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Connectedness and Continuous Sequences in Finite Topological Spaces

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Summary. First, equivalence conditions for connectedness are examined for a finite topological space (originated in [9]). Secondly, definitions of subspace, and components of the subspace of a finite topological space are given. Lastly, concepts of continuous finite sequence and minimum path of finite topological space are proposed.

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The articles [16], [5], [18], [13], [1], [19], [14], [3], [4], [2], [6], [12], [10], [15], [7], [11], [8], and [17] provide the terminology and notation for this paper.

1. Connectedness and Subspaces

In this paper F_1 denotes a non empty finite topology space and A, B, C denote subsets of F_1.

Let us consider F_1. One can check that $\emptyset_{(F_1)}$ is connected.

We now state two propositions:

(1) For all subsets A, B of F_1 holds $(A \cup B)^b = A^b \cup B^b$.

(2) $(\emptyset_{(F_1)})^b = \emptyset$.

Let us consider F_1. Observe that $(\emptyset_{(F_1)})^b$ is empty.

Next we state the proposition

(3) Let A be a subset of F_1. Suppose that for all subsets B, C of F_1 such that $A = B \cup C$ and $B \neq \emptyset$ and $C \neq \emptyset$ and B misses C holds B^b meets C and B meets C^b. Then A is connected.

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Let F_1 be a non empty finite topology space. We say that F_1 is connected if and only if:

(Def. 1) $\Omega_{(F_1)}$ is connected.

We now state four propositions:

(4) Let A be a subset of F_1. Suppose A is connected. Let A_2, B_2 be subsets of F_1. Suppose $A = A_2 \cup B_2$ and A_2 misses B_2 and A_2 and B_2 are separated. Then $A_2 = \emptyset_{(F_1)}$ or $B_2 = \emptyset_{(F_1)}$.

(5) Suppose F_1 is connected. Let A, B be subsets of F_1. Suppose $\Omega_{(F_1)} = A \cup B$ and A misses B and A and B are separated. Then $A = \emptyset_{(F_1)}$ or $B = \emptyset_{(F_1)}$.

(6) For all subsets A, B of F_1 such that F_1 is symmetric and A^b misses B holds A misses B^b.

(7) Let A be a subset of F_1. Suppose that

(i) F_1 is symmetric, and

(ii) for all subsets A_2, B_2 of F_1 such that $A = A_2 \cup B_2$ and A_2 misses B_2 and A_2 and B_2 are separated holds $A_2 = \emptyset_{(F_1)}$ or $B_2 = \emptyset_{(F_1)}$.

Then A is connected.

Let T be a finite topology space. A finite topology space is said to be a subspace of T if it satisfies the conditions (Def. 2).

(Def. 2)(i) The carrier of it $\subseteq$ the carrier of T,

(ii) dom (the neighbour-map of it) = the carrier of it, and

(iii) for every element x of it such that $x \in$ the carrier of it holds (the neighbour-map of it)(x) = (the neighbour-map of T)$(x) \cap$ the carrier of it.

Let T be a finite topology space. Note that there exists a subspace of T which is strict.

Let T be a non empty finite topology space. Note that there exists a subspace of T which is strict and non empty.

Let T be a non empty finite topology space and let P be a non empty subset of T. The functor $T{\restriction}P$ yields a strict non empty subspace of T and is defined as follows:

(Def. 3) $\Omega_{T\restriction P} = P$.

We now state the proposition

(8) For every non empty subspace X of F_1 such that F_1 is filled holds X is filled.

Let F_1 be a filled non empty finite topology space. Note that every non empty subspace of F_1 is filled.

Next we state a number of propositions:

(9) For every non empty subspace X of F_1 such that F_1 is symmetric holds X is symmetric.

(10) For every subspace X' of F_1 holds every subset of X' is a subset of F_1.

(11) For every subset P of F_1 holds P is closed iff $P^{\rm c}$ is open.

(12) Let A be a subset of F_1. Then A is open if and only if the following conditions are satisfied:

(i) for every element z of F_1 such that $U(z) \subseteq A$ holds $z \in A$, and

(ii) for every element x of F_1 such that $x \in A$ holds $U(x) \subseteq A$.

(13) Let X' be a non empty subspace of F_1, A be a subset of F_1, and A_1 be a subset of X'. If $A = A_1$, then ${A_1}^b = A^b \cap \Omega_{X'}$.

