

# Integral of Real-Valued Measurable Function<sup>1</sup>

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**Summary.** Based on [16], authors formalized the integral of an extended real valued measurable function in [12] before. However, the integral argued in [12] cannot be applied to real-valued functions unconditionally. Therefore, in this article we have formalized the integral of a real-value function.

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The papers [25], [11], [26], [1], [23], [24], [17], [18], [8], [27], [10], [2], [19], [7], [20], [6], [9], [3], [4], [5], [13], [14], [15], [22], [21], and [12] provide the terminology and notation for this paper.

## 1. THE MEASURABILITY OF REAL-VALUED FUNCTIONS

For simplicity, we follow the rules:  $X$  denotes a non empty set,  $Y$  denotes a set,  $S$  denotes a  $\sigma$ -field of subsets of  $X$ ,  $F$  denotes a function from  $\mathbb{N}$  into  $S$ ,  $f$ ,  $g$  denote partial functions from  $X$  to  $\mathbb{R}$ ,  $A$ ,  $B$  denote elements of  $S$ ,  $r$ ,  $s$  denote real numbers,  $a$  denotes a real number, and  $n$  denotes a natural number.

Let  $X$  be a non empty set, let  $f$  be a partial function from  $X$  to  $\mathbb{R}$ , and let  $a$  be a real number. The functor  $\text{LE-dom}(f, a)$  yields a subset of  $X$  and is defined as follows:

(Def. 1)  $\text{LE-dom}(f, a) = \text{LE-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a))$ .

The following three propositions are true:

(1)  $|\overline{\mathbb{R}}(f)| = \overline{\mathbb{R}}(|f|)$ .

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- (2) Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S$ ,  $f$  be a partial function from  $X$  to  $\overline{\mathbb{R}}$ , and  $r$  be a real number. Suppose  $\text{dom } f \in S$  and for every set  $x$  such that  $x \in \text{dom } f$  holds  $f(x) = r$ . Then  $f$  is simple function in  $S$ .
- (3) For every set  $x$  holds  $x \in \text{LEQ-dom}(f, a)$  iff  $x \in \text{dom } f$  and there exists a real number  $y$  such that  $y = f(x)$  and  $y < a$ .

Let us consider  $X, f, a$ . The functor  $\text{LEQ-dom}(f, a)$  yields a subset of  $X$  and is defined as follows:

$$\text{(Def. 2)} \quad \text{LEQ-dom}(f, a) = \text{LEQ-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a)).$$

We now state the proposition

- (4) For every set  $x$  holds  $x \in \text{LEQ-dom}(f, a)$  iff  $x \in \text{dom } f$  and there exists a real number  $y$  such that  $y = f(x)$  and  $y \leq a$ .

Let us consider  $X, f, a$ . The functor  $\text{GT-dom}(f, a)$  yielding a subset of  $X$  is defined as follows:

$$\text{(Def. 3)} \quad \text{GT-dom}(f, a) = \text{GT-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a)).$$

We now state the proposition

- (5) For every set  $x$  holds  $x \in \text{GT-dom}(f, r)$  iff  $x \in \text{dom } f$  and there exists a real number  $y$  such that  $y = f(x)$  and  $r < y$ .

Let us consider  $X, f, a$ . The functor  $\text{GTE-dom}(f, a)$  yields a subset of  $X$  and is defined as follows:

$$\text{(Def. 4)} \quad \text{GTE-dom}(f, a) = \text{GTE-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a)).$$

Next we state the proposition

- (6) For every set  $x$  holds  $x \in \text{GTE-dom}(f, r)$  iff  $x \in \text{dom } f$  and there exists a real number  $y$  such that  $y = f(x)$  and  $r \leq y$ .

