

Integral of Real-Valued Measurable Function¹

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Summary. Based on [16], authors formalized the integral of an extended real valued measurable function in [12] before. However, the integral argued in [12] cannot be applied to real-valued functions unconditionally. Therefore, in this article we have formalized the integral of a real-value function.

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The papers [25], [11], [26], [1], [23], [24], [17], [18], [8], [27], [10], [2], [19], [7], [20], [6], [9], [3], [4], [5], [13], [14], [15], [22], [21], and [12] provide the terminology and notation for this paper.

1. THE MEASURABILITY OF REAL-VALUED FUNCTIONS

For simplicity, we follow the rules: X denotes a non empty set, Y denotes a set, S denotes a σ -field of subsets of X , F denotes a function from \mathbb{N} into S , f , g denote partial functions from X to \mathbb{R} , A , B denote elements of S , r , s denote real numbers, a denotes a real number, and n denotes a natural number.

Let X be a non empty set, let f be a partial function from X to \mathbb{R} , and let a be a real number. The functor $\text{LE-dom}(f, a)$ yields a subset of X and is defined as follows:

(Def. 1) $\text{LE-dom}(f, a) = \text{LE-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$.

The following three propositions are true:

(1) $|\overline{\mathbb{R}}(f)| = \overline{\mathbb{R}}(|f|)$.

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- (2) Let X be a non empty set, S be a σ -field of subsets of X , M be a σ -measure on S , f be a partial function from X to $\overline{\mathbb{R}}$, and r be a real number. Suppose $\text{dom } f \in S$ and for every set x such that $x \in \text{dom } f$ holds $f(x) = r$. Then f is simple function in S .
- (3) For every set x holds $x \in \text{LEQ-dom}(f, a)$ iff $x \in \text{dom } f$ and there exists a real number y such that $y = f(x)$ and $y < a$.

Let us consider X, f, a . The functor $\text{LEQ-dom}(f, a)$ yields a subset of X and is defined as follows:

$$\text{(Def. 2)} \quad \text{LEQ-dom}(f, a) = \text{LEQ-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a)).$$

We now state the proposition

- (4) For every set x holds $x \in \text{LEQ-dom}(f, a)$ iff $x \in \text{dom } f$ and there exists a real number y such that $y = f(x)$ and $y \leq a$.

Let us consider X, f, a . The functor $\text{GT-dom}(f, a)$ yielding a subset of X is defined as follows:

$$\text{(Def. 3)} \quad \text{GT-dom}(f, a) = \text{GT-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a)).$$

We now state the proposition

- (5) For every set x holds $x \in \text{GT-dom}(f, r)$ iff $x \in \text{dom } f$ and there exists a real number y such that $y = f(x)$ and $r < y$.

Let us consider X, f, a . The functor $\text{GTE-dom}(f, a)$ yields a subset of X and is defined as follows:

$$\text{(Def. 4)} \quad \text{GTE-dom}(f, a) = \text{GTE-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a)).$$

Next we state the proposition

- (6) For every set x holds $x \in \text{GTE-dom}(f, r)$ iff $x \in \text{dom } f$ and there exists a real number y such that $y = f(x)$ and $r \leq y$.

Let us consider X, f, a . The functor $\text{EQ-dom}(f, a)$ yielding a subset of X is defined by:

$$\text{(Def. 5)} \quad \text{EQ-dom}(f, a) = \text{EQ-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a)).$$

The following propositions are true:

- (7) For every set x holds $x \in \text{EQ-dom}(f, r)$ iff $x \in \text{dom } f$ and there exists a real number y such that $y = f(x)$ and $r = y$.
- (8) If for every n holds $F(n) = Y \cap \text{GT-dom}(f, r - \frac{1}{n+1})$, then $Y \cap \text{GTE-dom}(f, r) = \bigcap \text{rng } F$.
- (9) If for every n holds $F(n) = Y \cap \text{LEQ-dom}(f, r + \frac{1}{n+1})$, then $Y \cap \text{LEQ-dom}(f, r) = \bigcap \text{rng } F$.
- (10) If for every n holds $F(n) = Y \cap \text{LEQ-dom}(f, r - \frac{1}{n+1})$, then $Y \cap \text{LE-dom}(f, r) = \bigcup \text{rng } F$.
- (11) If for every n holds $F(n) = Y \cap \text{GTE-dom}(f, r + \frac{1}{n+1})$, then $Y \cap \text{GT-dom}(f, r) = \bigcup \text{rng } F$.

