

## Several Differentiation Formulas of Special Functions. Part IV

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**Summary.** In this article, we give several differentiation formulas of special and composite functions including trigonometric function, polynomial function and logarithmic function.

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The notation and terminology used here are introduced in the following papers: [13], [15], [1], [16], [2], [4], [10], [11], [17], [5], [14], [12], [3], [7], [6], [9], and [8].

For simplicity, we adopt the following convention:  $x$ ,  $a$ ,  $b$ ,  $c$  denote real numbers,  $n$  denotes a natural number,  $Z$  denotes an open subset of  $\mathbb{R}$ , and  $f$ ,  $f_1$ ,  $f_2$  denote partial functions from  $\mathbb{R}$  to  $\mathbb{R}$ .

Next we state a number of propositions:

- (1) If  $x \in \text{dom}(\text{the function } \tan)$ , then  $(\text{the function } \cos)(x) \neq 0$ .
- (2) If  $x \in \text{dom}(\text{the function } \cot)$ , then  $(\text{the function } \sin)(x) \neq 0$ .
- (3) If  $Z \subseteq \text{dom}(\frac{f_1}{f_2})$ , then for every  $x$  such that  $x \in Z$  holds  $(\frac{f_1}{f_2})(x)_Z^n = \frac{f_1(x)_Z^n}{f_2(x)_Z^n}$ .
- (4) Suppose  $Z \subseteq \text{dom}(\frac{f_1}{f_2})$  and for every  $x$  such that  $x \in Z$  holds  $f_1(x) = x + a$  and  $f_2(x) = x - b$ . Then  $\frac{f_1}{f_2}$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(\frac{f_1}{f_2})'_{|Z}(x) = \frac{-a-b}{(x-b)^2}$ .
- (5) Suppose  $Z \subseteq \text{dom}((\text{the function } \ln) \cdot \frac{1}{f})$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = x$ . Then  $(\text{the function } \ln) \cdot \frac{1}{f}$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \ln) \cdot \frac{1}{f})'_{|Z}(x) = -\frac{1}{x}$ .
- (6) Suppose  $Z \subseteq \text{dom}((\text{the function } \tan) \cdot f)$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = a \cdot x + b$ . Then

- (i) (the function  $\tan$ )  $\cdot f$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \tan) \cdot f)'_{|Z}(x) = \frac{a}{(\text{the function } \cos)(a \cdot x + b)^2}$ .
- (7) Suppose  $Z \subseteq \text{dom}((\text{the function } \cot) \cdot f)$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = a \cdot x + b$ . Then
- (i) (the function  $\cot$ )  $\cdot f$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \cot) \cdot f)'_{|Z}(x) = -\frac{a}{(\text{the function } \sin)(a \cdot x + b)^2}$ .
- (8) Suppose  $Z \subseteq \text{dom}((\text{the function } \tan) \cdot \frac{1}{f})$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = x$ . Then
- (i) (the function  $\tan$ )  $\cdot \frac{1}{f}$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \tan) \cdot \frac{1}{f})'_{|Z}(x) = -\frac{1}{x^2 \cdot (\text{the function } \cos)(\frac{1}{x})^2}$ .
- (9) Suppose  $Z \subseteq \text{dom}((\text{the function } \cot) \cdot \frac{1}{f})$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = x$ . Then
- (i) (the function  $\cot$ )  $\cdot \frac{1}{f}$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \cot) \cdot \frac{1}{f})'_{|Z}(x) = \frac{1}{x^2 \cdot (\text{the function } \sin)(\frac{1}{x})^2}$ .
- (10) Suppose  $Z \subseteq \text{dom}((\text{the function } \tan) \cdot (f_1 + c f_2))$  and  $f_2 = \frac{2}{Z}$  and for every  $x$  such that  $x \in Z$  holds  $f_1(x) = a + b \cdot x$ . Then
- (i) (the function  $\tan$ )  $\cdot (f_1 + c f_2)$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \tan) \cdot (f_1 + c f_2))'_{|Z}(x) = \frac{b + 2 \cdot c \cdot x}{(\text{the function } \cos)(a + b \cdot x + c \cdot x^2)^2}$ .
- (11) Suppose  $Z \subseteq \text{dom}((\text{the function } \cot) \cdot (f_1 + c f_2))$  and  $f_2 = \frac{2}{Z}$  and for every  $x$  such that  $x \in Z$  holds  $f_1(x) = a + b \cdot x$ . Then
- (i) (the function  $\cot$ )  $\cdot (f_1 + c f_2)$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \cot) \cdot (f_1 + c f_2))'_{|Z}(x) = -\frac{b + 2 \cdot c \cdot x}{(\text{the function } \sin)(a + b \cdot x + c \cdot x^2)^2}$ .
- (12) Suppose  $Z \subseteq \text{dom}((\text{the function } \tan) \cdot (\text{the function } \exp))$ . Then
- (i) (the function  $\tan$ )  $\cdot (\text{the function } \exp)$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \tan) \cdot (\text{the function } \exp))'_{|Z}(x) = \frac{(\text{the function } \exp)(x)}{(\text{the function } \cos)((\text{the function } \exp)(x))^2}$ .
- (13) Suppose  $Z \subseteq \text{dom}((\text{the function } \cot) \cdot (\text{the function } \exp))$ . Then
- (i) (the function  $\cot$ )  $\cdot (\text{the function } \exp)$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \cot) \cdot (\text{the function } \exp))'_{|Z}(x) = -\frac{(\text{the function } \exp)(x)}{(\text{the function } \sin)((\text{the function } \exp)(x))^2}$ .
- (14) Suppose  $Z \subseteq \text{dom}((\text{the function } \tan) \cdot (\text{the function } \ln))$ . Then
- (i) (the function  $\tan$ )  $\cdot (\text{the function } \ln)$  is differentiable on  $Z$ , and