Let us consider  $X, f, a$ . The functor  $\text{EQ-dom}(f, a)$  yielding a subset of  $X$  is defined by:

$$\text{(Def. 5)} \quad \text{EQ-dom}(f, a) = \text{EQ-dom}(\overline{\mathbb{R}}(f), \overline{\mathbb{R}}(a)).$$

The following propositions are true:

- (7) For every set  $x$  holds  $x \in \text{EQ-dom}(f, r)$  iff  $x \in \text{dom } f$  and there exists a real number  $y$  such that  $y = f(x)$  and  $r = y$ .
- (8) If for every  $n$  holds  $F(n) = Y \cap \text{GT-dom}(f, r - \frac{1}{n+1})$ , then  $Y \cap \text{GTE-dom}(f, r) = \bigcap \text{rng } F$ .
- (9) If for every  $n$  holds  $F(n) = Y \cap \text{LEQ-dom}(f, r + \frac{1}{n+1})$ , then  $Y \cap \text{LEQ-dom}(f, r) = \bigcap \text{rng } F$ .
- (10) If for every  $n$  holds  $F(n) = Y \cap \text{LEQ-dom}(f, r - \frac{1}{n+1})$ , then  $Y \cap \text{LE-dom}(f, r) = \bigcup \text{rng } F$ .
- (11) If for every  $n$  holds  $F(n) = Y \cap \text{GTE-dom}(f, r + \frac{1}{n+1})$ , then  $Y \cap \text{GT-dom}(f, r) = \bigcup \text{rng } F$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $f$  be a partial function from  $X$  to  $\mathbb{R}$ , and let  $A$  be an element of  $S$ . We say that  $f$  is measurable on  $A$  if and only if:

(Def. 6)  $\overline{\mathbb{R}}(f)$  is measurable on  $A$ .

The following propositions are true:

- (12)  $f$  is measurable on  $A$  iff for every real number  $r$  holds  $A \cap \text{LE-dom}(f, r)$  is measurable on  $S$ .
- (13) Suppose  $A \subseteq \text{dom } f$ . Then  $f$  is measurable on  $A$  if and only if for every real number  $r$  holds  $A \cap \text{GTE-dom}(f, r)$  is measurable on  $S$ .
- (14)  $f$  is measurable on  $A$  iff for every real number  $r$  holds  $A \cap \text{LEQ-dom}(f, r)$  is measurable on  $S$ .
- (15) Suppose  $A \subseteq \text{dom } f$ . Then  $f$  is measurable on  $A$  if and only if for every real number  $r$  holds  $A \cap \text{GT-dom}(f, r)$  is measurable on  $S$ .
- (16) If  $B \subseteq A$  and  $f$  is measurable on  $A$ , then  $f$  is measurable on  $B$ .
- (17) If  $f$  is measurable on  $A$  and  $f$  is measurable on  $B$ , then  $f$  is measurable on  $A \cup B$ .
- (18) If  $f$  is measurable on  $A$  and  $A \subseteq \text{dom } f$ , then  $A \cap \text{GT-dom}(f, r) \cap \text{LE-dom}(f, s)$  is measurable on  $S$ .
- (19) If  $f$  is measurable on  $A$  and  $g$  is measurable on  $A$  and  $A \subseteq \text{dom } g$ , then  $A \cap \text{LE-dom}(f, r) \cap \text{GT-dom}(g, r)$  is measurable on  $S$ .
- (20)  $\overline{\mathbb{R}}(rf) = r \overline{\mathbb{R}}(f)$ .
- (21) If  $f$  is measurable on  $A$  and  $A \subseteq \text{dom } f$ , then  $rf$  is measurable on  $A$ .

## 2. THE MEASURABILITY OF $f + g$ AND $f - g$ FOR REAL-VALUED FUNCTIONS $f, g$

For simplicity, we adopt the following rules:  $X$  denotes a non empty set,  $S$  denotes a  $\sigma$ -field of subsets of  $X$ ,  $f, g$  denote partial functions from  $X$  to  $\mathbb{R}$ ,  $A$  denotes an element of  $S$ ,  $r$  denotes a real number, and  $p$  denotes a rational number.

