \in A$ holds $U(x) \subseteq A$.
- (13) Let X' be a non empty subspace of F_1 , A be a subset of F_1 , and A_1 be a subset of X' . If $A = A_1$, then $A_1^b = A^b \cap \Omega_{X'}$.
- (14) Let X' be a non empty subspace of F_1 , P_1, Q_1 be subsets of F_1 , and P, Q be subsets of X' . Suppose $P = P_1$ and $Q = Q_1$. If P and Q are separated, then P_1 and Q_1 are separated.
- (15) Let X' be a non empty subspace of F_1 , P, Q be subsets of F_1 , and P_1, Q_1 be subsets of X' . Suppose $P = P_1$ and $Q = Q_1$ and $P \cup Q \subseteq \Omega_{X'}$. If P and Q are separated, then P_1 and Q_1 are separated.
- (16) For every non empty subset A of F_1 holds A is connected iff $F_1 \upharpoonright A$ is connected.
- (17) Let F_1 be a filled non empty finite topology space and A be a non empty subset of F_1 . Suppose F_1 is symmetric. Then A is connected if and only if for all subsets P, Q of F_1 such that $A = P \cup Q$ and P misses Q and P and Q are separated holds $P = \emptyset_{(F_1)}$ or $Q = \emptyset_{(F_1)}$.
- (18) For every subset A of F_1 such that F_1 is filled and connected and $A \neq \emptyset$ and $A^c \neq \emptyset$ holds $A^\delta \neq \emptyset$.
- (19) For every subset A of F_1 such that F_1 is filled, symmetric, and connected and $A \neq \emptyset$ and $A^c \neq \emptyset$ holds $A^{\delta_i} \neq \emptyset$.
- (20) For every subset A of F_1 such that F_1 is filled, symmetric, and connected and $A \neq \emptyset$ and $A^c \neq \emptyset$ holds $A^{\delta_o} \neq \emptyset$.
- (21) For every subset A of F_1 holds A^{δ_i} misses A^{δ_o} .
- (22) For every filled non empty finite topology space F_1 and for every subset A of F_1 holds $A^{\delta_o} = A^b \setminus A$.
- (23) For all subsets A, B of F_1 such that A and B are separated holds A^{δ_o} misses B .
- (24) Let A, B be subsets of F_1 . Suppose F_1 is filled and A misses B and A^{δ_o} misses B and B^{δ_o} misses A . Then A and B are separated.
- (25) For every point x of F_1 holds $\{x\}$ is connected.

Let us consider F_1 and let x be a point of F_1 . Note that $\{x\}$ is connected.

Let F_1 be a non empty finite topology space and let A be a subset of F_1 .

We say that A is a component of F_1 if and only if:

- (Def. 4) A is connected and for every subset B of F_1 such that B is connected holds if $A \subseteq B$, then $A = B$.

One can prove the following propositions:

- (26) For every subset A of F_1 such that A is a component of F_1 holds $A \neq \emptyset_{(F_1)}$.
- (27) If A is closed and B is closed and A misses B , then A and B are separated.
- (28) If F_1 is filled and $\Omega_{(F_1)} = A \cup B$ and A and B are separated, then A is open and closed.
- (29) For all subsets A, B, A_1, B_1 of F_1 such that A and B are separated and $A_1 \subseteq A$ and $B_1 \subseteq B$ holds A_1 and B_1 are separated.
- (30) If A and B are separated and A and C are separated, then A and $B \cup C$ are separated.
- (31) Suppose that
- (i) F_1 is filled and symmetric, and
 - (ii) for all subsets A, B of F_1 such that $\Omega_{(F_1)} = A \cup B$ and $A \neq \emptyset_{(F_1)}$ and $B \neq \emptyset_{(F_1)}$ and A is closed and B is closed holds A meets B .
- Then F_1 is connected.
- (32) Suppose F_1 is connected. Let A, B be subsets of F_1 . Suppose $\Omega_{(F_1)} = A \cup B$ and $A \neq \emptyset_{(F_1)}$ and $B \neq \emptyset_{(F_1)}$ and A is closed and B is closed. Then A meets B .
- (33) If F_1 is filled and A is connected and $A \subseteq B \cup C$ and B and C are separated, then $A \subseteq B$ or $A \subseteq C$.
- (34) Let A, B be subsets of F_1 . Suppose F_1 is symmetric and A is connected and B is connected and A and B are not separated. Then $A \cup B$ is connected.
- (35) For all subsets A, C of F_1 such that F_1 is symmetric and C is connected and $C \subseteq A$ and $A \subseteq C^b$ holds A is connected.
- (36) For every subset C of F_1 such that F_1 is filled and symmetric and C is connected holds C^b is connected.
- (37) Suppose F_1 is filled, symmetric, and connected and A is connected and $\Omega_{(F_1)} \setminus A = B \cup C$ and B and C are separated. Then $A \cup B$ is connected.
- (38) Let X' be a non empty subspace of F_1 , A be a subset of F_1 , and B be a subset of X' . Suppose F_1 is symmetric and $A = B$. Then A is connected if and only if B is connected.
- (39) For every subset A of F_1 such that F_1 is filled and symmetric and A is a component of F_1 holds A is closed.
- (40) Let A, B be subsets of F_1 . Suppose F_1 is symmetric and A is a component of F_1 and B is a component of F_1 . Then $A = B$ or A and B are separated.
- (41) Let A, B be subsets of F_1 . Suppose F_1 is filled and symmetric and A is a component of F_1 and B is a component of F_1 . Then $A = B$ or A misses B .

