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Schur's Theorem on the Stability of Networks

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Summary. A complex polynomial is called a Hurwitz polynomial if all its roots have a real part smaller than zero. This kind of polynomial plays an all-dominant role in stability checks of electrical networks.

In this article we prove Schur's criterion [17] that allows to decide whether a polynomial $p(x)$ is Hurwitz without explicitly computing its roots: Schur's recursive algorithm successively constructs polynomials $p_i(x)$ of lesser degree by division with $x - c$, $\Re\{c\} < 0$, such that $p_i(x)$ is Hurwitz if and only if $p(x)$ is.

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The articles [20], [25], [26], [18], [13], [5], [6], [1], [22], [23], [21], [19], [24], [16], [4], [9], [2], [3], [15], [14], [7], [12], [10], [27], [11], and [8] provide the terminology and notation for this paper.

1. Preliminaries

One can prove the following propositions:

(1) Let L be an add-associative right zeroed right complementable associative commutative left unital distributive field-like non empty double loop structure and x be an element of L. If $x \neq 0_L$, then $-x^{-1} = (-x)^{-1}$.

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(2) Let L be an add-associative right zeroed right complementable associative commutative left unital field-like distributive non degenerated non empty double loop structure and k be an element of $\mathbb{N}$. Then $\mathrm{power}_L(-1_L, k) \neq 0_L$.

(3) Let L be an associative right unital non empty multiplicative loop structure, x be an element of L, and k_1, k_2 be elements of $\mathbb{N}$. Then $\mathrm{power}_L(x, k_1) \cdot \mathrm{power}_L(x, k_2) = \mathrm{power}_L(x, k_1 + k_2)$.

(4) Let L be an add-associative right zeroed right complementable left unital distributive non empty double loop structure and k be an element of $\mathbb{N}$. Then $\mathrm{power}_L(-1_L, 2 \cdot k) = 1_L$ and $\mathrm{power}_L(-1_L, 2 \cdot k + 1) = -1_L$.

(5) For every element z of $\mathbb{C}_\mathrm{F}$ and for every element k of $\mathbb{N}$ holds $\overline{\mathrm{power}_{\mathbb{C}_\mathrm{F}}(z, k)} = \mathrm{power}_{\mathbb{C}_\mathrm{F}}(\overline{z}, k)$.

(6) Let F, G be finite sequences of elements of $\mathbb{C}_\mathrm{F}$. Suppose $\mathrm{len}\, G = \mathrm{len}\, F$ and for every element i of $\mathbb{N}$ such that $i \in \mathrm{dom}\, G$ holds $G_i = \overline{F_i}$. Then $\sum G = \overline{\sum F}$.

(7) Let L be an add-associative right zeroed right complementable Abelian non empty loop structure and F_1, F_2 be finite sequences of elements of L. Suppose $\mathrm{len}\, F_1 = \mathrm{len}\, F_2$ and for every element i of $\mathbb{N}$ such that $i \in \mathrm{dom}\, F_1$ holds $(F_1)_i = -(F_2)_i$. Then $\sum F_1 = -\sum F_2$.

(8) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, x be an element of L, and F be a finite sequence of elements of L. Then $x \cdot \sum F = \sum (x \cdot F)$.

2. More on Polynomials

We now state four propositions:

(9) For every add-associative right zeroed right complementable non empty loop structure L holds $-\mathbf{0}.\, L = \mathbf{0}.\, L$.

(10) Let L be an add-associative right zeroed right complementable non empty loop structure and p be a polynomial of L. Then $--p = p$.

(11) Let L be an add-associative right zeroed right complementable Abelian distributive non empty double loop structure and p_1, p_2 be polynomials of L. Then $-(p_1 + p_2) = -p_1 + -p_2$.

(12) Let L be an add-associative right zeroed right complementable distributive Abelian non empty double loop structure and p_1, p_2 be polynomials of L. Then $-p_1 * p_2 = (-p_1) * p_2$ and $-p_1 * p_2 = p_1 * -p_2$.

Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, let F be a finite sequence of elements of Polynom-Ring L, and let i be an element of $\mathbb{N}$. The functor $\mathrm{Coeff}(F, i)$ yielding a finite sequence of elements of L is defined by the conditions (Def. 1).

