

Difference and Difference Quotient

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Summary. In this article, we give the definitions of forward difference, backward difference, central difference and difference quotient, and some of their important properties.

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The articles [2], [6], [1], [13], [16], [17], [14], [4], [5], [9], [8], [12], [18], [7], [15], [11], [10], [3], and [19] provide the terminology and notation for this paper.

For simplicity, we follow the rules: n, m, i are elements of \mathbb{N} , $h, r, r_1, r_2, x_0, x_1, x_2, x$ are real numbers, f is a partial function from \mathbb{R} to \mathbb{R} , and S is a sequence of partial functions from \mathbb{R} into \mathbb{R} .

Let f be a partial function from \mathbb{R} to \mathbb{R} and let h be a real number. The functor $\text{Shift}(f, h)$ yields a partial function from \mathbb{R} to \mathbb{R} and is defined by:

(Def. 1) $\text{dom Shift}(f, h) = -h + \text{dom } f$ and for every x such that $x \in -h + \text{dom } f$ holds $(\text{Shift}(f, h))(x) = f(x + h)$.

Let f be a function from \mathbb{R} into \mathbb{R} and let h be a real number. Then $\text{Shift}(f, h)$ is a function from \mathbb{R} into \mathbb{R} and it can be characterized by the condition:

(Def. 2) For every x holds $(\text{Shift}(f, h))(x) = f(x + h)$.

Let f be a partial function from \mathbb{R} to \mathbb{R} and let h be a real number. The functor $\text{fD}(f, h)$ yielding a partial function from \mathbb{R} to \mathbb{R} is defined as follows:

(Def. 3) $\text{fD}(f, h) = \text{Shift}(f, h) - f$.

Let f be a function from \mathbb{R} into \mathbb{R} and let h be a real number. Then $\text{fD}(f, h)$ is a function from \mathbb{R} into \mathbb{R} .

Let f be a partial function from \mathbb{R} to \mathbb{R} and let h be a real number. The functor $\text{bD}(f, h)$ yields a partial function from \mathbb{R} to \mathbb{R} and is defined by:

(Def. 4) $\text{bD}(f, h) = f - \text{Shift}(f, -h)$.

Let f be a function from \mathbb{R} into \mathbb{R} and let h be a real number. Then $\text{bD}(f, h)$ is a function from \mathbb{R} into \mathbb{R} .

We now state the proposition

$$(1) \quad \text{bD}(f, h) = -\text{fD}(f, -h).$$

Let f be a partial function from \mathbb{R} to \mathbb{R} and let h be a real number. The functor $\text{cD}(f, h)$ yielding a partial function from \mathbb{R} to \mathbb{R} is defined by:

(Def. 5) $\text{cD}(f, h) = \text{Shift}(f, \frac{h}{2}) - \text{Shift}(f, -\frac{h}{2})$.

Let f be a function from \mathbb{R} into \mathbb{R} and let h be a real number. Then $\text{cD}(f, h)$ is a function from \mathbb{R} into \mathbb{R} .

Let f be a partial function from \mathbb{R} to \mathbb{R} and let h be a real number. The forward difference of f and h yields a sequence of partial functions from \mathbb{R} into \mathbb{R} and is defined by the conditions (Def. 6).

(Def. 6)(i) (The forward difference of f and h)(0) = f , and
(ii) for every n holds (the forward difference of f and h)($n+1$) = $\text{fD}((\text{the forward difference of } f \text{ and } h)(n), h)$.

Let f be a partial function from \mathbb{R} to \mathbb{R} and let h be a real number. We introduce $\text{fdif}(f, h)$ as a synonym of the forward difference of f and h .

In the sequel f, f_1, f_2 denote functions from \mathbb{R} into \mathbb{R} .

The following propositions are true:

- (2) For every n holds $(\text{fdif}(f, h))(n)$ is a function from \mathbb{R} into \mathbb{R} .
- (3) For every x holds $(\text{fD}(f, h))(x) = f(x+h) - f(x)$.
- (4) For every x holds $(\text{bD}(f, h))(x) = f(x) - f(x-h)$.
- (5) For every x holds $(\text{cD}(f, h))(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2})$.
- (6) If f is constant, then for every x holds $(\text{fdif}(f, h))(n+1)(x) = 0$.
- (7) $(\text{fdif}(r f, h))(n+1)(x) = r \cdot (\text{fdif}(f, h))(n+1)(x)$.
- (8) $(\text{fdif}(f_1+f_2, h))(n+1)(x) = (\text{fdif}(f_1, h))(n+1)(x) + (\text{fdif}(f_2, h))(n+1)(x)$.
- (9) $(\text{fdif}(f_1-f_2, h))(n+1)(x) = (\text{fdif}(f_1, h))(n+1)(x) - (\text{fdif}(f_2, h))(n+1)(x)$.
- (10) If $f = r_1 f_1 + r_2 f_2$, then for every x holds $(\text{fdif}(f, h))(n+1)(x) = r_1 \cdot (\text{fdif}(f_1, h))(n+1)(x) + r_2 \cdot (\text{fdif}(f_2, h))(n+1)(x)$.
- (11) For every x holds $(\text{fdif}(f, h))(1)(x) = (\text{Shift}(f, h))(x) - f(x)$.