(14) Let X' be a non empty subspace of F_1, P_1, Q_1 be subsets of F_1, and P, Q be subsets of X'. Suppose $P = P_1$ and $Q = Q_1$. If P and Q are separated, then P_1 and Q_1 are separated.

(15) Let X' be a non empty subspace of F_1, P, Q be subsets of F_1, and P_1, Q_1 be subsets of X'. Suppose $P = P_1$ and $Q = Q_1$ and $P \cup Q \subseteq \Omega_{X'}$. If P and Q are separated, then P_1 and Q_1 are separated.

(16) For every non empty subset A of F_1 holds A is connected iff $F_1 \restriction A$ is connected.

(17) Let F_1 be a filled non empty finite topology space and A be a non empty subset of F_1. Suppose F_1 is symmetric. Then A is connected if and only if for all subsets P, Q of F_1 such that $A = P \cup Q$ and P misses Q and P and Q are separated holds $P = \emptyset_{(F_1)}$ or $Q = \emptyset_{(F_1)}$.

(18) For every subset A of F_1 such that F_1 is filled and connected and $A \neq \emptyset$ and $A^{\rm c} \neq \emptyset$ holds $A^\delta \neq \emptyset$.

(19) For every subset A of F_1 such that F_1 is filled, symmetric, and connected and $A \neq \emptyset$ and $A^{\rm c} \neq \emptyset$ holds $A^{\delta_i} \neq \emptyset$.

(20) For every subset A of F_1 such that F_1 is filled, symmetric, and connected and $A \neq \emptyset$ and $A^{\rm c} \neq \emptyset$ holds $A^{\delta_o} \neq \emptyset$.

(21) For every subset A of F_1 holds A^{δ_i} misses A^{δ_o}.

(22) For every filled non empty finite topology space F_1 and for every subset A of F_1 holds $A^{\delta_o} = A^b \setminus A$.

(23) For all subsets A, B of F_1 such that A and B are separated holds A^{δ_o} misses B.

(24) Let A, B be subsets of F_1. Suppose F_1 is filled and A misses B and A^{δ_o} misses B and B^{δ_o} misses A. Then A and B are separated.

(25) For every point x of F_1 holds $\{x\}$ is connected.

Let us consider F_1 and let x be a point of F_1. Note that $\{x\}$ is connected.

Let F_1 be a non empty finite topology space and let A be a subset of F_1. We say that A is a component of F_1 if and only if:

(Def. 4) A is connected and for every subset B of F_1 such that B is connected holds if $A \subseteq B$, then $A = B$.

One can prove the following propositions:

(26) For every subset A of F_1 such that A is a component of F_1 holds $A \neq \emptyset_{(F_1)}$.

(27) If A is closed and B is closed and A misses B, then A and B are separated.

(28) If F_1 is filled and $\Omega_{(F_1)} = A \cup B$ and A and B are separated, then A is open and closed.

(29) For all subsets A, B, A_1, B_1 of F_1 such that A and B are separated and $A_1 \subseteq A$ and $B_1 \subseteq B$ holds A_1 and B_1 are separated.

(30) If A and B are separated and A and C are separated, then A and $B \cup C$ are separated.

(31) Suppose that
 (i) F_1 is filled and symmetric, and
 (ii) for all subsets A, B of F_1 such that $\Omega_{(F_1)} = A \cup B$ and $A \neq \emptyset_{(F_1)}$ and $B \neq \emptyset_{(F_1)}$ and A is closed and B is closed holds A meets B.

 Then F_1 is connected.

(32) Suppose F_1 is connected. Let A, B be subsets of F_1. Suppose $\Omega_{(F_1)} = A \cup B$ and $A \neq \emptyset_{(F_1)}$ and $B \neq \emptyset_{(F_1)}$ and A is closed and B is closed. Then A meets B.

(33) If F_1 is filled and A is connected and $A \subseteq B \cup C$ and B and C are separated, then $A \subseteq B$ or $A \subseteq C$.

(34) Let A, B be subsets of F_1. Suppose F_1 is symmetric and A is connected and B is connected and A and B are not separated. Then $A \cup B$ is connected.

(35) For all subsets A, C of F_1 such that F_1 is symmetric and C is connected and $C \subseteq A$ and $A \subseteq C^b$ holds A is connected.

(36) For every subset C of F_1 such that F_1 is filled and symmetric and C is connected holds C^b is connected.

