

Multiplication of Polynomials using Discrete Fourier Transformation

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Summary. In this article we define the Discrete Fourier Transformation for univariate polynomials and show that multiplication of polynomials can be carried out by two Fourier Transformations with a vector multiplication in-between. Our proof follows the standard one found in the literature and uses Vandermonde matrices, see e.g. [27].

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The articles [20], [26], [28], [5], [6], [19], [12], [3], [18], [13], [25], [2], [4], [23], [8], [24], [14], [10], [11], [16], [7], [29], [22], [1], [15], [9], [21], and [17] provide the notation and terminology for this paper.

1. PRELIMINARIES

The following proposition is true

- (1) Let n be an element of \mathbb{N} , L be a unital integral domain-like non degenerated non empty double loop structure, and x be an element of L . If $x \neq 0_L$, then $x^n \neq 0_L$.

One can verify that every associative right unital add-associative right zeroed right complementable left distributive non empty double loop structure which is field-like is also integral domain-like.

The following four propositions are true:

- (2) Let L be an add-associative right zeroed right complementable associative commutative left unital field-like distributive non empty double loop structure and x, y be elements of L . If $x \neq 0_L$ and $y \neq 0_L$, then $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$.
- (3) Let L be an associative commutative left unital distributive field-like non empty double loop structure and z, z_1 be elements of L . If $z \neq 0_L$, then $z_1 = \frac{z_1 \cdot z}{z}$.
- (4) Let L be a left zeroed right zeroed add-associative right complementable non empty double loop structure, m be an element of \mathbb{N} , and s be a finite sequence of elements of L . Suppose $\text{len } s = m$ and for every element k of \mathbb{N} such that $1 \leq k$ and $k \leq m$ holds $s_k = 1_L$. Then $\sum s = m \cdot 1_L$.
- (5) Let L be an add-associative right zeroed right complementable associative commutative left unital distributive field-like non empty double loop structure, s be a finite sequence of elements of L , and q be an element of L . Suppose $q \neq 1_L$ and for every natural number i such that $1 \leq i$ and $i \leq \text{len } s$ holds $s(i) = q^{i-1}$. Then $\sum s = \frac{1_L - q^{\text{len } s}}{1_L - q}$.

Let L be a unital non empty double loop structure and let m be an element of \mathbb{N} . The functor m_L yielding an element of L is defined as follows:

(Def. 1) $m_L = m \cdot 1_L$.

Next we state several propositions:

- (6) Let L be a field and m, n, k be elements of \mathbb{N} . Suppose $m > 0$ and $n > 0$. Let M_1 be a matrix over L of dimension $m \times n$ and M_2 be a matrix over L of dimension $n \times k$. Then $(m_L \cdot M_1) \cdot M_2 = m_L \cdot (M_1 \cdot M_2)$.
- (7) Let L be a non empty zero structure, p be an algebraic sequence of L , and i be an element of \mathbb{N} . If $p(i) \neq 0_L$, then $\text{len } p \geq i + 1$.
- (8) For every non empty zero structure L and for every algebraic sequence s of L such that $\text{len } s > 0$ holds $s(\text{len } s - 1) \neq 0_L$.
- (9) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital integral domain-like non empty double loop structure and p, q be polynomials of L . If $\text{len } p > 0$ and $\text{len } q > 0$, then $\text{len}(p * q) \leq \text{len } p + \text{len } q$.
- (10) Let L be an associative non empty double loop structure, k, l be elements of L , and s_1 be a sequence of L . Then $k \cdot (l \cdot s_1) = (k \cdot l) \cdot s_1$.

2. MULTIPLICATION OF ALGEBRAIC SEQUENCES

Let L be a non empty double loop structure and let m_1, m_2 be sequences of L . The functor $m_1 \cdot m_2$ yields a sequence of L and is defined as follows:

(Def. 2) For every element i of \mathbb{N} holds $(m_1 \cdot m_2)(i) = m_1(i) \cdot m_2(i)$.

Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure and let m_1, m_2 be algebraic sequences of L . Observe that $m_1 \cdot m_2$ is finite-Support.

