

Several Differentiation Formulas of Special Functions. Part IV

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Summary. In this article, we give several differentiation formulas of special and composite functions including trigonometric function, polynomial function and logarithmic function.

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The notation and terminology used here are introduced in the following papers: [13], [15], [1], [16], [2], [4], [10], [11], [17], [5], [14], [12], [3], [7], [6], [9], and [8].

For simplicity, we adopt the following convention: x , a , b , c denote real numbers, n denotes a natural number, Z denotes an open subset of \mathbb{R} , and f , f_1 , f_2 denote partial functions from \mathbb{R} to \mathbb{R} .

Next we state a number of propositions:

- (1) If $x \in \text{dom}(\text{the function } \tan)$, then $(\text{the function } \cos)(x) \neq 0$.
- (2) If $x \in \text{dom}(\text{the function } \cot)$, then $(\text{the function } \sin)(x) \neq 0$.
- (3) If $Z \subseteq \text{dom}(\frac{f_1}{f_2})$, then for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})(x)_Z^n = \frac{f_1(x)_Z^n}{f_2(x)_Z^n}$.
- (4) Suppose $Z \subseteq \text{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x + a$ and $f_2(x) = x - b$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{|Z}(x) = \frac{-a-b}{(x-b)^2}$.
- (5) Suppose $Z \subseteq \text{dom}((\text{the function } \ln) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then $(\text{the function } \ln) \cdot \frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $((\text{the function } \ln) \cdot \frac{1}{f})'_{|Z}(x) = -\frac{1}{x}$.
- (6) Suppose $Z \subseteq \text{dom}((\text{the function } \tan) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then

- (i) (the function \tan) $\cdot f$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \tan) \cdot f)'_{|Z}(x) = \frac{a}{(\text{the function } \cos)(a \cdot x + b)^2}$.
- (7) Suppose $Z \subseteq \text{dom}((\text{the function } \cot) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
- (i) (the function \cot) $\cdot f$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \cot) \cdot f)'_{|Z}(x) = -\frac{a}{(\text{the function } \sin)(a \cdot x + b)^2}$.
- (8) Suppose $Z \subseteq \text{dom}((\text{the function } \tan) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- (i) (the function \tan) $\cdot \frac{1}{f}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \tan) \cdot \frac{1}{f})'_{|Z}(x) = -\frac{1}{x^2 \cdot (\text{the function } \cos)(\frac{1}{x})^2}$.
- (9) Suppose $Z \subseteq \text{dom}((\text{the function } \cot) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- (i) (the function \cot) $\cdot \frac{1}{f}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \cot) \cdot \frac{1}{f})'_{|Z}(x) = \frac{1}{x^2 \cdot (\text{the function } \sin)(\frac{1}{x})^2}$.
- (10) Suppose $Z \subseteq \text{dom}((\text{the function } \tan) \cdot (f_1 + c f_2))$ and $f_2 = \frac{2}{Z}$ and for every x such that $x \in Z$ holds $f_1(x) = a + b \cdot x$. Then
- (i) (the function \tan) $\cdot (f_1 + c f_2)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \tan) \cdot (f_1 + c f_2))'_{|Z}(x) = \frac{b+2 \cdot c \cdot x}{(\text{the function } \cos)(a+b \cdot x+c \cdot x^2)^2}$.
- (11) Suppose $Z \subseteq \text{dom}((\text{the function } \cot) \cdot (f_1 + c f_2))$ and $f_2 = \frac{2}{Z}$ and for every x such that $x \in Z$ holds $f_1(x) = a + b \cdot x$. Then
- (i) (the function \cot) $\cdot (f_1 + c f_2)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \cot) \cdot (f_1 + c f_2))'_{|Z}(x) = -\frac{b+2 \cdot c \cdot x}{(\text{the function } \sin)(a+b \cdot x+c \cdot x^2)^2}$.
- (12) Suppose $Z \subseteq \text{dom}((\text{the function } \tan) \cdot (\text{the function } \exp))$. Then
- (i) (the function \tan) $\cdot (\text{the function } \exp)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \tan) \cdot (\text{the function } \exp))'_{|Z}(x) = \frac{(\text{the function } \exp)(x)}{(\text{the function } \cos)((\text{the function } \exp)(x))^2}$.
- (13) Suppose $Z \subseteq \text{dom}((\text{the function } \cot) \cdot (\text{the function } \exp))$. Then
- (i) (the function \cot) $\cdot (\text{the function } \exp)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \cot) \cdot (\text{the function } \exp))'_{|Z}(x) = -\frac{(\text{the function } \exp)(x)}{(\text{the function } \sin)((\text{the function } \exp)(x))^2}$.
- (14) Suppose $Z \subseteq \text{dom}((\text{the function } \tan) \cdot (\text{the function } \ln))$. Then
- (i) (the function \tan) $\cdot (\text{the function } \ln)$ is differentiable on Z , and