Next we state several propositions:

- (22)  $\overline{\mathbb{R}}(f)$  is finite.
- (23)  $\overline{\mathbb{R}}(f + g) = \overline{\mathbb{R}}(f) + \overline{\mathbb{R}}(g)$  and  $\overline{\mathbb{R}}(f - g) = \overline{\mathbb{R}}(f) - \overline{\mathbb{R}}(g)$  and  $\text{dom } \overline{\mathbb{R}}(f + g) = \text{dom } \overline{\mathbb{R}}(f) \cap \text{dom } \overline{\mathbb{R}}(g)$  and  $\text{dom } \overline{\mathbb{R}}(f - g) = \text{dom } \overline{\mathbb{R}}(f) \cap \text{dom } \overline{\mathbb{R}}(g)$  and  $\text{dom } \overline{\mathbb{R}}(f + g) = \text{dom } f \cap \text{dom } g$  and  $\text{dom } \overline{\mathbb{R}}(f - g) = \text{dom } f \cap \text{dom } g$ .
- (24) For every function  $F$  from  $\mathbb{Q}$  into  $S$  such that for every  $p$  holds  $F(p) = A \cap \text{LE-dom}(f, p) \cap (A \cap \text{LE-dom}(g, r - p))$  holds  $A \cap \text{LE-dom}(f + g, r) = \bigcup \text{rng } F$ .

- (25) Suppose  $f$  is measurable on  $A$  and  $g$  is measurable on  $A$ . Then there exists a function  $F$  from  $\mathbb{Q}$  into  $S$  such that for every rational number  $p$  holds  $F(p) = A \cap \text{LE-dom}(f, p) \cap (A \cap \text{LE-dom}(g, r - p))$ .
- (26) If  $f$  is measurable on  $A$  and  $g$  is measurable on  $A$ , then  $f+g$  is measurable on  $A$ .
- (27)  $\overline{\mathbb{R}}(f) - \overline{\mathbb{R}}(g) = \overline{\mathbb{R}}(f) + \overline{\mathbb{R}}(-g)$ .
- (28)  $-\overline{\mathbb{R}}(f) = \overline{\mathbb{R}}((-1)f)$  and  $-\overline{\mathbb{R}}(f) = \overline{\mathbb{R}}(-f)$ .
- (29) If  $f$  is measurable on  $A$  and  $g$  is measurable on  $A$  and  $A \subseteq \text{dom } g$ , then  $f - g$  is measurable on  $A$ .

### 3. BASIC PROPERTIES OF REAL-VALUED FUNCTIONS, $\max_+ f$ AND $\max_- f$

In the sequel  $X$  denotes a non empty set,  $f$  denotes a partial function from  $X$  to  $\mathbb{R}$ , and  $r$  denotes a real number.

Next we state a number of propositions:

- (30)  $\max_+(\overline{\mathbb{R}}(f)) = \max_+(f)$  and  $\max_-(\overline{\mathbb{R}}(f)) = \max_-(f)$ .
- (31) For every element  $x$  of  $X$  holds  $0 \leq (\max_+(f))(x)$ .
- (32) For every element  $x$  of  $X$  holds  $0 \leq (\max_-(f))(x)$ .
- (33)  $\max_-(f) = \max_+(-f)$ .
- (34) For every set  $x$  such that  $x \in \text{dom } f$  and  $0 < (\max_+(f))(x)$  holds  $(\max_-(f))(x) = 0$ .
- (35) For every set  $x$  such that  $x \in \text{dom } f$  and  $0 < (\max_-(f))(x)$  holds  $(\max_+(f))(x) = 0$ .
- (36)  $\text{dom } f = \text{dom}(\max_+(f) - \max_-(f))$  and  $\text{dom } f = \text{dom}(\max_+(f) + \max_-(f))$ .
- (37) For every set  $x$  such that  $x \in \text{dom } f$  holds  $(\max_+(f))(x) = f(x)$  or  $(\max_+(f))(x) = 0$  but  $(\max_-(f))(x) = -f(x)$  or  $(\max_-(f))(x) = 0$ .
- (38) For every set  $x$  such that  $x \in \text{dom } f$  and  $(\max_+(f))(x) = f(x)$  holds  $(\max_-(f))(x) = 0$ .
- (39) For every set  $x$  such that  $x \in \text{dom } f$  and  $(\max_+(f))(x) = 0$  holds  $(\max_-(f))(x) = -f(x)$ .
- (40) For every set  $x$  such that  $x \in \text{dom } f$  and  $(\max_-(f))(x) = -f(x)$  holds  $(\max_+(f))(x) = 0$ .
- (41) For every set  $x$  such that  $x \in \text{dom } f$  and  $(\max_-(f))(x) = 0$  holds  $(\max_+(f))(x) = f(x)$ .
- (42)  $f = \max_+(f) - \max_-(f)$ .
- (43)  $|r| = |\overline{\mathbb{R}}(r)|$ .
- (44)  $\overline{\mathbb{R}}(|f|) = |\overline{\mathbb{R}}(f)|$ .