- (42) Let C be a subset of F_1 . Suppose F_1 is filled and symmetric and C is connected. Let S be a subset of F_1 . If S is a component of F_1 , then C misses S or $C \subseteq S$.

Let F_1 be a non empty finite topology space, let A be a non empty subset of F_1 , and let B be a subset of F_1 . We say that B is a component of A if and only if:

- (Def. 5) There exists a subset B_1 of $F_1 \upharpoonright A$ such that $B_1 = B$ and B_1 is a component of $F_1 \upharpoonright A$.

We now state the proposition

- (43) Let D be a non empty subset of F_1 . Suppose F_1 is filled and symmetric and $D = \Omega_{(F_1)} \setminus A$. Suppose F_1 is connected and A is connected and C is a component of D . Then $\Omega_{(F_1)} \setminus C$ is connected.

2. CONTINUOUS FINITE SEQUENCES AND MINIMUM PATH

Let us consider F_1 and let f be a finite sequence of elements of F_1 . We say that f is continuous if and only if the conditions (Def. 6) are satisfied.

- (Def. 6)(i) $1 \leq \text{len } f$, and
- (ii) for every natural number i and for every element x_1 of F_1 such that $1 \leq i$ and $i < \text{len } f$ and $x_1 = f(i)$ holds $f(i + 1) \in U(x_1)$.

Let us consider F_1 and let x be an element of F_1 . Observe that $\langle x \rangle$ is continuous.

One can prove the following two propositions:

- (44) Let f be a finite sequence of elements of F_1 and x, y be elements of F_1 . If f is continuous and $y = f(\text{len } f)$ and $x \in U(y)$, then $f \hat{\ } \langle x \rangle$ is continuous.
- (45) Let f, g be finite sequences of elements of F_1 . Suppose f is continuous and g is continuous and $g(1) \in U(f_{\text{len } f})$. Then $f \hat{\ } g$ is continuous.

Let us consider F_1 and let A be a subset of F_1 . We say that A is arcwise connected if and only if the condition (Def. 7) is satisfied.

- (Def. 7) Let x_1, x_2 be elements of F_1 . Suppose $x_1 \in A$ and $x_2 \in A$. Then there exists a finite sequence f of elements of F_1 such that f is continuous and $\text{rng } f \subseteq A$ and $f(1) = x_1$ and $f(\text{len } f) = x_2$.

Let us consider F_1 . Observe that $\emptyset_{(F_1)}$ is arcwise connected.

Let us consider F_1 and let x be an element of F_1 . One can verify that $\{x\}$ is arcwise connected.

The following three propositions are true:

- (46) For every subset A of F_1 such that F_1 is symmetric holds A is connected iff A is arcwise connected.
- (47) Let g be a finite sequence of elements of F_1 and k be a natural number. If g is continuous and $1 \leq k$, then $g \upharpoonright k$ is continuous.

- (48) Let g be a finite sequence of elements of F_1 and k be an element of \mathbb{N} .
If g is continuous and $k < \text{len } g$, then $g \downarrow k$ is continuous.

Let us consider F_1 , let g be a finite sequence of elements of F_1 , let A be a subset of F_1 , and let x_1, x_2 be elements of F_1 . We say that g is minimum path in A between x_1 and x_2 if and only if the conditions (Def. 8) are satisfied.

- (Def. 8)(i) g is continuous,
(ii) $\text{rng } g \subseteq A$,
(iii) $g(1) = x_1$,
(iv) $g(\text{len } g) = x_2$, and
(v) for every finite sequence h of elements of F_1 such that h is continuous and $\text{rng } h \subseteq A$ and $h(1) = x_1$ and $h(\text{len } h) = x_2$ holds $\text{len } g \leq \text{len } h$.