(Def. 1)(i) $\operatorname{len}\operatorname{Coeff}(F, i) = \operatorname{len} F$, and

(ii) for every element j of $\mathbb{N}$ such that $j \in \operatorname{dom}\operatorname{Coeff}(F, i)$ there exists a polynomial p of L such that $p = F(j)$ and $(\operatorname{Coeff}(F, i))(j) = p(i)$.

One can prove the following propositions:

(13) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, p be a polynomial of L, and F be a finite sequence of elements of Polynom-Ring L. If $p = \sum F$, then for every element i of $\mathbb{N}$ holds $p(i) = \sum \operatorname{Coeff}(F, i)$.

(14) Let L be an associative non empty double loop structure, p be a polynomial of L, and x_1, x_2 be elements of L. Then $x_1 \cdot (x_2 \cdot p) = (x_1 \cdot x_2) \cdot p$.

(15) Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure, p be a polynomial of L, and x be an element of L. Then $-x \cdot p = (-x) \cdot p$.

(16) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure, p be a polynomial of L, and x be an element of L. Then $-x \cdot p = x \cdot -p$.

(17) Let L be a left distributive non empty double loop structure, p be a polynomial of L, and x_1, x_2 be elements of L. Then $(x_1 + x_2) \cdot p = x_1 \cdot p + x_2 \cdot p$.

(18) Let L be a right distributive non empty double loop structure, p_1, p_2 be polynomials of L, and x be an element of L. Then $x \cdot (p_1 + p_2) = x \cdot p_1 + x \cdot p_2$.

(19) Let L be an add-associative right zeroed right complementable distributive commutative associative non empty double loop structure, p_1, p_2 be polynomials of L, and x be an element of L. Then $p_1 * (x \cdot p_2) = x \cdot (p_1 * p_2)$.

Let L be a non empty zero structure and let p be a polynomial of L. The functor degree(p) yields an integer and is defined by:

(Def. 2) $\operatorname{degree}(p) = \operatorname{len} p - 1$.

Let L be a non empty zero structure and let p be a polynomial of L. We introduce $\deg p$ as a synonym of degree(p).

We now state several propositions:

(20) For every non empty zero structure L and for every polynomial p of L holds $\deg p = -1$ iff $p = \mathbf{0}. L$.

(21) Let L be an add-associative right zeroed right complementable non empty loop structure and p_1, p_2 be polynomials of L. If $\deg p_1 \neq \deg p_2$, then $\deg(p_1 + p_2) = \max(\deg p_1, \deg p_2)$.

(22) Let L be an add-associative right zeroed right complementable Abelian non empty loop structure and p_1, p_2 be polynomials of L. Then $\deg(p_1 + p_2) \leq \max(\deg p_1, \deg p_2)$.

(23) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital integral domain-like non empty

double loop structure and p_1, p_2 be polynomials of L. If $p_1 \neq \mathbf{0}.\,L$ and $p_2 \neq \mathbf{0}.\,L$, then $\deg(p_1 * p_2) = \deg p_1 + \deg p_2$.

(24) Let L be an add-associative right zeroed right complementable unital non empty double loop structure and p be a polynomial of L such that $\deg p = 0$. Then p does not have roots.

Let L be a unital non empty double loop structure, let z be an element of L, and let k be an element of $\mathbb{N}$. The functor $\mathrm{rpoly}(k, z)$ yields a polynomial of L and is defined by:

(Def. 3) $\mathrm{rpoly}(k, z) = \mathbf{0}.\,L + \cdot [0 \longmapsto -\mathrm{power}_L(z,\ k), k \longmapsto 1_L]$.

One can prove the following propositions:

(25) Let L be a unital non empty double loop structure, z be an element of L, and k be an element of $\mathbb{N}$. If $k \neq 0$, then $(\mathrm{rpoly}(k, z))(0) = -\mathrm{power}_L(z,\ k)$ and $(\mathrm{rpoly}(k, z))(k) = 1_L$.

(26) Let L be a unital non empty double loop structure, z be an element of L, and i, k be elements of $\mathbb{N}$. If $i \neq 0$ and $i \neq k$, then $(\mathrm{rpoly}(k, z))(i) = 0_L$.

(27) Let L be a unital non degenerated non empty double loop structure, z be an element of L, and k be an element of $\mathbb{N}$. Then $\deg \mathrm{rpoly}(k, z) = k$.

(28) Let L be an add-associative right zeroed right complementable left unital commutative associative distributive field-like non degenerated non empty double loop structure and p be a polynomial of L. Then $\deg p = 1$ if and only if there exist elements x, z of L such that $x \neq 0_L$ and $p = x \cdot \mathrm{rpoly}(1, z)$.