$$(45) \quad |f| = \max_+(f) + \max_-(f).$$

#### 4. THE MEASURABILITY OF $\max_+ f, \max_- f$ AND $|f|$

In the sequel  $X$  denotes a non empty set,  $S$  denotes a  $\sigma$ -field of subsets of  $X$ ,  $f$  denotes a partial function from  $X$  to  $\mathbb{R}$ , and  $A$  denotes an element of  $S$ .

The following propositions are true:

- (46) If  $f$  is measurable on  $A$ , then  $\max_+(f)$  is measurable on  $A$ .
- (47) If  $f$  is measurable on  $A$  and  $A \subseteq \text{dom } f$ , then  $\max_-(f)$  is measurable on  $A$ .
- (48) If  $f$  is measurable on  $A$  and  $A \subseteq \text{dom } f$ , then  $|f|$  is measurable on  $A$ .

#### 5. THE DEFINITION AND THE MEASURABILITY OF A REAL-VALUED SIMPLE FUNCTION

For simplicity, we adopt the following rules:  $X$  is a non empty set,  $Y$  is a set,  $S$  is a  $\sigma$ -field of subsets of  $X$ ,  $f, g, h$  are partial functions from  $X$  to  $\mathbb{R}$ ,  $A$  is an element of  $S$ , and  $r$  is a real number.

Let us consider  $X, S, f$ . We say that  $f$  is simple function in  $S$  if and only if the condition (Def. 7) is satisfied.

- (Def. 7) There exists a finite sequence  $F$  of separated subsets of  $S$  such that
- (i)  $\text{dom } f = \bigcup \text{rng } F$ , and
  - (ii) for every natural number  $n$  and for all elements  $x, y$  of  $X$  such that  $n \in \text{dom } F$  and  $x \in F(n)$  and  $y \in F(n)$  holds  $f(x) = f(y)$ .

Next we state a number of propositions:

- (49)  $f$  is simple function in  $S$  iff  $\overline{\mathbb{R}}(f)$  is simple function in  $S$ .
- (50) If  $f$  is simple function in  $S$ , then  $f$  is measurable on  $A$ .
- (51) Let  $X$  be a set and  $f$  be a partial function from  $X$  to  $\mathbb{R}$ . Then  $f$  is non-negative if and only if for every set  $x$  holds  $0 \leq f(x)$ .
- (52) Let  $X$  be a set and  $f$  be a partial function from  $X$  to  $\mathbb{R}$ . If for every set  $x$  such that  $x \in \text{dom } f$  holds  $0 \leq f(x)$ , then  $f$  is non-negative.
- (53) Let  $X$  be a set and  $f$  be a partial function from  $X$  to  $\mathbb{R}$ . Then  $f$  is non-positive if and only if for every set  $x$  holds  $f(x) \leq 0$ .
- (54) If for every set  $x$  such that  $x \in \text{dom } f$  holds  $f(x) \leq 0$ , then  $f$  is non-positive.
- (55) If  $f$  is non-negative, then  $f|_Y$  is non-negative.
- (56) If  $f$  is non-negative and  $g$  is non-negative, then  $f + g$  is non-negative.
- (57) If  $f$  is non-negative, then if  $0 \leq r$ , then  $r f$  is non-negative and if  $r \leq 0$ , then  $r f$  is non-positive.

