

Schur's Theorem on the Stability of Networks

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Summary. A complex polynomial is called a Hurwitz polynomial if all its roots have a real part smaller than zero. This kind of polynomial plays an all-dominant role in stability checks of electrical networks.

In this article we prove Schur's criterion [17] that allows to decide whether a polynomial $p(x)$ is Hurwitz without explicitly computing its roots: Schur's recursive algorithm successively constructs polynomials $p_i(x)$ of lesser degree by division with $x - c$, $\Re\{c\} < 0$, such that $p_i(x)$ is Hurwitz if and only if $p(x)$ is.

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The articles [20], [25], [26], [18], [13], [5], [6], [1], [22], [23], [21], [19], [24], [16], [4], [9], [2], [3], [15], [14], [7], [12], [10], [27], [11], and [8] provide the terminology and notation for this paper.

1. PRELIMINARIES

One can prove the following propositions:

- (1) Let L be an add-associative right zeroed right complementable associative commutative left unital distributive field-like non empty double loop structure and x be an element of L . If $x \neq 0_L$, then $-x^{-1} = (-x)^{-1}$.

- (2) Let L be an add-associative right zeroed right complementable associative commutative left unital field-like distributive non degenerated non empty double loop structure and k be an element of \mathbb{N} . Then $\text{power}_L(-1_L, k) \neq 0_L$.
- (3) Let L be an associative right unital non empty multiplicative loop structure, x be an element of L , and k_1, k_2 be elements of \mathbb{N} . Then $\text{power}_L(x, k_1) \cdot \text{power}_L(x, k_2) = \text{power}_L(x, k_1 + k_2)$.
- (4) Let L be an add-associative right zeroed right complementable left unital distributive non empty double loop structure and k be an element of \mathbb{N} . Then $\text{power}_L(-1_L, 2 \cdot k) = 1_L$ and $\text{power}_L(-1_L, 2 \cdot k + 1) = -1_L$.
- (5) For every element z of \mathbb{C}_F and for every element k of \mathbb{N} holds $\overline{\text{power}_{\mathbb{C}_F}(z, k)} = \text{power}_{\mathbb{C}_F}(\overline{z}, k)$.
- (6) Let F, G be finite sequences of elements of \mathbb{C}_F . Suppose $\text{len } G = \text{len } F$ and for every element i of \mathbb{N} such that $i \in \text{dom } G$ holds $G_i = \overline{F_i}$. Then $\sum G = \overline{\sum F}$.
- (7) Let L be an add-associative right zeroed right complementable Abelian non empty loop structure and F_1, F_2 be finite sequences of elements of L . Suppose $\text{len } F_1 = \text{len } F_2$ and for every element i of \mathbb{N} such that $i \in \text{dom } F_1$ holds $(F_1)_i = -(F_2)_i$. Then $\sum F_1 = -\sum F_2$.
- (8) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, x be an element of L , and F be a finite sequence of elements of L . Then $x \cdot \sum F = \sum(x \cdot F)$.

2. MORE ON POLYNOMIALS

We now state four propositions:

- (9) For every add-associative right zeroed right complementable non empty loop structure L holds $-\mathbf{0} \cdot L = \mathbf{0} \cdot L$.
- (10) Let L be an add-associative right zeroed right complementable non empty loop structure and p be a polynomial of L . Then $--p = p$.
- (11) Let L be an add-associative right zeroed right complementable Abelian distributive non empty double loop structure and p_1, p_2 be polynomials of L . Then $-(p_1 + p_2) = -p_1 + -p_2$.
- (12) Let L be an add-associative right zeroed right complementable distributive Abelian non empty double loop structure and p_1, p_2 be polynomials of L . Then $-p_1 * p_2 = (-p_1) * p_2$ and $-p_1 * p_2 = p_1 * -p_2$.

Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, let F be a finite sequence of elements of Polynom-Ring L , and let i be an element of \mathbb{N} . The functor $\text{Coeff}(F, i)$ yielding a finite sequence of elements of L is defined by the conditions (Def. 1).

- (Def. 1)(i) $\text{len Coeff}(F, i) = \text{len } F$, and
(ii) for every element j of \mathbb{N} such that $j \in \text{dom Coeff}(F, i)$ there exists a polynomial p of L such that $p = F(j)$ and $(\text{Coeff}(F, i))(j) = p(i)$.