Let X be a non empty set, let S be a σ -field of subsets of X , let f be a partial function from X to \mathbb{R} , and let A be an element of S . We say that f is measurable on A if and only if:

(Def. 6) $\overline{\mathbb{R}}(f)$ is measurable on A .

The following propositions are true:

- (12) f is measurable on A iff for every real number r holds $A \cap \text{LE-dom}(f, r)$ is measurable on S .
- (13) Suppose $A \subseteq \text{dom } f$. Then f is measurable on A if and only if for every real number r holds $A \cap \text{GTE-dom}(f, r)$ is measurable on S .
- (14) f is measurable on A iff for every real number r holds $A \cap \text{LEQ-dom}(f, r)$ is measurable on S .
- (15) Suppose $A \subseteq \text{dom } f$. Then f is measurable on A if and only if for every real number r holds $A \cap \text{GT-dom}(f, r)$ is measurable on S .
- (16) If $B \subseteq A$ and f is measurable on A , then f is measurable on B .
- (17) If f is measurable on A and f is measurable on B , then f is measurable on $A \cup B$.
- (18) If f is measurable on A and $A \subseteq \text{dom } f$, then $A \cap \text{GT-dom}(f, r) \cap \text{LE-dom}(f, s)$ is measurable on S .
- (19) If f is measurable on A and g is measurable on A and $A \subseteq \text{dom } g$, then $A \cap \text{LE-dom}(f, r) \cap \text{GT-dom}(g, r)$ is measurable on S .
- (20) $\overline{\mathbb{R}}(rf) = r \overline{\mathbb{R}}(f)$.
- (21) If f is measurable on A and $A \subseteq \text{dom } f$, then rf is measurable on A .

2. THE MEASURABILITY OF $f + g$ AND $f - g$ FOR REAL-VALUED FUNCTIONS f, g

For simplicity, we adopt the following rules: X denotes a non empty set, S denotes a σ -field of subsets of X , f, g denote partial functions from X to \mathbb{R} , A denotes an element of S , r denotes a real number, and p denotes a rational number.

Next we state several propositions:

- (22) $\overline{\mathbb{R}}(f)$ is finite.
- (23) $\overline{\mathbb{R}}(f + g) = \overline{\mathbb{R}}(f) + \overline{\mathbb{R}}(g)$ and $\overline{\mathbb{R}}(f - g) = \overline{\mathbb{R}}(f) - \overline{\mathbb{R}}(g)$ and $\text{dom } \overline{\mathbb{R}}(f + g) = \text{dom } \overline{\mathbb{R}}(f) \cap \text{dom } \overline{\mathbb{R}}(g)$ and $\text{dom } \overline{\mathbb{R}}(f - g) = \text{dom } \overline{\mathbb{R}}(f) \cap \text{dom } \overline{\mathbb{R}}(g)$ and $\text{dom } \overline{\mathbb{R}}(f + g) = \text{dom } f \cap \text{dom } g$ and $\text{dom } \overline{\mathbb{R}}(f - g) = \text{dom } f \cap \text{dom } g$.
- (24) For every function F from \mathbb{Q} into S such that for every p holds $F(p) = A \cap \text{LE-dom}(f, p) \cap (A \cap \text{LE-dom}(g, r - p))$ holds $A \cap \text{LE-dom}(f + g, r) = \bigcup \text{rng } F$.

- (25) Suppose f is measurable on A and g is measurable on A . Then there exists a function F from \mathbb{Q} into S such that for every rational number p holds $F(p) = A \cap \text{LE-dom}(f, p) \cap (A \cap \text{LE-dom}(g, r - p))$.
- (26) If f is measurable on A and g is measurable on A , then $f+g$ is measurable on A .
- (27) $\overline{\mathbb{R}}(f) - \overline{\mathbb{R}}(g) = \overline{\mathbb{R}}(f) + \overline{\mathbb{R}}(-g)$.
- (28) $-\overline{\mathbb{R}}(f) = \overline{\mathbb{R}}((-1)f)$ and $-\overline{\mathbb{R}}(f) = \overline{\mathbb{R}}(-f)$.
- (29) If f is measurable on A and g is measurable on A and $A \subseteq \text{dom } g$, then $f - g$ is measurable on A .

3. BASIC PROPERTIES OF REAL-VALUED FUNCTIONS, $\max_+ f$ AND $\max_- f$

In the sequel X denotes a non empty set, f denotes a partial function from X to \mathbb{R} , and r denotes a real number.

