

# Schur's Theorem on the Stability of Networks

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**Summary.** A complex polynomial is called a Hurwitz polynomial if all its roots have a real part smaller than zero. This kind of polynomial plays an all-dominant role in stability checks of electrical networks.

In this article we prove Schur's criterion [17] that allows to decide whether a polynomial  $p(x)$  is Hurwitz without explicitly computing its roots: Schur's recursive algorithm successively constructs polynomials  $p_i(x)$  of lesser degree by division with  $x - c$ ,  $\Re\{c\} < 0$ , such that  $p_i(x)$  is Hurwitz if and only if  $p(x)$  is.

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The articles [20], [25], [26], [18], [13], [5], [6], [1], [22], [23], [21], [19], [24], [16], [4], [9], [2], [3], [15], [14], [7], [12], [10], [27], [11], and [8] provide the terminology and notation for this paper.

## 1. PRELIMINARIES

One can prove the following propositions:

- (1) Let  $L$  be an add-associative right zeroed right complementable associative commutative left unital distributive field-like non empty double loop structure and  $x$  be an element of  $L$ . If  $x \neq 0_L$ , then  $-x^{-1} = (-x)^{-1}$ .

- (2) Let  $L$  be an add-associative right zeroed right complementable associative commutative left unital field-like distributive non degenerated non empty double loop structure and  $k$  be an element of  $\mathbb{N}$ . Then  $\text{power}_L(-1_L, k) \neq 0_L$ .
- (3) Let  $L$  be an associative right unital non empty multiplicative loop structure,  $x$  be an element of  $L$ , and  $k_1, k_2$  be elements of  $\mathbb{N}$ . Then  $\text{power}_L(x, k_1) \cdot \text{power}_L(x, k_2) = \text{power}_L(x, k_1 + k_2)$ .
- (4) Let  $L$  be an add-associative right zeroed right complementable left unital distributive non empty double loop structure and  $k$  be an element of  $\mathbb{N}$ . Then  $\text{power}_L(-1_L, 2 \cdot k) = 1_L$  and  $\text{power}_L(-1_L, 2 \cdot k + 1) = -1_L$ .
- (5) For every element  $z$  of  $\mathbb{C}_F$  and for every element  $k$  of  $\mathbb{N}$  holds  $\overline{\text{power}_{\mathbb{C}_F}(z, k)} = \text{power}_{\mathbb{C}_F}(\overline{z}, k)$ .
- (6) Let  $F, G$  be finite sequences of elements of  $\mathbb{C}_F$ . Suppose  $\text{len } G = \text{len } F$  and for every element  $i$  of  $\mathbb{N}$  such that  $i \in \text{dom } G$  holds  $G_i = \overline{F_i}$ . Then  $\sum G = \overline{\sum F}$ .
- (7) Let  $L$  be an add-associative right zeroed right complementable Abelian non empty loop structure and  $F_1, F_2$  be finite sequences of elements of  $L$ . Suppose  $\text{len } F_1 = \text{len } F_2$  and for every element  $i$  of  $\mathbb{N}$  such that  $i \in \text{dom } F_1$  holds  $(F_1)_i = -(F_2)_i$ . Then  $\sum F_1 = -\sum F_2$ .
- (8) Let  $L$  be an add-associative right zeroed right complementable distributive non empty double loop structure,  $x$  be an element of  $L$ , and  $F$  be a finite sequence of elements of  $L$ . Then  $x \cdot \sum F = \sum(x \cdot F)$ .

## 2. MORE ON POLYNOMIALS

We now state four propositions:

- (9) For every add-associative right zeroed right complementable non empty loop structure  $L$  holds  $-\mathbf{0} \cdot L = \mathbf{0} \cdot L$ .
- (10) Let  $L$  be an add-associative right zeroed right complementable non empty loop structure and  $p$  be a polynomial of  $L$ . Then  $--p = p$ .
- (11) Let  $L$  be an add-associative right zeroed right complementable Abelian distributive non empty double loop structure and  $p_1, p_2$  be polynomials of  $L$ . Then  $-(p_1 + p_2) = -p_1 + -p_2$ .
- (12) Let  $L$  be an add-associative right zeroed right complementable distributive Abelian non empty double loop structure and  $p_1, p_2$  be polynomials of  $L$ . Then  $-p_1 * p_2 = (-p_1) * p_2$  and  $-p_1 * p_2 = p_1 * -p_2$ .