One can prove the following propositions:

- (13) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, p be a polynomial of L , and F be a finite sequence of elements of Polynom-Ring L . If $p = \sum F$, then for every element i of \mathbb{N} holds $p(i) = \sum \text{Coeff}(F, i)$.
- (14) Let L be an associative non empty double loop structure, p be a polynomial of L , and x_1, x_2 be elements of L . Then $x_1 \cdot (x_2 \cdot p) = (x_1 \cdot x_2) \cdot p$.
- (15) Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure, p be a polynomial of L , and x be an element of L . Then $-x \cdot p = (-x) \cdot p$.
- (16) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure, p be a polynomial of L , and x be an element of L . Then $-x \cdot p = x \cdot -p$.
- (17) Let L be a left distributive non empty double loop structure, p be a polynomial of L , and x_1, x_2 be elements of L . Then $(x_1 + x_2) \cdot p = x_1 \cdot p + x_2 \cdot p$.
- (18) Let L be a right distributive non empty double loop structure, p_1, p_2 be polynomials of L , and x be an element of L . Then $x \cdot (p_1 + p_2) = x \cdot p_1 + x \cdot p_2$.
- (19) Let L be an add-associative right zeroed right complementable distributive commutative associative non empty double loop structure, p_1, p_2 be polynomials of L , and x be an element of L . Then $p_1 * (x \cdot p_2) = x \cdot (p_1 * p_2)$.

Let L be a non empty zero structure and let p be a polynomial of L . The functor $\text{degree}(p)$ yields an integer and is defined by:

- (Def. 2) $\text{degree}(p) = \text{len } p - 1$.

Let L be a non empty zero structure and let p be a polynomial of L . We introduce $\text{deg } p$ as a synonym of $\text{degree}(p)$.

We now state several propositions:

- (20) For every non empty zero structure L and for every polynomial p of L holds $\text{deg } p = -1$ iff $p = \mathbf{0}$.
- (21) Let L be an add-associative right zeroed right complementable non empty loop structure and p_1, p_2 be polynomials of L . If $\text{deg } p_1 \neq \text{deg } p_2$, then $\text{deg}(p_1 + p_2) = \max(\text{deg } p_1, \text{deg } p_2)$.
- (22) Let L be an add-associative right zeroed right complementable Abelian non empty loop structure and p_1, p_2 be polynomials of L . Then $\text{deg}(p_1 + p_2) \leq \max(\text{deg } p_1, \text{deg } p_2)$.
- (23) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital integral domain-like non empty

double loop structure and p_1, p_2 be polynomials of L . If $p_1 \neq \mathbf{0}_L$ and $p_2 \neq \mathbf{0}_L$, then $\deg(p_1 * p_2) = \deg p_1 + \deg p_2$.

- (24) Let L be an add-associative right zeroed right complementable unital non empty double loop structure and p be a polynomial of L such that $\deg p = 0$. Then p does not have roots.

Let L be a unital non empty double loop structure, let z be an element of L , and let k be an element of \mathbb{N} . The functor $\text{rpoly}(k, z)$ yields a polynomial of L and is defined by:

(Def. 3) $\text{rpoly}(k, z) = \mathbf{0}_L + [0 \mapsto -\text{power}_L(z, k), k \mapsto 1_L]$.

One can prove the following propositions:

- (25) Let L be a unital non empty double loop structure, z be an element of L , and k be an element of \mathbb{N} . If $k \neq 0$, then $(\text{rpoly}(k, z))(0) = -\text{power}_L(z, k)$ and $(\text{rpoly}(k, z))(k) = 1_L$.
- (26) Let L be a unital non empty double loop structure, z be an element of L , and i, k be elements of \mathbb{N} . If $i \neq 0$ and $i \neq k$, then $(\text{rpoly}(k, z))(i) = 0_L$.
- (27) Let L be a unital non degenerated non empty double loop structure, z be an element of L , and k be an element of \mathbb{N} . Then $\deg \text{rpoly}(k, z) = k$.
- (28) Let L be an add-associative right zeroed right complementable left unital commutative associative distributive field-like non degenerated non empty double loop structure and p be a polynomial of L . Then $\deg p = 1$ if and only if there exist elements x, z of L such that $x \neq 0_L$ and $p = x \cdot \text{rpoly}(1, z)$.
- (29) Let L be an add-associative right zeroed right complementable Abelian unital non degenerated non empty double loop structure and x, z be elements of L . Then $\text{eval}(\text{rpoly}(1, z), x) = x - z$.
- (30) Let L be an add-associative right zeroed right complementable unital Abelian non degenerated non empty double loop structure and z be an element of L . Then z is a root of $\text{rpoly}(1, z)$.