- (58) If for every set  $x$  such that  $x \in \text{dom } f \cap \text{dom } g$  holds  $g(x) \leq f(x)$ , then  $f - g$  is non-negative.
- (59) If  $f$  is non-negative and  $g$  is non-negative and  $h$  is non-negative, then  $f + g + h$  is non-negative.
- (60) For every set  $x$  such that  $x \in \text{dom}(f + g + h)$  holds  $(f + g + h)(x) = f(x) + g(x) + h(x)$ .
- (61)  $\max_+(f)$  is non-negative and  $\max_-(f)$  is non-negative.
- (62)(i)  $\text{dom}(\max_+(f + g) + \max_-(f)) = \text{dom } f \cap \text{dom } g$ ,  
(ii)  $\text{dom}(\max_-(f + g) + \max_+(f)) = \text{dom } f \cap \text{dom } g$ ,  
(iii)  $\text{dom}(\max_+(f + g) + \max_-(f) + \max_-(g)) = \text{dom } f \cap \text{dom } g$ ,  
(iv)  $\text{dom}(\max_-(f + g) + \max_+(f) + \max_+(g)) = \text{dom } f \cap \text{dom } g$ ,  
(v)  $\max_+(f + g) + \max_-(f)$  is non-negative, and  
(vi)  $\max_-(f + g) + \max_+(f)$  is non-negative.
- (63)  $\max_+(f + g) + \max_-(f) + \max_-(g) = \max_-(f + g) + \max_+(f) + \max_+(g)$ .
- (64) If  $0 \leq r$ , then  $\max_+(rf) = r \max_+(f)$  and  $\max_-(rf) = r \max_-(f)$ .
- (65) If  $0 \leq r$ , then  $\max_+((-r)f) = r \max_-(f)$  and  $\max_-((-r)f) = r \max_+(f)$ .
- (66)  $\max_+(f|Y) = \max_+(f)|Y$  and  $\max_-(f|Y) = \max_-(f)|Y$ .
- (67) If  $Y \subseteq \text{dom}(f + g)$ , then  $\text{dom}((f + g)|Y) = Y$  and  $\text{dom}(f|Y + g|Y) = Y$  and  $(f + g)|Y = f|Y + g|Y$ .
- (68)  $\text{EQ-dom}(f, r) = f^{-1}(\{r\})$ .

## 6. LEMMAS FOR A REAL-VALUED MEASURABLE FUNCTION AND A SIMPLE FUNCTION

For simplicity, we use the following convention:  $X$  is a non empty set,  $S$  is a  $\sigma$ -field of subsets of  $X$ ,  $f, g$  are partial functions from  $X$  to  $\mathbb{R}$ ,  $A, B$  are elements of  $S$ , and  $r, s$  are real numbers.

We now state a number of propositions:

- (69) If  $f$  is measurable on  $A$  and  $A \subseteq \text{dom } f$ , then  $A \cap \text{GTE-dom}(f, r) \cap \text{LE-dom}(f, s)$  is measurable on  $S$ .
- (70) If  $f$  is simple function in  $S$ , then  $f|A$  is simple function in  $S$ .
- (71) If  $f$  is simple function in  $S$ , then  $\text{dom } f$  is an element of  $S$ .
- (72) If  $f$  is simple function in  $S$  and  $g$  is simple function in  $S$ , then  $f + g$  is simple function in  $S$ .
- (73) If  $f$  is simple function in  $S$ , then  $rf$  is simple function in  $S$ .
- (74) If for every set  $x$  such that  $x \in \text{dom}(f - g)$  holds  $g(x) \leq f(x)$ , then  $f - g$  is non-negative.

- (75) There exists a partial function  $f$  from  $X$  to  $\mathbb{R}$  such that  $f$  is simple function in  $S$  and  $\text{dom } f = A$  and for every set  $x$  such that  $x \in A$  holds  $f(x) = r$ .
- (76) If  $f$  is measurable on  $B$  and  $A = \text{dom } f \cap B$ , then  $f|_B$  is measurable on  $A$ .
- (77) If  $A \subseteq \text{dom } f$  and  $f$  is measurable on  $A$  and  $g$  is measurable on  $A$ , then  $\max_+(f + g) + \max_-(f)$  is measurable on  $A$ .
- (78) If  $A \subseteq \text{dom } f \cap \text{dom } g$  and  $f$  is measurable on  $A$  and  $g$  is measurable on  $A$ , then  $\max_-(f + g) + \max_+(f)$  is measurable on  $A$ .
- (79) If  $\text{dom } f \in S$  and  $\text{dom } g \in S$ , then  $\text{dom}(f + g) \in S$ .
- (80) If  $\text{dom } f = A$ , then  $f$  is measurable on  $B$  iff  $f$  is measurable on  $A \cap B$ .
- (81) Given an element  $A$  of  $S$  such that  $\text{dom } f = A$ . Let  $c$  be a real number and  $B$  be an element of  $S$ . If  $f$  is measurable on  $B$ , then  $cf$  is measurable on  $B$ .

