

The Catalan Numbers. Part II¹

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Summary. In this paper, we define sequence dominated by 0, in which every initial fragment contains more zeroes than ones. If $n \geq 2 \cdot m$ and $n > 0$, then the number of sequences dominated by 0 the length n including m of ones, is given by the formula

$$D(n, m) = \frac{n + 1 - 2 \cdot m}{n + 1 - m} \cdot \binom{n}{m}$$

and satisfies the recurrence relation

$$D(n, m) = D(n - 1, m) + \sum_{i=0}^{m-1} D(2 \cdot i, i) \cdot D(n - 2 \cdot (i + 1), m - (i + 1)).$$

Obviously, if $n = 2 \cdot m$, then we obtain the recurrence relation for the Catalan numbers (starting from 0)

$$C_{m+1} = \sum_{i=0}^{m-1} C_{i+1} \cdot C_{m-i}.$$

Using the above recurrence relation we can see that

$$\sum_{i=0}^{\infty} C_{i+1} \cdot x^i = 1 + \left(\sum_{i=0}^{\infty} C_{i+1} \cdot x^i \right)^2$$

where ($|x| < \frac{1}{4}$) and hence

$$\sum_{i=0}^{\infty} C_{i+1} \cdot x^i = \frac{1 - \sqrt{1 - 4 \cdot x}}{2 \cdot x}.$$

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The notation and terminology used here are introduced in the following papers: [2], [23], [7], [25], [19], [27], [5], [28], [9], [1], [26], [21], [6], [3], [14], [12], [16], [13], [20], [15], [8], [22], [11], [10], [18], [24], [17], and [4].

1. PRELIMINARIES

For simplicity, we adopt the following convention: x, D denote sets, i, j, k, l, m, n denote elements of \mathbb{N} , p, q denote finite 0-sequences of \mathbb{N} , p', q' denote finite 0-sequences, and p_1, q_1 denote finite 0-sequences of D .

Next we state several propositions:

- (1) $(p' \wedge q') \upharpoonright \text{dom } p' = p'$.
- (2) If $n \leq \text{dom } p'$, then $(p' \wedge q') \upharpoonright n = p' \upharpoonright n$.
- (3) If $n = \text{dom } p' + k$, then $(p' \wedge q') \upharpoonright n = p' \wedge (q' \upharpoonright k)$.
- (4) There exists q' such that $p' = (p' \upharpoonright n) \wedge q'$.
- (5) There exists q_1 such that $p_1 = (p_1 \upharpoonright n) \wedge q_1$.

Let us consider p . We say that p is dominated by 0 if and only if:

(Def. 1) $\text{rng } p \subseteq \{0, 1\}$ and for every k such that $k \leq \text{dom } p$ holds $2 \cdot \sum(p \upharpoonright k) \leq k$.

The following propositions are true:

- (6) If p is dominated by 0, then $2 \cdot \sum(p \upharpoonright k) \leq k$.
- (7) If p is dominated by 0, then $p(0) = 0$.

Let us consider k, m . Then $k \mapsto m$ is a finite 0-sequence of \mathbb{N} .

One can check that there exists a finite 0-sequence of \mathbb{N} which is empty and dominated by 0 and there exists a finite 0-sequence of \mathbb{N} which is non empty and dominated by 0.

The following propositions are true:

- (8) $n \mapsto 0$ is dominated by 0.
- (9) If $n \geq m$, then $(n \mapsto 0) \wedge (m \mapsto 1)$ is dominated by 0.
- (10) If p is dominated by 0, then $p \upharpoonright n$ is dominated by 0.
- (11) If p is dominated by 0 and q is dominated by 0, then $p \wedge q$ is dominated by 0.
- (12) If p is dominated by 0, then $2 \cdot \sum(p \upharpoonright (2 \cdot n + 1)) < 2 \cdot n + 1$.
- (13) If p is dominated by 0 and $n \leq \text{len } p - 2 \cdot \sum p$, then $p \wedge (n \mapsto 1)$ is dominated by 0.
- (14) If p is dominated by 0 and $n \leq (k + \text{len } p) - 2 \cdot \sum p$, then $(k \mapsto 0) \wedge p \wedge (n \mapsto 1)$ is dominated by 0.
- (15) If p is dominated by 0 and $2 \cdot \sum(p \upharpoonright k) = k$, then $k \leq \text{len } p$ and $\text{len}(p \upharpoonright k) = k$.
- (16) If p is dominated by 0 and $2 \cdot \sum(p \upharpoonright k) = k$ and $p = (p \upharpoonright k) \wedge q$, then q is dominated by 0.