We now state two propositions:

- (11) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and m_1, m_2 be algebraic sequences of L . Then $\text{len}(m_1 \cdot m_2) \leq \min(\text{len } m_1, \text{len } m_2)$.
- (12) Let L be an add-associative right zeroed right complementable distributive integral domain-like non empty double loop structure and m_1, m_2 be algebraic sequences of L . If $\text{len } m_1 = \text{len } m_2$, then $\text{len}(m_1 \cdot m_2) = \text{len } m_1$.

3. POWERS IN DOUBLE LOOP STRUCTURES

Let L be an associative commutative left unital distributive field-like non empty double loop structure, let a be an element of L , and let i be an integer. The functor a^i yielding an element of L is defined as follows:

(Def. 3) $a^i = \begin{cases} \text{power}_L(a, i), & \text{if } 0 \leq i, \\ \text{power}_L(a, |i|)^{-1}, & \text{otherwise.} \end{cases}$

Next we state a number of propositions:

- (13) Let L be an associative commutative left unital distributive field-like non empty double loop structure and x be an element of L . Then $x^0 = 1_L$.
- (14) Let L be an associative commutative left unital distributive field-like non empty double loop structure and x be an element of L . Then $x^1 = x$.
- (15) Let L be an associative commutative left unital distributive field-like non empty double loop structure and x be an element of L . Then $x^{-1} = x^{-1}$.
- (16) Let L be an associative commutative left unital distributive field-like non degenerated non empty double loop structure and i be an integer. Then $(1_L)^i = 1_L$.
- (17) Let L be an associative commutative left unital distributive field-like non empty double loop structure, x be an element of L , and n be an element of \mathbb{N} . Then $x^{n+1} = x^n \cdot x$ and $x^{n+1} = x \cdot x^n$.
- (18) Let L be an add-associative right zeroed right complementable associative commutative left unital distributive field-like non degenerated non empty double loop structure, i be an integer, and x be an element of L . If $x \neq 0_L$, then $(x^i)^{-1} = x^{-i}$.

- (19) For every field L and for every integer j and for every element x of L such that $x \neq 0_L$ holds $x^{j+1} = x^j \cdot x^1$.
- (20) For every field L and for every integer j and for every element x of L such that $x \neq 0_L$ holds $x^{j-1} = x^j \cdot x^{-1}$.
- (21) For every field L and for all integers i, j and for every element x of L such that $x \neq 0_L$ holds $x^i \cdot x^j = x^{i+j}$.
- (22) Let L be a field-like associative unital add-associative right zeroed right complementable left distributive commutative non degenerated non empty double loop structure, k be an element of \mathbb{N} , and x be an element of L . If $x \neq 0_L$, then $(x^{-1})^k = x^{-k}$.
- (23) Let L be a field and x be an element of L . Suppose $x \neq 0_L$. Let i, j, k be natural numbers. Then $x^{(i-1) \cdot (k-1)} \cdot x^{-(j-1) \cdot (k-1)} = x^{(i-j) \cdot (k-1)}$.
- (24) Let L be an associative commutative left unital distributive field-like non empty double loop structure, x be an element of L , and n, m be elements of \mathbb{N} . Then $x^{n \cdot m} = (x^n)^m$.
- (25) For every field L and for every element x of L such that $x \neq 0_L$ and for every integer i holds $(x^{-1})^i = (x^i)^{-1}$.
- (26) For every field L and for every element x of L such that $x \neq 0_L$ and for all integers i, j holds $x^{i \cdot j} = (x^i)^j$.
- (27) Let L be an associative commutative left unital distributive field-like non empty double loop structure, x be an element of L , and i, k be elements of \mathbb{N} . If $1 \leq k$, then $x^{i \cdot (k-1)} = (x^i)^{k-1}$.

4. CONVERSION BETWEEN ALGEBRAIC SEQUENCES AND MATRICES

Let m be a natural number, let L be a non empty zero structure, and let p be an algebraic sequence of L . The functor $\text{mConv}(p, m)$ yielding a matrix over L of dimension $m \times 1$ is defined as follows:

- (Def. 4) For every natural number i such that $1 \leq i$ and $i \leq m$ holds $(\text{mConv}(p, m))_{i,1} = p(i-1)$.