- (ii) for every  $x$  such that  $x \in Z$  holds ((the function  $\tan$ )  $\cdot$  (the function  $\ln$ ))'  $\Big|_Z(x) = \frac{1}{x \cdot (\text{the function } \cos)((\text{the function } \ln)(x))^2}$ .
- (15) Suppose  $Z \subseteq \text{dom}((\text{the function } \cot) \cdot (\text{the function } \ln))$ . Then
- (i) (the function  $\cot$ )  $\cdot$  (the function  $\ln$ ) is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds ((the function  $\cot$ )  $\cdot$  (the function  $\ln$ ))'  $\Big|_Z(x) = -\frac{1}{x \cdot (\text{the function } \sin)((\text{the function } \ln)(x))^2}$ .
- (16) Suppose  $Z \subseteq \text{dom}((\text{the function } \exp) \cdot (\text{the function } \tan))$ . Then
- (i) (the function  $\exp$ )  $\cdot$  (the function  $\tan$ ) is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds ((the function  $\exp$ )  $\cdot$  (the function  $\tan$ ))'  $\Big|_Z(x) = \frac{(\text{the function } \exp)((\text{the function } \tan)(x))}{(\text{the function } \cos)(x)^2}$ .
- (17) Suppose  $Z \subseteq \text{dom}((\text{the function } \exp) \cdot (\text{the function } \cot))$ . Then
- (i) (the function  $\exp$ )  $\cdot$  (the function  $\cot$ ) is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds ((the function  $\exp$ )  $\cdot$  (the function  $\cot$ ))'  $\Big|_Z(x) = -\frac{(\text{the function } \exp)((\text{the function } \cot)(x))}{(\text{the function } \sin)(x)^2}$ .
- (18) Suppose  $Z \subseteq \text{dom}((\text{the function } \ln) \cdot (\text{the function } \tan))$ . Then
- (i) (the function  $\ln$ )  $\cdot$  (the function  $\tan$ ) is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds ((the function  $\ln$ )  $\cdot$  (the function  $\tan$ ))'  $\Big|_Z(x) = \frac{1}{(\text{the function } \cos)(x) \cdot (\text{the function } \sin)(x)}$ .
- (19) Suppose  $Z \subseteq \text{dom}((\text{the function } \ln) \cdot (\text{the function } \cot))$ . Then
- (i) (the function  $\ln$ )  $\cdot$  (the function  $\cot$ ) is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds ((the function  $\ln$ )  $\cdot$  (the function  $\cot$ ))'  $\Big|_Z(x) = -\frac{1}{(\text{the function } \sin)(x) \cdot (\text{the function } \cos)(x)}$ .
- (20) Suppose  $Z \subseteq \text{dom}(\binom{n}{Z} \cdot (\text{the function } \tan))$  and  $1 \leq n$ . Then
- (i)  $\binom{n}{Z} \cdot (\text{the function } \tan)$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(\binom{n}{Z} \cdot (\text{the function } \tan))' \Big|_Z(x) = \frac{n \cdot (\text{the function } \sin)(x)_Z^{n-1}}{(\text{the function } \cos)(x)_Z^{n+1}}$ .
- (21) Suppose  $Z \subseteq \text{dom}(\binom{n}{Z} \cdot (\text{the function } \cot))$  and  $1 \leq n$ . Then
- (i)  $\binom{n}{Z} \cdot (\text{the function } \cot)$  is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(\binom{n}{Z} \cdot (\text{the function } \cot))' \Big|_Z(x) = -\frac{n \cdot (\text{the function } \cos)(x)_Z^{n-1}}{(\text{the function } \sin)(x)_Z^{n+1}}$ .
- (22) Suppose that
- (i)  $Z \subseteq \text{dom}((\text{the function } \tan) + \frac{1}{\text{the function } \cos})$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $1 + (\text{the function } \sin)(x) \neq 0$  and  $1 - (\text{the function } \sin)(x) \neq 0$ .
- Then
- (iii)  $(\text{the function } \tan) + \frac{1}{\text{the function } \cos}$  is differentiable on  $Z$ , and
- (iv) for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \tan) + \frac{1}{\text{the function } \cos})' \Big|_Z(x) = \frac{1}{1 - (\text{the function } \sin)(x)}$ .