Next we state a number of propositions:

- (30) $\max_+(\overline{\mathbb{R}}(f)) = \max_+(f)$ and $\max_-(\overline{\mathbb{R}}(f)) = \max_-(f)$.
- (31) For every element x of X holds $0 \leq (\max_+(f))(x)$.
- (32) For every element x of X holds $0 \leq (\max_-(f))(x)$.
- (33) $\max_-(f) = \max_+(-f)$.
- (34) For every set x such that $x \in \text{dom } f$ and $0 < (\max_+(f))(x)$ holds $(\max_-(f))(x) = 0$.
- (35) For every set x such that $x \in \text{dom } f$ and $0 < (\max_-(f))(x)$ holds $(\max_+(f))(x) = 0$.
- (36) $\text{dom } f = \text{dom}(\max_+(f) - \max_-(f))$ and $\text{dom } f = \text{dom}(\max_+(f) + \max_-(f))$.
- (37) For every set x such that $x \in \text{dom } f$ holds $(\max_+(f))(x) = f(x)$ or $(\max_+(f))(x) = 0$ but $(\max_-(f))(x) = -f(x)$ or $(\max_-(f))(x) = 0$.
- (38) For every set x such that $x \in \text{dom } f$ and $(\max_+(f))(x) = f(x)$ holds $(\max_-(f))(x) = 0$.
- (39) For every set x such that $x \in \text{dom } f$ and $(\max_+(f))(x) = 0$ holds $(\max_-(f))(x) = -f(x)$.
- (40) For every set x such that $x \in \text{dom } f$ and $(\max_-(f))(x) = -f(x)$ holds $(\max_+(f))(x) = 0$.
- (41) For every set x such that $x \in \text{dom } f$ and $(\max_-(f))(x) = 0$ holds $(\max_+(f))(x) = f(x)$.
- (42) $f = \max_+(f) - \max_-(f)$.
- (43) $|r| = |\overline{\mathbb{R}}(r)|$.
- (44) $\overline{\mathbb{R}}(|f|) = |\overline{\mathbb{R}}(f)|$.

$$(45) \quad |f| = \max_+(f) + \max_-(f).$$

4. THE MEASURABILITY OF $\max_+ f, \max_- f$ AND $|f|$

In the sequel X denotes a non empty set, S denotes a σ -field of subsets of X , f denotes a partial function from X to \mathbb{R} , and A denotes an element of S .

The following propositions are true:

- (46) If f is measurable on A , then $\max_+(f)$ is measurable on A .
- (47) If f is measurable on A and $A \subseteq \text{dom } f$, then $\max_-(f)$ is measurable on A .
- (48) If f is measurable on A and $A \subseteq \text{dom } f$, then $|f|$ is measurable on A .

5. THE DEFINITION AND THE MEASURABILITY OF A REAL-VALUED SIMPLE FUNCTION

For simplicity, we adopt the following rules: X is a non empty set, Y is a set, S is a σ -field of subsets of X , f, g, h are partial functions from X to \mathbb{R} , A is an element of S , and r is a real number.

Let us consider X, S, f . We say that f is simple function in S if and only if the condition (Def. 7) is satisfied.

- (Def. 7) There exists a finite sequence F of separated subsets of S such that
- (i) $\text{dom } f = \bigcup \text{rng } F$, and
 - (ii) for every natural number n and for all elements x, y of X such that $n \in \text{dom } F$ and $x \in F(n)$ and $y \in F(n)$ holds $f(x) = f(y)$.

Next we state a number of propositions:

- (49) f is simple function in S iff $\overline{\mathbb{R}}(f)$ is simple function in S .
- (50) If f is simple function in S , then f is measurable on A .
- (51) Let X be a set and f be a partial function from X to \mathbb{R} . Then f is non-negative if and only if for every set x holds $0 \leq f(x)$.
- (52) Let X be a set and f be a partial function from X to \mathbb{R} . If for every set x such that $x \in \text{dom } f$ holds $0 \leq f(x)$, then f is non-negative.
- (53) Let X be a set and f be a partial function from X to \mathbb{R} . Then f is non-positive if and only if for every set x holds $f(x) \leq 0$.
- (54) If for every set x such that $x \in \text{dom } f$ holds $f(x) \leq 0$, then f is non-positive.
- (55) If f is non-negative, then $f|_Y$ is non-negative.
- (56) If f is non-negative and g is non-negative, then $f + g$ is non-negative.
- (57) If f is non-negative, then if $0 \leq r$, then $r f$ is non-negative and if $r \leq 0$, then $r f$ is non-positive.