(37) Suppose F_1 is filled, symmetric, and connected and A is connected and $\Omega_{(F_1)} \setminus A = B \cup C$ and B and C are separated. Then $A \cup B$ is connected.

(38) Let X' be a non empty subspace of F_1, A be a subset of F_1, and B be a subset of X'. Suppose F_1 is symmetric and $A = B$. Then A is connected if and only if B is connected.

(39) For every subset A of F_1 such that F_1 is filled and symmetric and A is a component of F_1 holds A is closed.

(40) Let A, B be subsets of F_1. Suppose F_1 is symmetric and A is a component of F_1 and B is a component of F_1. Then $A = B$ or A and B are separated.

(41) Let A, B be subsets of F_1. Suppose F_1 is filled and symmetric and A is a component of F_1 and B is a component of F_1. Then $A = B$ or A misses B.

(42) Let C be a subset of F_1. Suppose F_1 is filled and symmetric and C is connected. Let S be a subset of F_1. If S is a component of F_1, then C misses S or $C \subseteq S$.

Let F_1 be a non empty finite topology space, let A be a non empty subset of F_1, and let B be a subset of F_1. We say that B is a component of A if and only if:

(Def. 5) There exists a subset B_1 of $F_1 \restriction A$ such that $B_1 = B$ and B_1 is a component of $F_1 \restriction A$.

We now state the proposition

(43) Let D be a non empty subset of F_1. Suppose F_1 is filled and symmetric and $D = \Omega_{(F_1)} \setminus A$. Suppose F_1 is connected and A is connected and C is a component of D. Then $\Omega_{(F_1)} \setminus C$ is connected.

2. Continuous Finite Sequences and Minimum Path

Let us consider F_1 and let f be a finite sequence of elements of F_1. We say that f is continuous if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) $1 \leq \operatorname{len} f$, and

(ii) for every natural number i and for every element x_1 of F_1 such that $1 \leq i$ and $i < \operatorname{len} f$ and $x_1 = f(i)$ holds $f(i+1) \in U(x_1)$.

Let us consider F_1 and let x be an element of F_1. Observe that $\langle x \rangle$ is continuous.

One can prove the following two propositions:

(44) Let f be a finite sequence of elements of F_1 and x, y be elements of F_1. If f is continuous and $y = f(\operatorname{len} f)$ and $x \in U(y)$, then $f \frown \langle x \rangle$ is continuous.

(45) Let f, g be finite sequences of elements of F_1. Suppose f is continuous and g is continuous and $g(1) \in U(f_{\operatorname{len} f})$. Then $f \frown g$ is continuous.

Let us consider F_1 and let A be a subset of F_1. We say that A is arcwise connected if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let x_1, x_2 be elements of F_1. Suppose $x_1 \in A$ and $x_2 \in A$. Then there exists a finite sequence f of elements of F_1 such that f is continuous and $\operatorname{rng} f \subseteq A$ and $f(1) = x_1$ and $f(\operatorname{len} f) = x_2$.

Let us consider F_1. Observe that $\emptyset_{(F_1)}$ is arcwise connected.

Let us consider F_1 and let x be an element of F_1. One can verify that $\{x\}$ is arcwise connected.

The following three propositions are true:

(46) For every subset A of F_1 such that F_1 is symmetric holds A is connected iff A is arcwise connected.

(47) Let g be a finite sequence of elements of F_1 and k be a natural number. If g is continuous and $1 \leq k$, then $g \restriction k$ is continuous.

(48) Let g be a finite sequence of elements of F_1 and k be an element of $\mathbb{N}$. If g is continuous and $k < \operatorname{len} g$, then $g_{\lfloor k}$ is continuous.

Let us consider F_1, let g be a finite sequence of elements of F_1, let A be a subset of F_1, and let x_1, x_2 be elements of F_1. We say that g is minimum path in A between x_1 and x_2 if and only if the conditions (Def. 8) are satisfied.

(Def. 8)(i) g is continuous,

(ii) $\operatorname{rng} g \subseteq A$,

(iii) $g(1) = x_1$,

(iv) $g(\operatorname{len} g) = x_2$, and

(v) for every finite sequence h of elements of F_1 such that h is continuous and $\operatorname{rng} h \subseteq A$ and $h(1) = x_1$ and $h(\operatorname{len} h) = x_2$ holds $\operatorname{len} g \leq \operatorname{len} h$.

One can prove the following propositions:

(49) For every subset A of F_1 and for every element x of F_1 such that $x \in A$ holds $\langle x \rangle$ is minimum path in A between x and x.