7. THE INTEGRAL OF A REAL-VALUED FUNCTION

For simplicity, we follow the rules:  $X$  is a non empty set,  $S$  is a  $\sigma$ -field of subsets of  $X$ ,  $M$  is a  $\sigma$ -measure on  $S$ ,  $f, g$  are partial functions from  $X$  to  $\mathbb{R}$ ,  $r$  is a real number, and  $E, A, B$  are elements of  $S$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $f$  be a partial function from  $X$  to  $\mathbb{R}$ . The functor  $\int f \, dM$  yields an element of  $\overline{\mathbb{R}}$  and is defined by:

(Def. 8)  $\int f \, dM = \int \overline{\mathbb{R}}(f) \, dM.$

The following propositions are true:

- (82) If there exists an element  $A$  of  $S$  such that  $A = \text{dom } f$  and  $f$  is measurable on  $A$  and  $f$  is non-negative, then  $\int f \, dM = \int^+ \overline{\mathbb{R}}(f) \, dM.$
- (83) If  $f$  is simple function in  $S$  and  $f$  is non-negative, then  $\int f \, dM = \int^+ \overline{\mathbb{R}}(f) \, dM$  and  $\int f \, dM = \int' \overline{\mathbb{R}}(f) \, dM.$
- (84) If there exists an element  $A$  of  $S$  such that  $A = \text{dom } f$  and  $f$  is measurable on  $A$  and  $f$  is non-negative, then  $0 \leq \int f \, dM.$
- (85) Suppose there exists an element  $E$  of  $S$  such that  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and  $f$  is non-negative and  $A$  misses  $B$ . Then  $\int f|_{(A \cup B)} \, dM = \int f|_A \, dM + \int f|_B \, dM.$
- (86) If there exists an element  $E$  of  $S$  such that  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and  $f$  is non-negative, then  $0 \leq \int f|_A \, dM.$
- (87) Suppose there exists an element  $E$  of  $S$  such that  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and  $f$  is non-negative and  $A \subseteq B$ . Then  $\int f|_A \, dM \leq \int f|_B \, dM.$

- (88) If there exists an element  $E$  of  $S$  such that  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and  $M(A) = 0$ , then  $\int f \upharpoonright A \, dM = 0$ .
- (89) If  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and  $M(A) = 0$ , then  $\int f \upharpoonright (E \setminus A) \, dM = \int f \, dM$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $f$  be a partial function from  $X$  to  $\mathbb{R}$ . We say that  $f$  is integrable on  $M$  if and only if:

(Def. 9)  $\overline{\mathbb{R}}(f)$  is integrable on  $M$ .

We now state a number of propositions:

- (90) If  $f$  is integrable on  $M$ , then  $-\infty < \int f \, dM$  and  $\int f \, dM < +\infty$ .
- (91) If  $f$  is integrable on  $M$ , then  $f \upharpoonright A$  is integrable on  $M$ .
- (92) If  $f$  is integrable on  $M$  and  $A$  misses  $B$ , then  $\int f \upharpoonright (A \cup B) \, dM = \int f \upharpoonright A \, dM + \int f \upharpoonright B \, dM$ .
- (93) If  $f$  is integrable on  $M$  and  $B = \text{dom } f \setminus A$ , then  $f \upharpoonright A$  is integrable on  $M$  and  $\int f \, dM = \int f \upharpoonright A \, dM + \int f \upharpoonright B \, dM$ .
- (94) Given an element  $A$  of  $S$  such that  $A = \text{dom } f$  and  $f$  is measurable on  $A$ . Then  $f$  is integrable on  $M$  if and only if  $|f|$  is integrable on  $M$ .
- (95) If  $f$  is integrable on  $M$ , then  $|\int f \, dM| \leq \int |f| \, dM$ .
- (96) Suppose that
- (i) there exists an element  $A$  of  $S$  such that  $A = \text{dom } f$  and  $f$  is measurable on  $A$ ,
  - (ii)  $\text{dom } f = \text{dom } g$ ,
  - (iii)  $g$  is integrable on  $M$ , and
  - (iv) for every element  $x$  of  $X$  such that  $x \in \text{dom } f$  holds  $|f(x)| \leq g(x)$ .
- Then  $f$  is integrable on  $M$  and  $\int |f| \, dM \leq \int g \, dM$ .
- (97) If  $\text{dom } f \in S$  and  $0 \leq r$  and for every set  $x$  such that  $x \in \text{dom } f$  holds  $f(x) = r$ , then  $\int f \, dM = \overline{\mathbb{R}}(r) \cdot M(\text{dom } f)$ .
- (98) Suppose  $f$  is integrable on  $M$  and  $g$  is integrable on  $M$  and  $f$  is non-negative and  $g$  is non-negative. Then  $f + g$  is integrable on  $M$ .
- (99) If  $f$  is integrable on  $M$  and  $g$  is integrable on  $M$ , then  $\text{dom}(f + g) \in S$ .
- (100) If  $f$  is integrable on  $M$  and  $g$  is integrable on  $M$ , then  $f + g$  is integrable on  $M$ .
- (101) Suppose  $f$  is integrable on  $M$  and  $g$  is integrable on  $M$ . Then there exists an element  $E$  of  $S$  such that  $E = \text{dom } f \cap \text{dom } g$  and  $\int f + g \, dM = \int f \upharpoonright E \, dM + \int g \upharpoonright E \, dM$ .
- (102) If  $f$  is integrable on  $M$ , then  $rf$  is integrable on  $M$  and  $\int rf \, dM = \overline{\mathbb{R}}(r) \cdot \int f \, dM$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , let  $f$  be a partial function from  $X$  to  $\mathbb{R}$ , and let  $B$  be an



element of  $S$ . The functor  $\int_B f \, dM$  yielding an element of  $\overline{\mathbb{R}}$  is defined by:

$$\text{(Def. 10)} \quad \int_B f \, dM = \int f \upharpoonright B \, dM.$$

Next we state two propositions:

- (103) Suppose  $f$  is integrable on  $M$  and  $g$  is integrable on  $M$  and  $B \subseteq \text{dom}(f + g)$ . Then  $f + g$  is integrable on  $M$  and  $\int_B f + g \, dM = \int_B f \, dM + \int_B g \, dM$ .
- (104) If  $f$  is integrable on  $M$  and  $f$  is measurable on  $B$ , then  $f \upharpoonright B$  is integrable on  $M$  and  $\int_B r f \, dM = \overline{\mathbb{R}}(r) \cdot \int_B f \, dM$ .

#### REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [3] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(1):163–171, 1991.
- [4] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(1):173–183, 1991.
- [5] Józef Białas. The  $\sigma$ -additive measure theory. *Formalized Mathematics*, 2(2):263–270, 1991.
- [6] Józef Białas. Some properties of the intervals. *Formalized Mathematics*, 5(1):21–26, 1996.
- [7] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [10] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [11] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [12] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(2):53–70, 2006.
- [13] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Formalized Mathematics*, 9(3):491–494, 2001.
- [14] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(3):495–500, 2001.
- [15] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. The measurability of extended real valued functions. *Formalized Mathematics*, 9(3):525–529, 2001.
- [16] P. R. Halmos. *Measure Theory*. Springer-Verlag, 1987.
- [17] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [18] Andrzej Kondracki. Basic properties of rational numbers. *Formalized Mathematics*, 1(5):841–845, 1990.
- [19] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [20] Jarosław Kotowicz and Yuji Sakai. Properties of partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 3(2):279–288, 1992.
- [21] Andrzej Nędzusiak.  $\sigma$ -fields and probability. *Formalized Mathematics*, 1(2):401–407, 1990.
- [22] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [23] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
- [24] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(1):115–122, 1990.
- [25] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [26] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.

- [27] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

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