(23) Suppose that

- (i)  $Z \subseteq \text{dom}(\text{the function } \tan - \frac{1}{\text{the function } \cos})$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $1 - (\text{the function } \sin)(x) \neq 0$  and  $1 + (\text{the function } \sin)(x) \neq 0$ .

Then

- (iii)  $(\text{the function } \tan - \frac{1}{\text{the function } \cos})$  is differentiable on  $Z$ , and
  - (iv) for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \tan - \frac{1}{\text{the function } \cos})'_{|Z}(x) = \frac{1}{1 + (\text{the function } \sin)(x)}}$ .
- (24) Suppose  $Z \subseteq \text{dom}(\text{the function } \tan - \text{id}_Z)$ . Then
- (i)  $(\text{the function } \tan - \text{id}_Z)$  is differentiable on  $Z$ , and
  - (ii) for every  $x$  such that  $x \in Z$  holds  $((\text{the function } \tan - \text{id}_Z)'_{|Z}(x) = \frac{(\text{the function } \sin)(x)^2}{(\text{the function } \cos)(x)^2}$ .
- (25) Suppose  $Z \subseteq \text{dom}(-\text{the function } \cot - \text{id}_Z)$ . Then
- (i)  $-\text{the function } \cot - \text{id}_Z$  is differentiable on  $Z$ , and
  - (ii) for every  $x$  such that  $x \in Z$  holds  $(-\text{the function } \cot - \text{id}_Z)'_{|Z}(x) = \frac{(\text{the function } \cos)(x)^2}{(\text{the function } \sin)(x)^2}$ .
- (26) Suppose  $Z \subseteq \text{dom}(\frac{1}{a}((\text{the function } \tan) \cdot f) - \text{id}_Z)$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = a \cdot x$  and  $a \neq 0$ . Then
- (i)  $\frac{1}{a}((\text{the function } \tan) \cdot f) - \text{id}_Z$  is differentiable on  $Z$ , and
  - (ii) for every  $x$  such that  $x \in Z$  holds  $(\frac{1}{a}((\text{the function } \tan) \cdot f) - \text{id}_Z)'_{|Z}(x) = \frac{(\text{the function } \sin)(a \cdot x)^2}{(\text{the function } \cos)(a \cdot x)^2}$ .
- (27) Suppose  $Z \subseteq \text{dom}((-\frac{1}{a})((\text{the function } \cot) \cdot f) - \text{id}_Z)$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = a \cdot x$  and  $a \neq 0$ . Then
- (i)  $(-\frac{1}{a})((\text{the function } \cot) \cdot f) - \text{id}_Z$  is differentiable on  $Z$ , and
  - (ii) for every  $x$  such that  $x \in Z$  holds  $((-\frac{1}{a})((\text{the function } \cot) \cdot f) - \text{id}_Z)'_{|Z}(x) = \frac{(\text{the function } \cos)(a \cdot x)^2}{(\text{the function } \sin)(a \cdot x)^2}$ .
- (28) Suppose  $Z \subseteq \text{dom}(f(\text{the function } \tan))$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = a \cdot x + b$ . Then
- (i)  $f(\text{the function } \tan)$  is differentiable on  $Z$ , and
  - (ii) for every  $x$  such that  $x \in Z$  holds  $(f(\text{the function } \tan))'_{|Z}(x) = \frac{a \cdot (\text{the function } \sin)(x)}{(\text{the function } \cos)(x)} + \frac{a \cdot x + b}{(\text{the function } \cos)(x)^2}$ .
- (29) Suppose  $Z \subseteq \text{dom}(f(\text{the function } \cot))$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = a \cdot x + b$ . Then
- (i)  $f(\text{the function } \cot)$  is differentiable on  $Z$ , and
  - (ii) for every  $x$  such that  $x \in Z$  holds  $(f(\text{the function } \cot))'_{|Z}(x) = \frac{a \cdot (\text{the function } \cos)(x)}{(\text{the function } \sin)(x)} - \frac{a \cdot x + b}{(\text{the function } \sin)(x)^2}$ .
- (30) Suppose  $Z \subseteq \text{dom}(\text{the function } \exp(\text{the function } \tan))$ . Then
- (i)  $(\text{the function } \exp(\text{the function } \tan))$  is differentiable on  $Z$ , and