- (ii) for every x such that $x \in Z$ holds ((the function \tan) \cdot (the function \ln))' $\Big|_Z(x) = \frac{1}{x \cdot (\text{the function } \cos)((\text{the function } \ln)(x))^2}$.
- (15) Suppose $Z \subseteq \text{dom}((\text{the function } \cot) \cdot (\text{the function } \ln))$. Then
- (i) (the function \cot) \cdot (the function \ln) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds ((the function \cot) \cdot (the function \ln))' $\Big|_Z(x) = -\frac{1}{x \cdot (\text{the function } \sin)((\text{the function } \ln)(x))^2}$.
- (16) Suppose $Z \subseteq \text{dom}((\text{the function } \exp) \cdot (\text{the function } \tan))$. Then
- (i) (the function \exp) \cdot (the function \tan) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds ((the function \exp) \cdot (the function \tan))' $\Big|_Z(x) = \frac{(\text{the function } \exp)((\text{the function } \tan)(x))}{(\text{the function } \cos)(x)^2}$.
- (17) Suppose $Z \subseteq \text{dom}((\text{the function } \exp) \cdot (\text{the function } \cot))$. Then
- (i) (the function \exp) \cdot (the function \cot) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds ((the function \exp) \cdot (the function \cot))' $\Big|_Z(x) = -\frac{(\text{the function } \exp)((\text{the function } \cot)(x))}{(\text{the function } \sin)(x)^2}$.
- (18) Suppose $Z \subseteq \text{dom}((\text{the function } \ln) \cdot (\text{the function } \tan))$. Then
- (i) (the function \ln) \cdot (the function \tan) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds ((the function \ln) \cdot (the function \tan))' $\Big|_Z(x) = \frac{1}{(\text{the function } \cos)(x) \cdot (\text{the function } \sin)(x)}$.
- (19) Suppose $Z \subseteq \text{dom}((\text{the function } \ln) \cdot (\text{the function } \cot))$. Then
- (i) (the function \ln) \cdot (the function \cot) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds ((the function \ln) \cdot (the function \cot))' $\Big|_Z(x) = -\frac{1}{(\text{the function } \sin)(x) \cdot (\text{the function } \cos)(x)}$.
- (20) Suppose $Z \subseteq \text{dom}(\binom{n}{Z} \cdot (\text{the function } \tan))$ and $1 \leq n$. Then
- (i) $\binom{n}{Z} \cdot (\text{the function } \tan)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\binom{n}{Z} \cdot (\text{the function } \tan))' \Big|_Z(x) = \frac{n \cdot (\text{the function } \sin)(x)_Z^{n-1}}{(\text{the function } \cos)(x)_Z^{n+1}}$.
- (21) Suppose $Z \subseteq \text{dom}(\binom{n}{Z} \cdot (\text{the function } \cot))$ and $1 \leq n$. Then
- (i) $\binom{n}{Z} \cdot (\text{the function } \cot)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\binom{n}{Z} \cdot (\text{the function } \cot))' \Big|_Z(x) = -\frac{n \cdot (\text{the function } \cos)(x)_Z^{n-1}}{(\text{the function } \sin)(x)_Z^{n+1}}$.
- (22) Suppose that
- (i) $Z \subseteq \text{dom}((\text{the function } \tan) + \frac{1}{\text{the function } \cos})$, and
- (ii) for every x such that $x \in Z$ holds $1 + (\text{the function } \sin)(x) \neq 0$ and $1 - (\text{the function } \sin)(x) \neq 0$.
- Then
- (iii) $(\text{the function } \tan) + \frac{1}{\text{the function } \cos}$ is differentiable on Z , and
- (iv) for every x such that $x \in Z$ holds $((\text{the function } \tan) + \frac{1}{\text{the function } \cos})' \Big|_Z(x) = \frac{1}{1 - (\text{the function } \sin)(x)}$.

(23) Suppose that

- (i) $Z \subseteq \text{dom}(\text{the function } \tan - \frac{1}{\text{the function } \cos})$, and
- (ii) for every x such that $x \in Z$ holds $1 - (\text{the function } \sin)(x) \neq 0$ and $1 + (\text{the function } \sin)(x) \neq 0$.