- (17) If p is dominated by 0 and $2 \cdot \sum(p \upharpoonright k) = k$ and $k = n + 1$, then $p \upharpoonright k = (p \upharpoonright n) \wedge (1 \mapsto 1)$.
- (18) Let given m, p . Suppose $m = \min^*\{n : 2 \cdot \sum(p \upharpoonright n) = n \wedge n > 0\}$ and $m > 0$ and p is dominated by 0. Then there exists q such that $p \upharpoonright m = (1 \mapsto 0) \wedge q \wedge (1 \mapsto 1)$ and q is dominated by 0.
- (19) Let given p . Suppose $\text{rng } p \subseteq \{0, 1\}$ and p is not dominated by 0. Then there exists k such that $2 \cdot k + 1 = \min^*\{n : 2 \cdot \sum(p \upharpoonright n) > n\}$ and $2 \cdot k + 1 \leq \text{dom } p$ and $k = \sum(p \upharpoonright (2 \cdot k))$ and $p(2 \cdot k) = 1$.
- (20) Let given p, q, k . Suppose $p \upharpoonright (2 \cdot k + \text{len } q) = (k \mapsto 0) \wedge q \wedge (k \mapsto 1)$ and q is dominated by 0 and $2 \cdot \sum q = \text{len } q$ and $k > 0$. Then $\min^*\{n : 2 \cdot \sum(p \upharpoonright n) = n \wedge n > 0\} = 2 \cdot k + \text{len } q$.
- (21) Let given p . Suppose p is dominated by 0 and $\{i : 2 \cdot \sum(p \upharpoonright i) = i \wedge i > 0\} = \emptyset$ and $\text{len } p > 0$. Then there exists q such that $p = \langle 0 \rangle \wedge q$ and q is dominated by 0.
- (22) If p is dominated by 0, then $\langle 0 \rangle \wedge p$ is dominated by 0 and $\{i : 2 \cdot \sum(\langle \langle 0 \rangle \wedge p \rangle \upharpoonright i) = i \wedge i > 0\} = \emptyset$.
- (23) If $\text{rng } p \subseteq \{0, 1\}$ and p is not dominated by 0 and $2 \cdot k + 1 = \min^*\{n : 2 \cdot \sum(p \upharpoonright n) > n\}$, then $p \upharpoonright (2 \cdot k)$ is dominated by 0.

2. THE RECURRENCE RELATION FOR THE CATALAN NUMBERS

Let n, m be natural numbers. The functor $\text{Domin}_0(n, m)$ yields a subset of $\{0, 1\}^\omega$ and is defined as follows:

- (Def. 2) $x \in \text{Domin}_0(n, m)$ iff there exists a finite 0-sequence p of \mathbb{N} such that $p = x$ and p is dominated by 0 and $\text{dom } p = n$ and $\sum p = m$.

Next we state two propositions:

- (24) $p \in \text{Domin}_0(n, m)$ iff p is dominated by 0 and $\text{dom } p = n$ and $\sum p = m$.
- (25) $\text{Domin}_0(n, m) \subseteq \text{Choose}(n, m, 1, 0)$.

Let us consider n, m . One can check that $\text{Domin}_0(n, m)$ is finite.

One can prove the following propositions:

- (26) $\text{Domin}_0(n, m)$ is empty iff $2 \cdot m > n$.
- (27) $\text{Domin}_0(n, 0) = \{n \mapsto 0\}$.
- (28) $\text{card } \text{Domin}_0(n, 0) = 1$.
- (29) Let given p, n . Suppose $\text{rng } p \subseteq \{0, n\}$. Then there exists q such that $\text{len } p = \text{len } q$ and $\text{rng } q \subseteq \{0, n\}$ and for every k such that $k \leq \text{len } p$ holds $\sum(p \upharpoonright k) + \sum(q \upharpoonright k) = n \cdot k$ and for every k such that $k \in \text{len } p$ holds $q(k) = n - p(k)$.
- (30) If $m \leq n$, then $\binom{n}{m} > 0$.