- (ii) for every  $x$  such that  $x \in Z$  holds ((the function exp) (the function tan))'  $\upharpoonright_Z(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)} + \frac{(\text{the function exp})(x)}{(\text{the function cos})(x)^2}$ .
- (31) Suppose  $Z \subseteq \text{dom}((\text{the function exp}) (\text{the function cot}))$ . Then
- (i) ((the function exp) (the function cot)) is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds ((the function exp) (the function cot))'  $\upharpoonright_Z(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)} - \frac{(\text{the function exp})(x)}{(\text{the function sin})(x)^2}$ .
- (32) Suppose  $Z \subseteq \text{dom}((\text{the function ln}) (\text{the function tan}))$ . Then
- (i) ((the function ln) (the function tan)) is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds ((the function ln) (the function tan))'  $\upharpoonright_Z(x) = \frac{\frac{(\text{the function sin})(x)}{(\text{the function cos})(x)}}{x} + \frac{(\text{the function ln})(x)}{(\text{the function cos})(x)^2}$ .
- (33) Suppose  $Z \subseteq \text{dom}((\text{the function ln}) (\text{the function cot}))$ . Then
- (i) ((the function ln) (the function cot)) is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds ((the function ln) (the function cot))'  $\upharpoonright_Z(x) = \frac{\frac{(\text{the function cos})(x)}{(\text{the function sin})(x)}}{x} - \frac{(\text{the function ln})(x)}{(\text{the function sin})(x)^2}$ .
- (34) Suppose  $Z \subseteq \text{dom}(\frac{1}{f} (\text{the function tan}))$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = x$ . Then
- (i)  $\frac{1}{f}$  (the function tan) is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(\frac{1}{f} (\text{the function tan}))' \upharpoonright_Z(x) = -\frac{\frac{(\text{the function sin})(x)}{(\text{the function cos})(x)}}{x^2} + \frac{\frac{1}{x}}{(\text{the function cos})(x)^2}$ .
- (35) Suppose  $Z \subseteq \text{dom}(\frac{1}{f} (\text{the function cot}))$  and for every  $x$  such that  $x \in Z$  holds  $f(x) = x$ . Then
- (i)  $\frac{1}{f}$  (the function cot) is differentiable on  $Z$ , and
- (ii) for every  $x$  such that  $x \in Z$  holds  $(\frac{1}{f} (\text{the function cot}))' \upharpoonright_Z(x) = -\frac{\frac{(\text{the function cos})(x)}{(\text{the function sin})(x)}}{x^2} - \frac{\frac{1}{x}}{(\text{the function sin})(x)^2}$ .

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