Then

- (iii) $\text{the function } \tan - \frac{1}{\text{the function } \cos}$ is differentiable on Z , and
 - (iv) for every x such that $x \in Z$ holds $(\text{the function } \tan - \frac{1}{\text{the function } \cos})'_{|Z}(x) = \frac{1}{1 + (\text{the function } \sin)(x)}$.
- (24) Suppose $Z \subseteq \text{dom}(\text{the function } \tan - \text{id}_Z)$. Then
- (i) $\text{the function } \tan - \text{id}_Z$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\text{the function } \tan - \text{id}_Z)'_{|Z}(x) = \frac{(\text{the function } \sin)(x)^2}{(\text{the function } \cos)(x)^2}$.
- (25) Suppose $Z \subseteq \text{dom}(-\text{the function } \cot - \text{id}_Z)$. Then
- (i) $-\text{the function } \cot - \text{id}_Z$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(-\text{the function } \cot - \text{id}_Z)'_{|Z}(x) = \frac{(\text{the function } \cos)(x)^2}{(\text{the function } \sin)(x)^2}$.
- (26) Suppose $Z \subseteq \text{dom}(\frac{1}{a}((\text{the function } \tan) \cdot f) - \text{id}_Z)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x$ and $a \neq 0$. Then
- (i) $\frac{1}{a}((\text{the function } \tan) \cdot f) - \text{id}_Z$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(\frac{1}{a}((\text{the function } \tan) \cdot f) - \text{id}_Z)'_{|Z}(x) = \frac{(\text{the function } \sin)(a \cdot x)^2}{(\text{the function } \cos)(a \cdot x)^2}$.
- (27) Suppose $Z \subseteq \text{dom}((-\frac{1}{a})((\text{the function } \cot) \cdot f) - \text{id}_Z)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x$ and $a \neq 0$. Then
- (i) $(-\frac{1}{a})((\text{the function } \cot) \cdot f) - \text{id}_Z$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $((-\frac{1}{a})((\text{the function } \cot) \cdot f) - \text{id}_Z)'_{|Z}(x) = \frac{(\text{the function } \cos)(a \cdot x)^2}{(\text{the function } \sin)(a \cdot x)^2}$.
- (28) Suppose $Z \subseteq \text{dom}(f(\text{the function } \tan))$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
- (i) $f(\text{the function } \tan)$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(f(\text{the function } \tan))'_{|Z}(x) = \frac{a \cdot (\text{the function } \sin)(x)}{(\text{the function } \cos)(x)} + \frac{a \cdot x + b}{(\text{the function } \cos)(x)^2}$.
- (29) Suppose $Z \subseteq \text{dom}(f(\text{the function } \cot))$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
- (i) $f(\text{the function } \cot)$ is differentiable on Z , and
 - (ii) for every x such that $x \in Z$ holds $(f(\text{the function } \cot))'_{|Z}(x) = \frac{a \cdot (\text{the function } \cos)(x)}{(\text{the function } \sin)(x)} - \frac{a \cdot x + b}{(\text{the function } \sin)(x)^2}$.
- (30) Suppose $Z \subseteq \text{dom}(\text{the function } \exp(\text{the function } \tan))$. Then
- (i) $\text{the function } \exp(\text{the function } \tan)$ is differentiable on Z , and

- (ii) for every x such that $x \in Z$ holds ((the function exp) (the function tan))' $\upharpoonright_Z(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)} + \frac{(\text{the function exp})(x)}{(\text{the function cos})(x)^2}$.
- (31) Suppose $Z \subseteq \text{dom}((\text{the function exp}) (\text{the function cot}))$. Then
- (i) (the function exp) (the function cot) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds ((the function exp) (the function cot))' $\upharpoonright_Z(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)} - \frac{(\text{the function exp})(x)}{(\text{the function sin})(x)^2}$.
- (32) Suppose $Z \subseteq \text{dom}((\text{the function ln}) (\text{the function tan}))$. Then
- (i) (the function ln) (the function tan) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds ((the function ln) (the function tan))' $\upharpoonright_Z(x) = \frac{\frac{(\text{the function sin})(x)}{(\text{the function cos})(x)}}{x} + \frac{(\text{the function ln})(x)}{(\text{the function cos})(x)^2}$.
- (33) Suppose $Z \subseteq \text{dom}((\text{the function ln}) (\text{the function cot}))$. Then
- (i) (the function ln) (the function cot) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds ((the function ln) (the function cot))' $\upharpoonright_Z(x) = \frac{\frac{(\text{the function cos})(x)}{(\text{the function sin})(x)}}{x} - \frac{(\text{the function ln})(x)}{(\text{the function sin})(x)^2}$.
- (34) Suppose $Z \subseteq \text{dom}(\frac{1}{f} (\text{the function tan}))$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- (i) $\frac{1}{f}$ (the function tan) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\frac{1}{f} (\text{the function tan}))' \upharpoonright_Z(x) = -\frac{\frac{(\text{the function sin})(x)}{(\text{the function cos})(x)}}{x^2} + \frac{\frac{1}{x}}{(\text{the function cos})(x)^2}$.
- (35) Suppose $Z \subseteq \text{dom}(\frac{1}{f} (\text{the function cot}))$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- (i) $\frac{1}{f}$ (the function cot) is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\frac{1}{f} (\text{the function cot}))' \upharpoonright_Z(x) = -\frac{\frac{(\text{the function cos})(x)}{(\text{the function sin})(x)}}{x^2} - \frac{\frac{1}{x}}{(\text{the function sin})(x)^2}$.

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