We now state two propositions:

- (28) Let m be a natural number. Suppose $m > 0$. Let L be a non empty zero structure and p be an algebraic sequence of L . Then $\text{len mConv}(p, m) = m$ and $\text{width mConv}(p, m) = 1$ and for every natural number i such that $i < m$ holds $(\text{mConv}(p, m))_{i+1,1} = p(i)$.
- (29) Let m be a natural number. Suppose $m > 0$. Let L be a non empty zero structure, a be an algebraic sequence of L , and M be a matrix over L of dimension $m \times 1$. Suppose that for every natural number i such that $i < m$ holds $M_{i+1,1} = a(i)$. Then $\text{mConv}(a, m) = M$.

Let L be a non empty zero structure and let M be a matrix over L . The functor $\text{aConv } M$ yielding an algebraic sequence of L is defined by the conditions (Def. 5).

- (Def. 5)(i) For every natural number i such that $i < \text{len } M$ holds $(\text{aConv } M)(i) = M_{i+1,1}$, and
 (ii) for every natural number i such that $i \geq \text{len } M$ holds $(\text{aConv } M)(i) = 0_L$.

5. PRIMITIVE ROOTS, DFT AND VANDERMONDE MATRIX

Let L be a unital non empty double loop structure, let x be an element of L , and let n be an element of \mathbb{N} . We say that x is primitive root of degree n if and only if:

- (Def. 6) $n \neq 0$ and $x^n = 1_L$ and for every element i of \mathbb{N} such that $0 < i$ and $i < n$ holds $x^i \neq 1_L$.

We now state three propositions:

- (30) Let L be a unital add-associative right zeroed right complementable right distributive non degenerated non empty double loop structure and n be an element of \mathbb{N} . Then 0_L is !not primitive root of degree n .
 (31) Let L be an add-associative right zeroed right complementable associative commutative unital distributive field-like non degenerated non empty double loop structure, m be an element of \mathbb{N} , and x be an element of L . If x is primitive root of degree m , then x^{-1} is primitive root of degree m .
 (32) Let L be an add-associative right zeroed right complementable associative commutative left unital distributive field-like non degenerated non empty double loop structure, m be an element of \mathbb{N} , and x be an element of L . Suppose x is primitive root of degree m . Let i, j be natural numbers. If $1 \leq i$ and $i \leq m$ and $1 \leq j$ and $j \leq m$ and $i \neq j$, then $x^{i-j} \neq 1_L$.

Let m be a natural number, let L be a unital non empty double loop structure, let p be a polynomial of L , and let x be an element of L . The functor $\text{DFT}(p, x, m)$ yielding an algebraic sequence of L is defined by the conditions (Def. 7).

- (Def. 7)(i) For every element i of \mathbb{N} such that $i < m$ holds $(\text{DFT}(p, x, m))(i) = \text{eval}(p, x^i)$, and
 (ii) for every element i of \mathbb{N} such that $i \geq m$ holds $(\text{DFT}(p, x, m))(i) = 0_L$.

The following propositions are true:

- (33) Let m be a natural number, L be a unital non empty double loop structure, and x be an element of L . Then $\text{DFT}(\mathbf{0}.L, x, m) = \mathbf{0}.L$.
 (34) Let m be a natural number, L be a field, p, q be polynomials of L , and x be an element of L . Then $\text{DFT}(p, x, m) \cdot \text{DFT}(q, x, m) = \text{DFT}(p * q, x, m)$.

Let L be an associative commutative left unital distributive field-like non empty double loop structure, let m be a natural number, and let x be an element of L . The functor $\text{Vandermonde}(x, m)$ yielding a matrix over L of dimension m is defined as follows:

- (Def. 8) For all natural numbers i, j such that $1 \leq i$ and $i \leq m$ and $1 \leq j$ and $j \leq m$ holds $(\text{Vandermonde}(x, m))_{i,j} = x^{(i-1) \cdot (j-1)}$.