Let  $L$  be an add-associative right zeroed right complementable distributive non empty double loop structure, let  $F$  be a finite sequence of elements of Polynom-Ring  $L$ , and let  $i$  be an element of  $\mathbb{N}$ . The functor  $\text{Coeff}(F, i)$  yielding a finite sequence of elements of  $L$  is defined by the conditions (Def. 1).

- (Def. 1)(i)  $\text{len Coeff}(F, i) = \text{len } F$ , and  
(ii) for every element  $j$  of  $\mathbb{N}$  such that  $j \in \text{dom Coeff}(F, i)$  there exists a polynomial  $p$  of  $L$  such that  $p = F(j)$  and  $(\text{Coeff}(F, i))(j) = p(i)$ .

One can prove the following propositions:

- (13) Let  $L$  be an add-associative right zeroed right complementable distributive non empty double loop structure,  $p$  be a polynomial of  $L$ , and  $F$  be a finite sequence of elements of Polynom-Ring  $L$ . If  $p = \sum F$ , then for every element  $i$  of  $\mathbb{N}$  holds  $p(i) = \sum \text{Coeff}(F, i)$ .
- (14) Let  $L$  be an associative non empty double loop structure,  $p$  be a polynomial of  $L$ , and  $x_1, x_2$  be elements of  $L$ . Then  $x_1 \cdot (x_2 \cdot p) = (x_1 \cdot x_2) \cdot p$ .
- (15) Let  $L$  be an add-associative right zeroed right complementable left distributive non empty double loop structure,  $p$  be a polynomial of  $L$ , and  $x$  be an element of  $L$ . Then  $-x \cdot p = (-x) \cdot p$ .
- (16) Let  $L$  be an add-associative right zeroed right complementable right distributive non empty double loop structure,  $p$  be a polynomial of  $L$ , and  $x$  be an element of  $L$ . Then  $-x \cdot p = x \cdot -p$ .
- (17) Let  $L$  be a left distributive non empty double loop structure,  $p$  be a polynomial of  $L$ , and  $x_1, x_2$  be elements of  $L$ . Then  $(x_1 + x_2) \cdot p = x_1 \cdot p + x_2 \cdot p$ .
- (18) Let  $L$  be a right distributive non empty double loop structure,  $p_1, p_2$  be polynomials of  $L$ , and  $x$  be an element of  $L$ . Then  $x \cdot (p_1 + p_2) = x \cdot p_1 + x \cdot p_2$ .
- (19) Let  $L$  be an add-associative right zeroed right complementable distributive commutative associative non empty double loop structure,  $p_1, p_2$  be polynomials of  $L$ , and  $x$  be an element of  $L$ . Then  $p_1 * (x \cdot p_2) = x \cdot (p_1 * p_2)$ .

Let  $L$  be a non empty zero structure and let  $p$  be a polynomial of  $L$ . The functor  $\text{degree}(p)$  yields an integer and is defined by:

- (Def. 2)  $\text{degree}(p) = \text{len } p - 1$ .

Let  $L$  be a non empty zero structure and let  $p$  be a polynomial of  $L$ . We introduce  $\text{deg } p$  as a synonym of  $\text{degree}(p)$ .

We now state several propositions:

- (20) For every non empty zero structure  $L$  and for every polynomial  $p$  of  $L$  holds  $\text{deg } p = -1$  iff  $p = \mathbf{0}$ .
- (21) Let  $L$  be an add-associative right zeroed right complementable non empty loop structure and  $p_1, p_2$  be polynomials of  $L$ . If  $\text{deg } p_1 \neq \text{deg } p_2$ , then  $\text{deg}(p_1 + p_2) = \max(\text{deg } p_1, \text{deg } p_2)$ .
- (22) Let  $L$  be an add-associative right zeroed right complementable Abelian non empty loop structure and  $p_1, p_2$  be polynomials of  $L$ . Then  $\text{deg}(p_1 + p_2) \leq \max(\text{deg } p_1, \text{deg } p_2)$ .
- (23) Let  $L$  be an add-associative right zeroed right complementable distributive commutative associative left unital integral domain-like non empty

double loop structure and  $p_1, p_2$  be polynomials of  $L$ . If  $p_1 \neq \mathbf{0}_L$  and  $p_2 \neq \mathbf{0}_L$ , then  $\deg(p_1 * p_2) = \deg p_1 + \deg p_2$ .