- (58) If for every set x such that $x \in \text{dom } f \cap \text{dom } g$ holds $g(x) \leq f(x)$, then $f - g$ is non-negative.
- (59) If f is non-negative and g is non-negative and h is non-negative, then $f + g + h$ is non-negative.
- (60) For every set x such that $x \in \text{dom}(f + g + h)$ holds $(f + g + h)(x) = f(x) + g(x) + h(x)$.
- (61) $\max_+(f)$ is non-negative and $\max_-(f)$ is non-negative.
- (62)(i) $\text{dom}(\max_+(f + g) + \max_-(f)) = \text{dom } f \cap \text{dom } g$,
(ii) $\text{dom}(\max_-(f + g) + \max_+(f)) = \text{dom } f \cap \text{dom } g$,
(iii) $\text{dom}(\max_+(f + g) + \max_-(f) + \max_-(g)) = \text{dom } f \cap \text{dom } g$,
(iv) $\text{dom}(\max_-(f + g) + \max_+(f) + \max_+(g)) = \text{dom } f \cap \text{dom } g$,
(v) $\max_+(f + g) + \max_-(f)$ is non-negative, and
(vi) $\max_-(f + g) + \max_+(f)$ is non-negative.
- (63) $\max_+(f + g) + \max_-(f) + \max_-(g) = \max_-(f + g) + \max_+(f) + \max_+(g)$.
- (64) If $0 \leq r$, then $\max_+(rf) = r \max_+(f)$ and $\max_-(rf) = r \max_-(f)$.
- (65) If $0 \leq r$, then $\max_+((-r)f) = r \max_-(f)$ and $\max_-((-r)f) = r \max_+(f)$.
- (66) $\max_+(f|Y) = \max_+(f)|Y$ and $\max_-(f|Y) = \max_-(f)|Y$.
- (67) If $Y \subseteq \text{dom}(f + g)$, then $\text{dom}((f + g)|Y) = Y$ and $\text{dom}(f|Y + g|Y) = Y$ and $(f + g)|Y = f|Y + g|Y$.
- (68) $\text{EQ-dom}(f, r) = f^{-1}(\{r\})$.

6. LEMMAS FOR A REAL-VALUED MEASURABLE FUNCTION AND A SIMPLE FUNCTION

For simplicity, we use the following convention: X is a non empty set, S is a σ -field of subsets of X , f, g are partial functions from X to \mathbb{R} , A, B are elements of S , and r, s are real numbers.

We now state a number of propositions:

- (69) If f is measurable on A and $A \subseteq \text{dom } f$, then $A \cap \text{GTE-dom}(f, r) \cap \text{LE-dom}(f, s)$ is measurable on S .
- (70) If f is simple function in S , then $f|A$ is simple function in S .
- (71) If f is simple function in S , then $\text{dom } f$ is an element of S .
- (72) If f is simple function in S and g is simple function in S , then $f + g$ is simple function in S .
- (73) If f is simple function in S , then rf is simple function in S .
- (74) If for every set x such that $x \in \text{dom}(f - g)$ holds $g(x) \leq f(x)$, then $f - g$ is non-negative.

- (75) There exists a partial function f from X to \mathbb{R} such that f is simple function in S and $\text{dom } f = A$ and for every set x such that $x \in A$ holds $f(x) = r$.
- (76) If f is measurable on B and $A = \text{dom } f \cap B$, then $f|_B$ is measurable on A .
- (77) If $A \subseteq \text{dom } f$ and f is measurable on A and g is measurable on A , then $\max_+(f + g) + \max_-(f)$ is measurable on A .
- (78) If $A \subseteq \text{dom } f \cap \text{dom } g$ and f is measurable on A and g is measurable on A , then $\max_-(f + g) + \max_+(f)$ is measurable on A .
- (79) If $\text{dom } f \in S$ and $\text{dom } g \in S$, then $\text{dom}(f + g) \in S$.
- (80) If $\text{dom } f = A$, then f is measurable on B iff f is measurable on $A \cap B$.
- (81) Given an element A of S such that $\text{dom } f = A$. Let c be a real number and B be an element of S . If f is measurable on B , then cf is measurable on B .

7. THE INTEGRAL OF A REAL-VALUED FUNCTION

For simplicity, we follow the rules: X is a non empty set, S is a σ -field of subsets of X , M is a σ -measure on S , f, g are partial functions from X to \mathbb{R} , r is a real number, and E, A, B are elements of S .