Let L be an associative commutative left unital distributive field-like non empty double loop structure, let m be a natural number, and let x be an element of L . We introduce $\text{VM}(x, m)$ as a synonym of $\text{Vandermonde}(x, m)$.

One can prove the following propositions:

- (35) Let L be a field and m, n be natural numbers. Suppose $m > 0$. Let M be

$$\text{a matrix over } L \text{ of dimension } m \times n. \text{ Then } \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{L}^{m \times m} \cdot M = M.$$

- (36) Let L be a field and m be an element of \mathbb{N} . Suppose $0 < m$. Let u, v, u_1 be matrices over L of dimension m . Suppose that for all natural numbers i, j such that $1 \leq i$ and $i \leq m$ and $1 \leq j$ and $j \leq m$ holds $(u \cdot v)_{i,j} = m_L \cdot (u_1)_{i,j}$. Then $u \cdot v = m_L \cdot u_1$.

- (37) Let L be a field, x be an element of L , s be a finite sequence of elements of L , and i, j, m be elements of \mathbb{N} . Suppose that x is primitive root of degree m and $1 \leq i$ and $i \leq m$ and $1 \leq j$ and $j \leq m$ and $\text{len } s = m$ and for every natural number k such that $1 \leq k$ and $k \leq m$ holds $s_k = x^{(i-j) \cdot (k-1)}$. Then $(\text{VM}(x, m) \cdot \text{VM}(x^{-1}, m))_{i,j} = \sum s$.

- (38) Let L be a field, m, i, j be elements of \mathbb{N} , and x be an element of L . Suppose $i \neq j$ and $1 \leq i$ and $i \leq m$ and $1 \leq j$ and $j \leq m$ and x is primitive root of degree m . Then $(\text{VM}(x, m) \cdot \text{VM}(x^{-1}, m))_{i,j} = 0_L$.

- (39) Let L be a field and m be an element of \mathbb{N} . Suppose $m > 0$. Let x be an element of L . If x is primitive root of degree m , then $\text{VM}(x, m) \cdot$

$$\text{VM}(x^{-1}, m) = m_L \cdot \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{L}^{m \times m}.$$

- (40) Let L be a field, m be an element of \mathbb{N} , and x be an element of L . If $m > 0$ and x is primitive root of degree m , then $\text{VM}(x, m) \cdot \text{VM}(x^{-1}, m) = \text{VM}(x^{-1}, m) \cdot \text{VM}(x, m)$.

6. DFT-MULTIPLICATION OF POLYNOMIALS

We now state four propositions:

- (41) Let L be a field, p be a polynomial of L , and m be an element of \mathbb{N} . Suppose $m > 0$ and $\text{len } p \leq m$. Let x be an element of L and i be an element of \mathbb{N} . If $i < m$, then $(\text{DFT}(p, x, m))(i) = (\text{VM}(x, m) \cdot \text{mConv}(p, m))_{i+1,1}$.
- (42) Let L be a field, p be a polynomial of L , and m be a natural number. If $0 < m$ and $\text{len } p \leq m$, then for every element x of L holds $\text{DFT}(p, x, m) = \text{aConv}(\text{VM}(x, m) \cdot \text{mConv}(p, m))$.
- (43) Let L be a field, p, q be polynomials of L , and m be an element of \mathbb{N} . Suppose $m > 0$ and $\text{len } p \leq m$ and $\text{len } q \leq m$. Let x be an element of L . If x is primitive root of degree $2 \cdot m$, then $\text{DFT}(\text{DFT}(p * q, x, 2 \cdot m), x^{-1}, 2 \cdot m) = (2 \cdot m)_L \cdot (p * q)$.
- (44) Let L be a field, p, q be polynomials of L , and m be an element of \mathbb{N} . Suppose $m > 0$ and $\text{len } p \leq m$ and $\text{len } q \leq m$. Let x be an element of L . Suppose x is primitive root of degree $2 \cdot m$. If $(2 \cdot m)_L \neq 0_L$, then $((2 \cdot m)_L)^{-1} \cdot \text{DFT}(\text{DFT}(p, x, 2 \cdot m) \cdot \text{DFT}(q, x, 2 \cdot m), x^{-1}, 2 \cdot m) = p * q$.

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