Let L be a unital non empty double loop structure, let z be an element of L , and let k be an element of \mathbb{N} . The functor $\text{qpoly}(k, z)$ yielding a polynomial of L is defined by the conditions (Def. 4).

- (Def. 4)(i) For every element i of \mathbb{N} such that $i < k$ holds $(\text{qpoly}(k, z))(i) = \text{power}_L(z, k - i - 1)$, and
- (ii) for every element i of \mathbb{N} such that $i \geq k$ holds $(\text{qpoly}(k, z))(i) = 0_L$.

Next we state three propositions:

- (31) Let L be a unital non degenerated non empty double loop structure, z be an element of L , and k be an element of \mathbb{N} . If $k \geq 1$, then $\deg \text{qpoly}(k, z) = k - 1$.
- (32) Let L be an add-associative right zeroed right complementable left distributive unital commutative non empty double loop structure, z be an

element of L , and k be an element of \mathbb{N} . If $k > 1$, then $\text{rpoly}(1, z) * \text{qpoly}(k, z) = \text{rpoly}(k, z)$.

- (33) Let L be an Abelian add-associative right zeroed right complementable unital associative distributive commutative non empty double loop structure, p be a polynomial of L , and z be an element of L . If z is a root of p , then there exists a polynomial s of L such that $p = \text{rpoly}(1, z) * s$.

3. DIVISION OF POLYNOMIALS

Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let p, s be polynomials of L . Let us assume that $s \neq \mathbf{0}_L$. The functor $p \div s$ yields a polynomial of L and is defined by:

- (Def. 5) There exists a polynomial t of L such that $p = (p \div s) * s + t$ and $\text{deg } t < \text{deg } s$.

Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let p, s be polynomials of L . The functor $p \bmod s$ yielding a polynomial of L is defined by:

- (Def. 6) $p \bmod s = p - (p \div s) * s$.

Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let p, s be polynomials of L . The predicate $s \mid p$ is defined by:

- (Def. 7) $p \bmod s = \mathbf{0}_L$.

One can prove the following three propositions:

- (34) Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and p, s be polynomials of L . Suppose $s \neq \mathbf{0}_L$. Then $s \mid p$ if and only if there exists a polynomial t of L such that $t * s = p$.
- (35) Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non degenerated non empty double loop structure, p be a polynomial of L , and z be an element of L . If z is a root of p , then $\text{rpoly}(1, z) \mid p$.
- (36) Let L be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non degenerated non empty double loop structure, p be a polynomial of L , and z be an element of L . If $p \neq \mathbf{0}_L$ and z is a root of p , then $\text{deg}(p \div \text{rpoly}(1, z)) = \text{deg } p - 1$.

4. SCHUR'S THEOREM

Let f be a polynomial of \mathbb{C}_F . We say that f is Hurwitz if and only if:

(Def. 8) For every element z of \mathbb{C}_F such that z is a root of f holds $\Re(z) < 0$.

We now state several propositions:

- (37) $\mathbf{0}(\mathbb{C}_F)$ is non Hurwitz.
- (38) For every element x of \mathbb{C}_F such that $x \neq 0_{\mathbb{C}_F}$ holds $x \cdot \mathbf{1}(\mathbb{C}_F)$ is Hurwitz.
- (39) For all elements x, z of \mathbb{C}_F such that $x \neq 0_{\mathbb{C}_F}$ holds $x \cdot \text{rpoly}(1, z)$ is Hurwitz iff $\Re(z) < 0$.
- (40) Let f be a polynomial of \mathbb{C}_F and z be an element of \mathbb{C}_F . If $z \neq 0_{\mathbb{C}_F}$, then f is Hurwitz iff $z \cdot f$ is Hurwitz.
- (41) For all polynomials f, g of \mathbb{C}_F holds $f * g$ is Hurwitz iff f is Hurwitz and g is Hurwitz.

Let f be a polynomial of \mathbb{C}_F . The functor \overline{f} yielding a polynomial of \mathbb{C}_F is defined by:

(Def. 9) For every element i of \mathbb{N} holds $\overline{f}(i) = \text{power}_{\mathbb{C}_F}(-1_{\mathbb{C}_F}, i) \cdot \overline{f(i)}$.