- (24) Let  $L$  be an add-associative right zeroed right complementable unital non empty double loop structure and  $p$  be a polynomial of  $L$  such that  $\deg p = 0$ . Then  $p$  does not have roots.

Let  $L$  be a unital non empty double loop structure, let  $z$  be an element of  $L$ , and let  $k$  be an element of  $\mathbb{N}$ . The functor  $\text{rpoly}(k, z)$  yields a polynomial of  $L$  and is defined by:

(Def. 3)  $\text{rpoly}(k, z) = \mathbf{0}_L + [0 \mapsto -\text{power}_L(z, k), k \mapsto 1_L]$ .

One can prove the following propositions:

- (25) Let  $L$  be a unital non empty double loop structure,  $z$  be an element of  $L$ , and  $k$  be an element of  $\mathbb{N}$ . If  $k \neq 0$ , then  $(\text{rpoly}(k, z))(0) = -\text{power}_L(z, k)$  and  $(\text{rpoly}(k, z))(k) = 1_L$ .
- (26) Let  $L$  be a unital non empty double loop structure,  $z$  be an element of  $L$ , and  $i, k$  be elements of  $\mathbb{N}$ . If  $i \neq 0$  and  $i \neq k$ , then  $(\text{rpoly}(k, z))(i) = 0_L$ .
- (27) Let  $L$  be a unital non degenerated non empty double loop structure,  $z$  be an element of  $L$ , and  $k$  be an element of  $\mathbb{N}$ . Then  $\deg \text{rpoly}(k, z) = k$ .
- (28) Let  $L$  be an add-associative right zeroed right complementable left unital commutative associative distributive field-like non degenerated non empty double loop structure and  $p$  be a polynomial of  $L$ . Then  $\deg p = 1$  if and only if there exist elements  $x, z$  of  $L$  such that  $x \neq 0_L$  and  $p = x \cdot \text{rpoly}(1, z)$ .
- (29) Let  $L$  be an add-associative right zeroed right complementable Abelian unital non degenerated non empty double loop structure and  $x, z$  be elements of  $L$ . Then  $\text{eval}(\text{rpoly}(1, z), x) = x - z$ .
- (30) Let  $L$  be an add-associative right zeroed right complementable unital Abelian non degenerated non empty double loop structure and  $z$  be an element of  $L$ . Then  $z$  is a root of  $\text{rpoly}(1, z)$ .

Let  $L$  be a unital non empty double loop structure, let  $z$  be an element of  $L$ , and let  $k$  be an element of  $\mathbb{N}$ . The functor  $\text{qpoly}(k, z)$  yielding a polynomial of  $L$  is defined by the conditions (Def. 4).

- (Def. 4)(i) For every element  $i$  of  $\mathbb{N}$  such that  $i < k$  holds  $(\text{qpoly}(k, z))(i) = \text{power}_L(z, k - i - 1)$ , and
- (ii) for every element  $i$  of  $\mathbb{N}$  such that  $i \geq k$  holds  $(\text{qpoly}(k, z))(i) = 0_L$ .

Next we state three propositions:

- (31) Let  $L$  be a unital non degenerated non empty double loop structure,  $z$  be an element of  $L$ , and  $k$  be an element of  $\mathbb{N}$ . If  $k \geq 1$ , then  $\deg \text{qpoly}(k, z) = k - 1$ .
- (32) Let  $L$  be an add-associative right zeroed right complementable left distributive unital commutative non empty double loop structure,  $z$  be an

element of  $L$ , and  $k$  be an element of  $\mathbb{N}$ . If  $k > 1$ , then  $\text{rpoly}(1, z) * \text{qpoly}(k, z) = \text{rpoly}(k, z)$ .