(50) Let A be a subset of F_1. Then A is arcwise connected if and only if for all elements x_1, x_2 of F_1 such that $x_1 \in A$ and $x_2 \in A$ holds there exists a finite sequence of elements of F_1 which is minimum path in A between x_1 and x_2.

(51) Let A be a subset of F_1 and x_1, x_2 be elements of F_1. Given a finite sequence f of elements of F_1 such that f is continuous and $\operatorname{rng} f \subseteq A$ and $f(1) = x_1$ and $f(\operatorname{len} f) = x_2$. Then there exists a finite sequence of elements of F_1 which is minimum path in A between x_1 and x_2.

(52) Let g be a finite sequence of elements of F_1, A be a subset of F_1, x_1, x_2 be elements of F_1, and k be an element of $\mathbb{N}$. Suppose g is minimum path in A between x_1 and x_2 and $1 \leq k$ and $k \leq \operatorname{len} g$. Then $g \restriction k$ is continuous and $\operatorname{rng}(g \restriction k) \subseteq A$ and $(g \restriction k)(1) = x_1$ and $(g \restriction k)(\operatorname{len}(g \restriction k)) = g_k$.

(53) Let g be a finite sequence of elements of F_1, A be a subset of F_1, x_1, x_2 be elements of F_1, and k be an element of $\mathbb{N}$. Suppose g is minimum path in A between x_1 and x_2 and $k < \operatorname{len} g$. Then $g_{\lfloor k}$ is continuous and $\operatorname{rng}(g_{\lfloor k}) \subseteq A$ and $g_{\lfloor k}(1) = g_{1+k}$ and $g_{\lfloor k}(\operatorname{len}(g_{\lfloor k})) = x_2$.

(54) Let g be a finite sequence of elements of F_1, A be a subset of F_1, and x_1, x_2 be elements of F_1. Suppose g is minimum path in A between x_1 and x_2. Let k be a natural number. If $1 \leq k$ and $k \leq \operatorname{len} g$, then $g \restriction k$ is minimum path in A between x_1 and g_k.

(55) Let g be a finite sequence of elements of F_1, A be a subset of F_1, and x_1, x_2 be elements of F_1. If g is minimum path in A between x_1 and x_2, then g is one-to-one.

Let us consider F_1 and let f be a finite sequence of elements of F_1. We say that f is inversely continuous if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) $1 \leq \operatorname{len} f$, and

(ii) for all natural numbers i, j and for every element y of F_1 such that $1 \leq i$ and $i \leq \operatorname{len} f$ and $1 \leq j$ and $j \leq \operatorname{len} f$ and $y = f(i)$ and $i \neq j$ and $f(j) \in U(y)$ holds $i = j + 1$ or $j = i + 1$.

We now state three propositions:

(56) Let g be a finite sequence of elements of F_1, A be a subset of F_1, and x_1, x_2 be elements of F_1. Suppose g is minimum path in A between x_1 and x_2 and F_1 is symmetric. Then g is inversely continuous.

(57) Let g be a finite sequence of elements of F_1, A be a subset of F_1, and x_1, x_2 be elements of F_1. Suppose g is minimum path in A between x_1 and x_2 and F_1 is filled and symmetric and $x_1 \neq x_2$. Then

(i) for every natural number i such that $1 < i$ and $i < \operatorname{len} g$ holds $\operatorname{rng} g \cap U(g_i) = \{g(i -' 1), g(i), g(i+1)\}$,

(ii) $\operatorname{rng} g \cap U(g_1) = \{g(1), g(2)\}$, and

(iii) $\operatorname{rng} g \cap U(g_{\operatorname{len} g}) = \{g(\operatorname{len} g -' 1), g(\operatorname{len} g)\}$.

(58) Let g be a finite sequence of elements of F_1, A be a non empty subset of F_1, x_1, x_2 be elements of F_1, and B_0 be a subset of $F_1 \restriction A$. Suppose g is minimum path in A between x_1 and x_2 and F_1 is filled and symmetric and $x_1 \neq x_2$ and $B_0 = \{x_1\}$. Let i be an element of $\mathbb{N}$. If $i < \operatorname{len} g$, then $g(i+1) \in \operatorname{Finf}(B_0, i)$ and if $i \geq 1$, then $g(i+1) \notin \operatorname{Finf}(B_0, i -' 1)$.

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