Let f be a partial function from \mathbb{R} to \mathbb{R} and let h be a real number. The backward difference of f and h yielding a sequence of partial functions from \mathbb{R} into \mathbb{R} is defined by the conditions (Def. 7).

- (Def. 7)(i) (The backward difference of f and h)(0) = f , and
(ii) for every n holds (the backward difference of f and h)($n+1$) = $\text{bD}((\text{the backward difference of } f \text{ and } h)(n), h)$.

Let f be a partial function from \mathbb{R} to \mathbb{R} and let h be a real number. We introduce $\text{bdif}(f, h)$ as a synonym of the backward difference of f and h .

We now state several propositions:

- (12) For every n holds $(\text{bdif}(f, h))(n)$ is a function from \mathbb{R} into \mathbb{R} .
(13) If f is constant, then for every x holds $(\text{bdif}(f, h))(n+1)(x) = 0$.
(14) $(\text{bdif}(r f, h))(n+1)(x) = r \cdot (\text{bdif}(f, h))(n+1)(x)$.
(15) $(\text{bdif}(f_1 + f_2, h))(n+1)(x) = (\text{bdif}(f_1, h))(n+1)(x) + (\text{bdif}(f_2, h))(n+1)(x)$.
(16) $(\text{bdif}(f_1 - f_2, h))(n+1)(x) = (\text{bdif}(f_1, h))(n+1)(x) - (\text{bdif}(f_2, h))(n+1)(x)$.
(17) If $f = r_1 f_1 + r_2 f_2$, then for every x holds $(\text{bdif}(f, h))(n+1)(x) = r_1 \cdot (\text{bdif}(f_1, h))(n+1)(x) + r_2 \cdot (\text{bdif}(f_2, h))(n+1)(x)$.
(18) $(\text{bdif}(f, h))(1)(x) = f(x) - (\text{Shift}(f, -h))(x)$.

Let f be a partial function from \mathbb{R} to \mathbb{R} and let h be a real number. The central difference of f and h yielding a sequence of partial functions from \mathbb{R} into \mathbb{R} is defined by the conditions (Def. 8).

- (Def. 8)(i) (The central difference of f and h)(0) = f , and
(ii) for every n holds (the central difference of f and h)($n+1$) = $\text{cD}((\text{the central difference of } f \text{ and } h)(n), h)$.

Let f be a partial function from \mathbb{R} to \mathbb{R} and let h be a real number. We introduce $\text{cdif}(f, h)$ as a synonym of the central difference of f and h .

One can prove the following propositions:

- (19) For every n holds $(\text{cdif}(f, h))(n)$ is a function from \mathbb{R} into \mathbb{R} .
(20) If f is constant, then for every x holds $(\text{cdif}(f, h))(n+1)(x) = 0$.
(21) $(\text{cdif}(r f, h))(n+1)(x) = r \cdot (\text{cdif}(f, h))(n+1)(x)$.
(22) $(\text{cdif}(f_1 + f_2, h))(n+1)(x) = (\text{cdif}(f_1, h))(n+1)(x) + (\text{cdif}(f_2, h))(n+1)(x)$.
(23) $(\text{cdif}(f_1 - f_2, h))(n+1)(x) = (\text{cdif}(f_1, h))(n+1)(x) - (\text{cdif}(f_2, h))(n+1)(x)$.
(24) If $f = r_1 f_1 + r_2 f_2$, then for every x holds $(\text{cdif}(f, h))(n+1)(x) = r_1 \cdot (\text{cdif}(f_1, h))(n+1)(x) + r_2 \cdot (\text{cdif}(f_2, h))(n+1)(x)$.
(25) $(\text{cdif}(f, h))(1)(x) = (\text{Shift}(f, \frac{h}{2}))(x) - (\text{Shift}(f, -\frac{h}{2}))(x)$.
(26) $(\text{fdif}(f, h))(n)(x) = (\text{bdif}(f, h))(n)(x + n \cdot h)$.
(27) $(\text{fdif}(f, h))(2 \cdot n)(x) = (\text{cdif}(f, h))(2 \cdot n)(x + n \cdot h)$.
(28) $(\text{fdif}(f, h))(2 \cdot n + 1)(x) = (\text{cdif}(f, h))(2 \cdot n + 1)(x + n \cdot h + \frac{h}{2})$.