- (33) Let  $L$  be an Abelian add-associative right zeroed right complementable unital associative distributive commutative non empty double loop structure,  $p$  be a polynomial of  $L$ , and  $z$  be an element of  $L$ . If  $z$  is a root of  $p$ , then there exists a polynomial  $s$  of  $L$  such that  $p = \text{rpoly}(1, z) * s$ .

### 3. DIVISION OF POLYNOMIALS

Let  $L$  be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let  $p, s$  be polynomials of  $L$ . Let us assume that  $s \neq \mathbf{0}_L$ . The functor  $p \div s$  yields a polynomial of  $L$  and is defined by:

- (Def. 5) There exists a polynomial  $t$  of  $L$  such that  $p = (p \div s) * s + t$  and  $\deg t < \deg s$ .

Let  $L$  be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let  $p, s$  be polynomials of  $L$ . The functor  $p \bmod s$  yielding a polynomial of  $L$  is defined by:

- (Def. 6)  $p \bmod s = p - (p \div s) * s$ .

Let  $L$  be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and let  $p, s$  be polynomials of  $L$ . The predicate  $s \mid p$  is defined by:

- (Def. 7)  $p \bmod s = \mathbf{0}_L$ .

One can prove the following three propositions:

- (34) Let  $L$  be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non empty double loop structure and  $p, s$  be polynomials of  $L$ . Suppose  $s \neq \mathbf{0}_L$ . Then  $s \mid p$  if and only if there exists a polynomial  $t$  of  $L$  such that  $t * s = p$ .
- (35) Let  $L$  be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non degenerated non empty double loop structure,  $p$  be a polynomial of  $L$ , and  $z$  be an element of  $L$ . If  $z$  is a root of  $p$ , then  $\text{rpoly}(1, z) \mid p$ .
- (36) Let  $L$  be an Abelian add-associative right zeroed right complementable left unital associative commutative distributive field-like non degenerated non empty double loop structure,  $p$  be a polynomial of  $L$ , and  $z$  be an element of  $L$ . If  $p \neq \mathbf{0}_L$  and  $z$  is a root of  $p$ , then  $\deg(p \div \text{rpoly}(1, z)) = \deg p - 1$ .

## 4. SCHUR'S THEOREM

Let  $f$  be a polynomial of  $\mathbb{C}_F$ . We say that  $f$  is Hurwitz if and only if:

(Def. 8) For every element  $z$  of  $\mathbb{C}_F$  such that  $z$  is a root of  $f$  holds  $\Re(z) < 0$ .

We now state several propositions:

- (37)  $\mathbf{0}(\mathbb{C}_F)$  is non Hurwitz.
- (38) For every element  $x$  of  $\mathbb{C}_F$  such that  $x \neq 0_{\mathbb{C}_F}$  holds  $x \cdot \mathbf{1}(\mathbb{C}_F)$  is Hurwitz.
- (39) For all elements  $x, z$  of  $\mathbb{C}_F$  such that  $x \neq 0_{\mathbb{C}_F}$  holds  $x \cdot \text{rpoly}(1, z)$  is Hurwitz iff  $\Re(z) < 0$ .
- (40) Let  $f$  be a polynomial of  $\mathbb{C}_F$  and  $z$  be an element of  $\mathbb{C}_F$ . If  $z \neq 0_{\mathbb{C}_F}$ , then  $f$  is Hurwitz iff  $z \cdot f$  is Hurwitz.
- (41) For all polynomials  $f, g$  of  $\mathbb{C}_F$  holds  $f * g$  is Hurwitz iff  $f$  is Hurwitz and  $g$  is Hurwitz.

Let  $f$  be a polynomial of  $\mathbb{C}_F$ . The functor  $\overline{f}$  yielding a polynomial of  $\mathbb{C}_F$  is defined by:

(Def. 9) For every element  $i$  of  $\mathbb{N}$  holds  $\overline{f}(i) = \text{power}_{\mathbb{C}_F}(-1_{\mathbb{C}_F}, i) \cdot \overline{f(i)}$ .