We now state several propositions:

- (42) For every polynomial f of \mathbb{C}_F holds $\deg \overline{f} = \deg f$.
- (43) For every polynomial f of \mathbb{C}_F holds $\overline{\overline{f}} = f$.
- (44) For every polynomial f of \mathbb{C}_F and for every element z of \mathbb{C}_F holds $\overline{z \cdot f} = \overline{z} \cdot \overline{f}$.
- (45) For every polynomial f of \mathbb{C}_F holds $\overline{-f} = -\overline{f}$.
- (46) For all polynomials f, g of \mathbb{C}_F holds $\overline{f + g} = \overline{f} + \overline{g}$.
- (47) For all polynomials f, g of \mathbb{C}_F holds $\overline{f * g} = \overline{f} * \overline{g}$.
- (48) For all elements x, z of \mathbb{C}_F holds $\text{eval}(\overline{\text{rpoly}(1, z)}, x) = -x - \overline{z}$.
- (49) For every polynomial f of \mathbb{C}_F such that f is Hurwitz and for every element x of \mathbb{C}_F such that $\Re(x) \geq 0$ holds $0 < |\text{eval}(f, x)|$.
- (50) Let f be a polynomial of \mathbb{C}_F . Suppose $\deg f \geq 1$ and f is Hurwitz. Let x be an element of \mathbb{C}_F . Then
 - (i) if $\Re(x) < 0$, then $|\text{eval}(f, x)| < |\text{eval}(\overline{f}, x)|$,
 - (ii) if $\Re(x) > 0$, then $|\text{eval}(f, x)| > |\text{eval}(\overline{f}, x)|$, and
 - (iii) if $\Re(x) = 0$, then $|\text{eval}(f, x)| = |\text{eval}(\overline{f}, x)|$.

Let f be a polynomial of \mathbb{C}_F and let z be an element of \mathbb{C}_F . The functor $F * (f, z)$ yields a polynomial of \mathbb{C}_F and is defined as follows:

(Def. 10) $F * (f, z) = \text{eval}(\overline{f}, z) \cdot f - \text{eval}(f, z) \cdot \overline{f}$.

We now state four propositions:

- (51) Let a, b be elements of \mathbb{C}_F . Suppose $|a| > |b|$. Let f be a polynomial of \mathbb{C}_F . If $\deg f \geq 1$, then f is Hurwitz iff $a \cdot f - b \cdot \overline{f}$ is Hurwitz.

- (52) Let f be a polynomial of \mathbb{C}_F . Suppose $\deg f \geq 1$. Let r_1 be an element of \mathbb{C}_F . If $\Re(r_1) < 0$, then if f is Hurwitz, then $F * (f, r_1) \div \text{rpoly}(1, r_1)$ is Hurwitz.
- (53) Let f be a polynomial of \mathbb{C}_F . Suppose $\deg f \geq 1$. Given an element r_1 of \mathbb{C}_F such that $\Re(r_1) < 0$ and $|\text{eval}(f, r_1)| \geq |\text{eval}(\bar{f}, r_1)|$. Then f is non Hurwitz.
- (54) Let f be a polynomial of \mathbb{C}_F . Suppose $\deg f \geq 1$. Let r_1 be an element of \mathbb{C}_F . Suppose $\Re(r_1) < 0$ and $|\text{eval}(f, r_1)| < |\text{eval}(\bar{f}, r_1)|$. Then f is Hurwitz if and only if $F * (f, r_1) \div \text{rpoly}(1, r_1)$ is Hurwitz.

REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(1):175–180, 1990.
- [4] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [7] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(3):521–527, 1990.
- [8] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(2):335–342, 1990.
- [9] Anna Justyna Milewska. The field of complex numbers. *Formalized Mathematics*, 9(2):265–269, 2001.
- [10] Robert Milewski. The evaluation of polynomials. *Formalized Mathematics*, 9(2):391–395, 2001.
- [11] Robert Milewski. Fundamental theorem of algebra. *Formalized Mathematics*, 9(3):461–470, 2001.
- [12] Robert Milewski. The ring of polynomials. *Formalized Mathematics*, 9(2):339–346, 2001.
- [13] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(1):3–11, 1991.
- [14] Michał Muzalewski and Lesław W. Szcerba. Construction of finite sequences over ring and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(1):97–104, 1991.
- [15] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(1):111–115, 1991.
- [16] Piotr Rudnicki and Andrzej Trybulec. Multivariate polynomials with arbitrary number of variables. *Formalized Mathematics*, 9(1):95–110, 2001.
- [17] J. Schur. Über algebraische Gleichungen, die nur Wurzeln mit negativen Realteilen besitzen. *Zeitschrift für angewandte Mathematik und Mechanik*, 1:307–311, 1921.
- [18] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
- [19] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [20] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [21] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [22] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(5):821–827, 1990.
- [23] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(3):575–579, 1990.
- [24] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [26] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.

- [27] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

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