- (31) If $2 \cdot (m + 1) \leq n$, then $\text{card}(\text{Choose}(n, m + 1, 1, 0) \setminus \text{Domin}_0(n, m + 1)) = \text{card} \text{Choose}(n, m, 1, 0)$.
- (32) If $2 \cdot (m + 1) \leq n$, then $\text{card} \text{Domin}_0(n, m + 1) = \binom{n}{m+1} - \binom{n}{m}$.
- (33) If $2 \cdot m \leq n$, then $\text{card} \text{Domin}_0(n, m) = \frac{(n+1)-2 \cdot m}{(n+1)-m} \cdot \binom{n}{m}$.
- (34) $\text{card} \text{Domin}_0(2 + k, 1) = k + 1$.
- (35) $\text{card} \text{Domin}_0(4 + k, 2) = \frac{(k+1) \cdot (k+4)}{2}$.
- (36) $\text{card} \text{Domin}_0(6 + k, 3) = \frac{(k+1) \cdot (k+5) \cdot (k+6)}{6}$.
- (37) $\text{card} \text{Domin}_0(2 \cdot n, n) = \frac{\binom{2 \cdot n}{n}}{n+1}$.
- (38) $\text{card} \text{Domin}_0(2 \cdot n, n) = \text{Catalan}(n + 1)$.

Let us consider D . A functional non empty set is said to be a set of ω -sequences of D if:

(Def. 3) For every x such that $x \in$ it holds x is a finite 0-sequence of D .

Let us consider D . Then D^ω is a set of ω -sequences of D . Let F be a set of ω -sequences of D . We see that the element of F is a finite 0-sequence of D .

In the sequel p_2 denotes an element of \mathbb{N}^ω .

We now state several propositions:

- (39) $\overline{\overline{\{p_2 : \text{dom } p_2 = 2 \cdot n \wedge p_2 \text{ is dominated by } 0\}}} = \binom{2 \cdot n}{n}$.
- (40) Let given n, m, k, j, l . Suppose $j = n - 2 \cdot (k + 1)$ and $l = m - (k + 1)$. Then $\overline{\overline{\{p_2 : p_2 \in \text{Domin}_0(n, m) \wedge 2 \cdot (k + 1) = \min^* \{i : 2 \cdot \sum (p_2 \upharpoonright i) = i \wedge i > 0\}\}}} = \text{card} \text{Domin}_0(2 \cdot k, k) \cdot \text{card} \text{Domin}_0(j, l)$.
- (41) Let given n, m . Suppose $2 \cdot m \leq n$. Then there exists a finite 0-sequence C_1 of \mathbb{N} such that $\overline{\overline{\{p_2 : p_2 \in \text{Domin}_0(n, m) \wedge \{i : 2 \cdot \sum (p_2 \upharpoonright i) = i \wedge i > 0\} \neq \emptyset\}}} = \sum C_1$ and $\text{dom } C_1 = m$ and for every j such that $j < m$ holds $C_1(j) = \text{card} \text{Domin}_0(2 \cdot j, j) \cdot \text{card} \text{Domin}_0(n - ' 2 \cdot (j + 1), m - ' (j + 1))$.
- (42) For every n such that $n > 0$ holds $\text{Domin}_0(2 \cdot n, n) = \{p_2 : p_2 \in \text{Domin}_0(2 \cdot n, n) \wedge \{i : 2 \cdot \sum (p_2 \upharpoonright i) = i \wedge i > 0\} \neq \emptyset\}$.
- (43) Let given n . Suppose $n > 0$. Then there exists a finite 0-sequence C_2 of \mathbb{N} such that $\sum C_2 = \text{Catalan}(n + 1)$ and $\text{dom } C_2 = n$ and for every j such that $j < n$ holds $C_2(j) = \text{Catalan}(j + 1) \cdot \text{Catalan}(n - ' j)$.
- (44) $\overline{\overline{\{p_2 : p_2 \in \text{Domin}_0(n + 1, m) \wedge \{i : 2 \cdot \sum (p_2 \upharpoonright i) = i \wedge i > 0\} = \emptyset\}}} = \text{card} \text{Domin}_0(n, m)$.
- (45) Let given n, m . Suppose $2 \cdot m \leq n$. Then there exists a finite 0-sequence C_1 of \mathbb{N} such that $\text{card} \text{Domin}_0(n, m) = \sum C_1 + \text{card} \text{Domin}_0(n - ' 1, m)$ and $\text{dom } C_1 = m$ and for every j such that $j < m$ holds $C_1(j) = \text{card} \text{Domin}_0(2 \cdot j, j) \cdot \text{card} \text{Domin}_0(n - ' 2 \cdot (j + 1), m - ' (j + 1))$.
- (46) For all n, k there exists p such that $\sum p = \text{card} \text{Domin}_0(2 \cdot n + 2 + k, n + 1)$ and $\text{dom } p = k + 1$ and for every i such that $i \leq k$ holds $p(i) =$

card $\text{Domin}_0(2 \cdot n + 1 + i, n)$.

3. CAUCHY PRODUCT

We use the following convention: s_1, s_2, s_3 denote sequences of real numbers, r denotes a real number, and F_1, F_2, F_3 denote finite 0-sequences of \mathbb{R} .