We now state several propositions:

- (42) For every polynomial  $f$  of  $\mathbb{C}_F$  holds  $\deg \overline{f} = \deg f$ .
- (43) For every polynomial  $f$  of  $\mathbb{C}_F$  holds  $\overline{\overline{f}} = f$ .
- (44) For every polynomial  $f$  of  $\mathbb{C}_F$  and for every element  $z$  of  $\mathbb{C}_F$  holds  $\overline{z \cdot f} = \overline{z} \cdot \overline{f}$ .
- (45) For every polynomial  $f$  of  $\mathbb{C}_F$  holds  $\overline{-f} = -\overline{f}$ .
- (46) For all polynomials  $f, g$  of  $\mathbb{C}_F$  holds  $\overline{f + g} = \overline{f} + \overline{g}$ .
- (47) For all polynomials  $f, g$  of  $\mathbb{C}_F$  holds  $\overline{f * g} = \overline{f} * \overline{g}$ .
- (48) For all elements  $x, z$  of  $\mathbb{C}_F$  holds  $\text{eval}(\overline{\text{rpoly}(1, z)}, x) = -x - \overline{z}$ .
- (49) For every polynomial  $f$  of  $\mathbb{C}_F$  such that  $f$  is Hurwitz and for every element  $x$  of  $\mathbb{C}_F$  such that  $\Re(x) \geq 0$  holds  $0 < |\text{eval}(f, x)|$ .
- (50) Let  $f$  be a polynomial of  $\mathbb{C}_F$ . Suppose  $\deg f \geq 1$  and  $f$  is Hurwitz. Let  $x$  be an element of  $\mathbb{C}_F$ . Then
  - (i) if  $\Re(x) < 0$ , then  $|\text{eval}(f, x)| < |\text{eval}(\overline{f}, x)|$ ,
  - (ii) if  $\Re(x) > 0$ , then  $|\text{eval}(f, x)| > |\text{eval}(\overline{f}, x)|$ , and
  - (iii) if  $\Re(x) = 0$ , then  $|\text{eval}(f, x)| = |\text{eval}(\overline{f}, x)|$ .

Let  $f$  be a polynomial of  $\mathbb{C}_F$  and let  $z$  be an element of  $\mathbb{C}_F$ . The functor  $F * (f, z)$  yields a polynomial of  $\mathbb{C}_F$  and is defined as follows:

(Def. 10)  $F * (f, z) = \text{eval}(\overline{f}, z) \cdot f - \text{eval}(f, z) \cdot \overline{f}$ .

We now state four propositions:

- (51) Let  $a, b$  be elements of  $\mathbb{C}_F$ . Suppose  $|a| > |b|$ . Let  $f$  be a polynomial of  $\mathbb{C}_F$ . If  $\deg f \geq 1$ , then  $f$  is Hurwitz iff  $a \cdot f - b \cdot \overline{f}$  is Hurwitz.

- (52) Let  $f$  be a polynomial of  $\mathbb{C}_F$ . Suppose  $\deg f \geq 1$ . Let  $r_1$  be an element of  $\mathbb{C}_F$ . If  $\Re(r_1) < 0$ , then if  $f$  is Hurwitz, then  $F * (f, r_1) \div \text{rpoly}(1, r_1)$  is Hurwitz.
- (53) Let  $f$  be a polynomial of  $\mathbb{C}_F$ . Suppose  $\deg f \geq 1$ . Given an element  $r_1$  of  $\mathbb{C}_F$  such that  $\Re(r_1) < 0$  and  $|\text{eval}(f, r_1)| \geq |\text{eval}(\bar{f}, r_1)|$ . Then  $f$  is non Hurwitz.
- (54) Let  $f$  be a polynomial of  $\mathbb{C}_F$ . Suppose  $\deg f \geq 1$ . Let  $r_1$  be an element of  $\mathbb{C}_F$ . Suppose  $\Re(r_1) < 0$  and  $|\text{eval}(f, r_1)| < |\text{eval}(\bar{f}, r_1)|$ . Then  $f$  is Hurwitz if and only if  $F * (f, r_1) \div \text{rpoly}(1, r_1)$  is Hurwitz.

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