(29) Let L be an add-associative right zeroed right complementable Abelian unital non degenerated non empty double loop structure and x, z be elements of L. Then $\mathrm{eval}(\mathrm{rpoly}(1, z), x) = x - z$.

(30) Let L be an add-associative right zeroed right complementable unital Abelian non degenerated non empty double loop structure and z be an element of L. Then z is a root of $\mathrm{rpoly}(1, z)$.

Let L be a unital non empty double loop structure, let z be an element of L, and let k be an element of $\mathbb{N}$. The functor $\mathrm{qpoly}(k, z)$ yielding a polynomial of L is defined by the conditions (Def. 4).

(Def. 4)(i) For every element i of $\mathbb{N}$ such that $i < k$ holds $(\mathrm{qpoly}(k, z))(i) = \mathrm{power}_L(z,\ k - i - 1)$, and

(ii) for every element i of $\mathbb{N}$ such that $i \geq k$ holds $(\mathrm{qpoly}(k, z))(i) = 0_L$.

Next we state three propositions:

(31) Let L be a unital non degenerated non empty double loop structure, z be an element of L, and k be an element of $\mathbb{N}$. If $k \geq 1$, then $\deg \mathrm{qpoly}(k, z) = k - 1$.

(32) Let L be an add-associative right zeroed right complementable left distributive unital commutative non empty double loop structure, z be an

element of L, and k be an element of $\mathbb{N}$. If $k > 1$, then $\mathrm{rpoly}(1, z) * \mathrm{qpoly}(k, z) = \mathrm{rpoly}(k, z)$.

(33) Let L be an Abelian add-associative right zeroed right complementable unital associative distributive commutative non empty double loop structure, p be a polynomial of L, and z be an element of L. If z is a root of p, then there exists a polynomial s of L such that $p = \mathrm{rpoly}(1, z) * s$.

3. Division of Polynomials

Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let p, s be polynomials of L. Let us assume that $s \neq \mathbf{0}. L$. The functor $p \div s$ yields a polynomial of L and is defined by:

(Def. 5) There exists a polynomial t of L such that $p = (p \div s) * s + t$ and $\deg t < \deg s$.

Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let p, s be polynomials of L. The functor $p \bmod s$ yielding a polynomial of L is defined by:

(Def. 6) $p \bmod s = p - (p \div s) * s$.

Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let p, s be polynomials of L. The predicate $s \mid p$ is defined by:

(Def. 7) $p \bmod s = \mathbf{0}. L$.

One can prove the following three propositions:

(34) Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and p, s be polynomials of L. Suppose $s \neq \mathbf{0}. L$. Then $s \mid p$ if and only if there exists a polynomial t of L such that $t * s = p$.

(35) Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non degenerated non empty double loop structure, p be a polynomial of L, and z be an element of L. If z is a root of p, then $\mathrm{rpoly}(1, z) \mid p$.

(36) Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non degenerated non empty double loop structure, p be a polynomial of L, and z be an element of L. If $p \neq \mathbf{0}. L$ and z is a root of p, then $\deg(p \div \mathrm{rpoly}(1, z)) = \deg p - 1$.

4. Schur's Theorem

Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$. We say that f is Hurwitz if and only if:

(Def. 8) For every element z of $\mathbb{C}_{\mathrm{F}}$ such that z is a root of f holds $\Re(z) < 0$.

We now state several propositions:

(37) $\mathbf{0}.(\mathbb{C}_{\mathrm{F}})$ is non Hurwitz.

(38) For every element x of $\mathbb{C}_{\mathrm{F}}$ such that $x \neq 0_{\mathbb{C}_{\mathrm{F}}}$ holds $x \cdot \mathbf{1}.(\mathbb{C}_{\mathrm{F}})$ is Hurwitz.

(39) For all elements x, z of $\mathbb{C}_{\mathrm{F}}$ such that $x \neq 0_{\mathbb{C}_{\mathrm{F}}}$ holds $x \cdot \mathrm{rpoly}(1, z)$ is Hurwitz iff $\Re(z) < 0$.

(40) Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$ and z be an element of $\mathbb{C}_{\mathrm{F}}$. If $z \neq 0_{\mathbb{C}_{\mathrm{F}}}$, then f is Hurwitz iff $z \cdot f$ is Hurwitz.