One can prove the following propositions:

- (49) For every subset A of F_1 and for every element x of F_1 such that $x \in A$ holds $\langle x \rangle$ is minimum path in A between x and x .
- (50) Let A be a subset of F_1 . Then A is arcwise connected if and only if for all elements x_1, x_2 of F_1 such that $x_1 \in A$ and $x_2 \in A$ holds there exists a finite sequence of elements of F_1 which is minimum path in A between x_1 and x_2 .
- (51) Let A be a subset of F_1 and x_1, x_2 be elements of F_1 . Given a finite sequence f of elements of F_1 such that f is continuous and $\text{rng } f \subseteq A$ and $f(1) = x_1$ and $f(\text{len } f) = x_2$. Then there exists a finite sequence of elements of F_1 which is minimum path in A between x_1 and x_2 .
- (52) Let g be a finite sequence of elements of F_1 , A be a subset of F_1 , x_1, x_2 be elements of F_1 , and k be an element of \mathbb{N} . Suppose g is minimum path in A between x_1 and x_2 and $1 \leq k$ and $k \leq \text{len } g$. Then $g \uparrow k$ is continuous and $\text{rng}(g \uparrow k) \subseteq A$ and $(g \uparrow k)(1) = x_1$ and $(g \uparrow k)(\text{len}(g \uparrow k)) = g_k$.
- (53) Let g be a finite sequence of elements of F_1 , A be a subset of F_1 , x_1, x_2 be elements of F_1 , and k be an element of \mathbb{N} . Suppose g is minimum path in A between x_1 and x_2 and $k < \text{len } g$. Then $g \downarrow k$ is continuous and $\text{rng}(g \downarrow k) \subseteq A$ and $g \downarrow k(1) = g_{1+k}$ and $g \downarrow k(\text{len}(g \downarrow k)) = x_2$.
- (54) Let g be a finite sequence of elements of F_1 , A be a subset of F_1 , and x_1, x_2 be elements of F_1 . Suppose g is minimum path in A between x_1 and x_2 . Let k be a natural number. If $1 \leq k$ and $k \leq \text{len } g$, then $g \uparrow k$ is minimum path in A between x_1 and g_k .
- (55) Let g be a finite sequence of elements of F_1 , A be a subset of F_1 , and x_1, x_2 be elements of F_1 . If g is minimum path in A between x_1 and x_2 , then g is one-to-one.

Let us consider F_1 and let f be a finite sequence of elements of F_1 . We say that f is inversely continuous if and only if the conditions (Def. 9) are satisfied.

- (Def. 9)(i) $1 \leq \text{len } f$, and

- (ii) for all natural numbers i, j and for every element y of F_1 such that $1 \leq i$ and $i \leq \text{len } f$ and $1 \leq j$ and $j \leq \text{len } f$ and $y = f(i)$ and $i \neq j$ and $f(j) \in U(y)$ holds $i = j + 1$ or $j = i + 1$.

We now state three propositions:

- (56) Let g be a finite sequence of elements of F_1 , A be a subset of F_1 , and x_1, x_2 be elements of F_1 . Suppose g is minimum path in A between x_1 and x_2 and F_1 is symmetric. Then g is inversely continuous.
- (57) Let g be a finite sequence of elements of F_1 , A be a subset of F_1 , and x_1, x_2 be elements of F_1 . Suppose g is minimum path in A between x_1 and x_2 and F_1 is filled and symmetric and $x_1 \neq x_2$. Then
- (i) for every natural number i such that $1 < i$ and $i < \text{len } g$ holds $\text{rng } g \cap U(g_i) = \{g(i-1), g(i), g(i+1)\}$,
- (ii) $\text{rng } g \cap U(g_1) = \{g(1), g(2)\}$, and
- (iii) $\text{rng } g \cap U(g_{\text{len } g}) = \{g(\text{len } g - 1), g(\text{len } g)\}$.
- (58) Let g be a finite sequence of elements of F_1 , A be a non empty subset of F_1 , x_1, x_2 be elements of F_1 , and B_0 be a subset of $F_1 \setminus A$. Suppose g is minimum path in A between x_1 and x_2 and F_1 is filled and symmetric and $x_1 \neq x_2$ and $B_0 = \{x_1\}$. Let i be an element of \mathbb{N} . If $i < \text{len } g$, then $g(i+1) \in \text{Finf}(B_0, i)$ and if $i \geq 1$, then $g(i+1) \notin \text{Finf}(B_0, i-1)$.

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