Let f be a function from \mathbb{R} into \mathbb{R} and let us consider x_0, x_1 . The functor $\Delta(f, x_0, x_1)$ yielding a real number is defined as follows:

- (Def. 9)(i) $\Delta(f, x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$ if $x_0 \neq x_1$,
(ii) $x_0 \neq x_1$, otherwise.

Let x_0, x_1, x_2 be real numbers and let f be a function from \mathbb{R} into \mathbb{R} . The functor $[!f, x_0, x_1, x_2!]$ yielding a real number is defined as follows:

- (Def. 10)(i) $[!f, x_0, x_1, x_2!] = \frac{\Delta(f, x_0, x_1) - \Delta(f, x_1, x_2)}{x_0 - x_2}$ if $x_0 \neq x_2$,
(ii) $x_0 \neq x_2$, otherwise.

Let x_0, x_1, x_2, x_3 be real numbers and let f be a function from \mathbb{R} into \mathbb{R} . The functor $[!f, x_0, x_1, x_2, x_3!]$ yielding a real number is defined by:

- (Def. 11)(i) $[!f, x_0, x_1, x_2, x_3!] = \frac{[!f, x_0, x_1, x_2!] - [!f, x_1, x_2, x_3!]}{x_0 - x_3}$ if $x_0 \neq x_3$,
(ii) $x_0 \neq x_3$, otherwise.

We now state several propositions:

- (29) If $x_0 \neq x_1$, then $\Delta(f, x_0, x_1) = \Delta(f, x_1, x_0)$.
(30) If f is constant and $x_0 \neq x_1$, then $\Delta(f, x_0, x_1) = 0$.
(31) If $x_0 \neq x_1$, then $\Delta(r f, x_0, x_1) = r \cdot \Delta(f, x_0, x_1)$.
(32) If $x_0 \neq x_1$, then $\Delta(f_1 + f_2, x_0, x_1) = \Delta(f_1, x_0, x_1) + \Delta(f_2, x_0, x_1)$.
(33) If $x_0 \neq x_1$, then $\Delta(r_1 f_1 + r_2 f_2, x_0, x_1) = r_1 \cdot \Delta(f_1, x_0, x_1) + r_2 \cdot \Delta(f_2, x_0, x_1)$.
(34) If $x_0 \neq x_1$ and $x_0 \neq x_2$ and $x_1 \neq x_2$, then $[!f, x_0, x_1, x_2!] = [!f, x_1, x_2, x_0!]$
and $[!f, x_0, x_1, x_2!] = [!f, x_2, x_1, x_0!]$.
(35) If $x_0 \neq x_1$ and $x_0 \neq x_2$ and $x_1 \neq x_2$, then $[!f, x_0, x_1, x_2!] = [!f, x_2, x_0, x_1!]$
and $[!f, x_0, x_1, x_2!] = [!f, x_1, x_0, x_2!]$.
(36) $(\text{fdif}((\text{fdif}(f, h))(m), h))(n)(x) = (\text{fdif}(f, h))(m + n)(x)$.

Let us consider S . We say that S is sequence-yielding if and only if:

- (Def. 12) For every n holds $S(n)$ is a sequence of real numbers.

Let us note that there exists a sequence of partial functions from \mathbb{R} into \mathbb{R} which is sequence-yielding.

A seq sequence is a sequence-yielding sequence of partial functions from \mathbb{R} into \mathbb{R} .

Let S be a seq sequence and let us consider n . Then $S(n)$ is a sequence of real numbers.

In the sequel S denotes a seq sequence.

Next we state the proposition

- (37) Suppose that for every n and for every i such that $i \leq n$ holds
 $S(n)(i) = \binom{n}{i} \cdot (\text{fdif}(f_1, h))(i)(x) \cdot (\text{fdif}(f_2, h))(n - i)(x + i \cdot h)$.
Then $(\text{fdif}(f_1 f_2, h))(1)(x) = \sum_{\kappa=0}^1 S(1)(\kappa)$ and $(\text{fdif}(f_1 f_2, h))(2)(x) = \sum_{\kappa=0}^2 S(2)(\kappa)$.

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