(41) For all polynomials f, g of $\mathbb{C}_{\mathrm{F}}$ holds $f * g$ is Hurwitz iff f is Hurwitz and g is Hurwitz.

Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$. The functor $\overline{f}$ yielding a polynomial of $\mathbb{C}_{\mathrm{F}}$ is defined by:

(Def. 9) For every element i of $\mathbb{N}$ holds $\overline{f}(i) = \mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(-1_{\mathbb{C}_{\mathrm{F}}}, i) \cdot \overline{f(i)}$.

We now state several propositions:

(42) For every polynomial f of $\mathbb{C}_{\mathrm{F}}$ holds $\deg \overline{f} = \deg f$.

(43) For every polynomial f of $\mathbb{C}_{\mathrm{F}}$ holds $\overline{\overline{f}} = f$.

(44) For every polynomial f of $\mathbb{C}_{\mathrm{F}}$ and for every element z of $\mathbb{C}_{\mathrm{F}}$ holds $\overline{z \cdot f} = \overline{z} \cdot \overline{f}$.

(45) For every polynomial f of $\mathbb{C}_{\mathrm{F}}$ holds $\overline{-f} = -\overline{f}$.

(46) For all polynomials f, g of $\mathbb{C}_{\mathrm{F}}$ holds $\overline{f + g} = \overline{f} + \overline{g}$.

(47) For all polynomials f, g of $\mathbb{C}_{\mathrm{F}}$ holds $\overline{f * g} = \overline{f} * \overline{g}$.

(48) For all elements x, z of $\mathbb{C}_{\mathrm{F}}$ holds $\mathrm{eval}(\overline{\mathrm{rpoly}(1, z)}, x) = -x - \overline{z}$.

(49) For every polynomial f of $\mathbb{C}_{\mathrm{F}}$ such that f is Hurwitz and for every element x of $\mathbb{C}_{\mathrm{F}}$ such that $\Re(x) \geq 0$ holds $0 < |\,\mathrm{eval}(f, x)|$.

(50) Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$. Suppose $\deg f \geq 1$ and f is Hurwitz. Let x be an element of $\mathbb{C}_{\mathrm{F}}$. Then

(i) if $\Re(x) < 0$, then $|\,\mathrm{eval}(f, x)| < |\,\mathrm{eval}(\overline{f}, x)|$,

(ii) if $\Re(x) > 0$, then $|\,\mathrm{eval}(f, x)| > |\,\mathrm{eval}(\overline{f}, x)|$, and

(iii) if $\Re(x) = 0$, then $|\,\mathrm{eval}(f, x)| = |\,\mathrm{eval}(\overline{f}, x)|$.

Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$ and let z be an element of $\mathbb{C}_{\mathrm{F}}$. The functor $F * (f, z)$ yields a polynomial of $\mathbb{C}_{\mathrm{F}}$ and is defined as follows:

(Def. 10) $F * (f, z) = \mathrm{eval}(\overline{f}, z) \cdot f - \mathrm{eval}(f, z) \cdot \overline{f}$.

We now state four propositions:

(51) Let a, b be elements of $\mathbb{C}_{\mathrm{F}}$. Suppose $|a| > |b|$. Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$. If $\deg f \geq 1$, then f is Hurwitz iff $a \cdot f - b \cdot \overline{f}$ is Hurwitz.

(52) Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$. Suppose $\deg f \geq 1$. Let r_1 be an element of $\mathbb{C}_{\mathrm{F}}$. If $\Re(r_1) < 0$, then if f is Hurwitz, then $F * (f, r_1) \div \mathrm{rpoly}(1, r_1)$ is Hurwitz.

(53) Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$. Suppose $\deg f \geq 1$. Given an element r_1 of $\mathbb{C}_{\mathrm{F}}$ such that $\Re(r_1) < 0$ and $|\,\mathrm{eval}(f, r_1)| \geq |\,\mathrm{eval}(\overline{f}, r_1)|$. Then f is non Hurwitz.

(54) Let f be a polynomial of $\mathbb{C}_{\mathrm{F}}$. Suppose $\deg f \geq 1$. Let r_1 be an element of $\mathbb{C}_{\mathrm{F}}$. Suppose $\Re(r_1) < 0$ and $|\,\mathrm{eval}(f, r_1)| < |\,\mathrm{eval}(\overline{f}, r_1)|$. Then f is Hurwitz if and only if $F * (f, r_1) \div \mathrm{rpoly}(1, r_1)$ is Hurwitz.

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