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , and let f be a partial function from X to \mathbb{R} . The functor $\int f \, dM$ yields an element of $\overline{\mathbb{R}}$ and is defined by:

(Def. 8) $\int f \, dM = \int \overline{\mathbb{R}}(f) \, dM.$

The following propositions are true:

- (82) If there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A and f is non-negative, then $\int f \, dM = \int^+ \overline{\mathbb{R}}(f) \, dM.$
- (83) If f is simple function in S and f is non-negative, then $\int f \, dM = \int^+ \overline{\mathbb{R}}(f) \, dM$ and $\int f \, dM = \int' \overline{\mathbb{R}}(f) \, dM.$
- (84) If there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A and f is non-negative, then $0 \leq \int f \, dM.$
- (85) Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and f is non-negative and A misses B . Then $\int f|_{(A \cup B)} \, dM = \int f|_A \, dM + \int f|_B \, dM.$
- (86) If there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and f is non-negative, then $0 \leq \int f|_A \, dM.$
- (87) Suppose there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and f is non-negative and $A \subseteq B$. Then $\int f|_A \, dM \leq \int f|_B \, dM.$

- (88) If there exists an element E of S such that $E = \text{dom } f$ and f is measurable on E and $M(A) = 0$, then $\int f \upharpoonright A \, dM = 0$.
- (89) If $E = \text{dom } f$ and f is measurable on E and $M(A) = 0$, then $\int f \upharpoonright (E \setminus A) \, dM = \int f \, dM$.

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , and let f be a partial function from X to \mathbb{R} . We say that f is integrable on M if and only if:

(Def. 9) $\overline{\mathbb{R}}(f)$ is integrable on M .

We now state a number of propositions:

- (90) If f is integrable on M , then $-\infty < \int f \, dM$ and $\int f \, dM < +\infty$.
- (91) If f is integrable on M , then $f \upharpoonright A$ is integrable on M .
- (92) If f is integrable on M and A misses B , then $\int f \upharpoonright (A \cup B) \, dM = \int f \upharpoonright A \, dM + \int f \upharpoonright B \, dM$.
- (93) If f is integrable on M and $B = \text{dom } f \setminus A$, then $f \upharpoonright A$ is integrable on M and $\int f \, dM = \int f \upharpoonright A \, dM + \int f \upharpoonright B \, dM$.
- (94) Given an element A of S such that $A = \text{dom } f$ and f is measurable on A . Then f is integrable on M if and only if $|f|$ is integrable on M .
- (95) If f is integrable on M , then $|\int f \, dM| \leq \int |f| \, dM$.
- (96) Suppose that
- (i) there exists an element A of S such that $A = \text{dom } f$ and f is measurable on A ,
 - (ii) $\text{dom } f = \text{dom } g$,
 - (iii) g is integrable on M , and
 - (iv) for every element x of X such that $x \in \text{dom } f$ holds $|f(x)| \leq g(x)$.
- Then f is integrable on M and $\int |f| \, dM \leq \int g \, dM$.
- (97) If $\text{dom } f \in S$ and $0 \leq r$ and for every set x such that $x \in \text{dom } f$ holds $f(x) = r$, then $\int f \, dM = \overline{\mathbb{R}}(r) \cdot M(\text{dom } f)$.
- (98) Suppose f is integrable on M and g is integrable on M and f is non-negative and g is non-negative. Then $f + g$ is integrable on M .
- (99) If f is integrable on M and g is integrable on M , then $\text{dom}(f + g) \in S$.
- (100) If f is integrable on M and g is integrable on M , then $f + g$ is integrable on M .
- (101) Suppose f is integrable on M and g is integrable on M . Then there exists an element E of S such that $E = \text{dom } f \cap \text{dom } g$ and $\int f + g \, dM = \int f \upharpoonright E \, dM + \int g \upharpoonright E \, dM$.
- (102) If f is integrable on M , then rf is integrable on M and $\int rf \, dM = \overline{\mathbb{R}}(r) \cdot \int f \, dM$.

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , let f be a partial function from X to \mathbb{R} , and let B be an

element of S . The functor $\int_B f \, dM$ yielding an element of $\overline{\mathbb{R}}$ is defined by:

$$\text{(Def. 10)} \quad \int_B f \, dM = \int f \upharpoonright B \, dM.$$

Next we state two propositions:

- (103) Suppose f is integrable on M and g is integrable on M and $B \subseteq \text{dom}(f + g)$. Then $f + g$ is integrable on M and $\int_B f + g \, dM = \int_B f \, dM + \int_B g \, dM$.
- (104) If f is integrable on M and f is measurable on B , then $f \upharpoonright B$ is integrable on M and $\int_B r f \, dM = \overline{\mathbb{R}}(r) \cdot \int_B f \, dM$.

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