Let us consider F_1 . The functor $\sum F_1$ yields a real number and is defined as follows:

(Def. 4) $\sum F_1 = +_{\mathbb{R}} \odot F_1$.

Let us consider F_1, x . Then $F_1(x)$ is a real number.

Let s_1, s_2 be sequences of real numbers. The functor $s_1 (\#) s_2$ yields a sequence of real numbers and is defined by the condition (Def. 5).

(Def. 5) Let k be a natural number. Then there exists a finite 0-sequence F_1 of \mathbb{R} such that $\text{dom } F_1 = k + 1$ and for every n such that $n \in k + 1$ holds $F_1(n) = s_1(n) \cdot s_2(k -' n)$ and $\sum F_1 = (s_1 (\#) s_2)(k)$.

Let us notice that the functor $s_1 (\#) s_2$ is commutative.

One can prove the following propositions:

(47) For all F_1, n such that $n \in \text{dom } F_1$ holds $\sum(F_1 \upharpoonright n) + F_1(n) = \sum(F_1 \upharpoonright (n + 1))$.

(48) For all F_2, F_3 such that $\text{dom } F_2 = \text{dom } F_3$ and for every n such that $n \in \text{len } F_2$ holds $F_2(n) = F_3(\text{len } F_2 -' (1 + n))$ holds $\sum F_2 = \sum F_3$.

(49) For all F_2, F_3 such that $\text{dom } F_2 = \text{dom } F_3$ and for every n such that $n \in \text{len } F_2$ holds $F_2(n) = r \cdot F_3(n)$ holds $\sum F_2 = r \cdot \sum F_3$.

(50) $s_1 (\#) r s_2 = r (s_1 (\#) s_2)$.

(51) $s_1 (\#) (s_2 + s_3) = (s_1 (\#) s_2) + (s_1 (\#) s_3)$.

(52) $(s_1 (\#) s_2)(0) = s_1(0) \cdot s_2(0)$.

(53) For all s_1, s_2, n there exists F_1 such that $(\sum_{\alpha=0}^{\kappa} (s_1 (\#) s_2)(\alpha))_{\kappa \in \mathbb{N}}(n) = \sum F_1$ and $\text{dom } F_1 = n + 1$ and for every i such that $i \in n + 1$ holds $F_1(i) = s_1(i) \cdot (\sum_{\alpha=0}^{\kappa} (s_2)(\alpha))_{\kappa \in \mathbb{N}}(n -' i)$.

(54) Let given s_1, s_2, n . Suppose s_2 is summable. Then there exists F_1 such that $(\sum_{\alpha=0}^{\kappa} (s_1 (\#) s_2)(\alpha))_{\kappa \in \mathbb{N}}(n) = \sum s_2 \cdot (\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}(n) - \sum F_1$ and $\text{dom } F_1 = n + 1$ and for every i such that $i \in n + 1$ holds $F_1(i) = s_1(i) \cdot \sum (s_2 \upharpoonright ((n -' i) + 1))$.

(55) For every F_1 there exists a finite 0-sequence a_1 of \mathbb{R} such that $\text{dom } a_1 = \text{dom } F_1$ and $|\sum F_1| \leq \sum a_1$ and for every i such that $i \in \text{dom } a_1$ holds $a_1(i) = |F_1(i)|$.

(56) For every s_1 such that s_1 is summable there exists r such that $0 < r$ and for every k holds $|\sum (s_1 \upharpoonright k)| < r$.

- (57) For all s_1, n, m such that $n \leq m$ and for every i holds $s_1(i) \geq 0$ holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}(m)$.
- (58) For all s_1, s_2 such that s_1 is absolutely summable and s_2 is summable holds $s_1 (\#) s_2$ is summable and $\sum (s_1 (\#) s_2) = \sum s_1 \cdot \sum s_2$.
- (59) If $p = F_1$, then $\sum p = \sum F_1$.

4. THE GENERATING FUNCTION FOR THE CATALAN NUMBERS

Next we state the proposition

- (60) Let given r . Then there exists a sequence C_2 of real numbers such that
- for every n holds $C_2(n) = \text{Catalan}(n+1) \cdot r^n$, and
 - if $|r| < \frac{1}{4}$, then C_2 is absolutely summable and $\sum C_2 = 1 + r \cdot (\sum C_2)^2$ and $\sum C_2 = \frac{2}{1+\sqrt{1-4r}}$ and if $r \neq 0$, then $\sum C_2 = \frac{1-\sqrt{1-4r}}{2 \cdot r}$.

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