

The Quaternion Numbers

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Summary. In this article, we define the set \mathbb{H} of quaternion numbers as the set of all ordered sequences $q = \langle x, y, w, z \rangle$ where x, y, w and z are real numbers. The addition, difference and multiplication of the quaternion numbers are also defined. We define the real and imaginary parts of q and denote this by $x = \Re(q)$, $y = \Im_1(q)$, $w = \Im_2(q)$, $z = \Im_3(q)$. We define the addition, difference, multiplication again and denote this operation by real and three imaginary parts. We define the conjugate of q denoted by $q^{*'}$ and the absolute value of q denoted by $|q|$. We also give some properties of quaternion numbers.

MML identifier: QUATERNI, version: 7.8.03 4.75.958

The articles [14], [16], [2], [1], [12], [17], [4], [5], [6], [13], [3], [11], [7], [8], [15], [18], [9], and [10] provide the terminology and notation for this paper.

We use the following convention: $a, b, c, d, x, y, w, z, x_1, x_2, x_3, x_4$ denote sets and A denotes a non empty set.

The functor \mathbb{H} is defined by:

(Def. 1) $\mathbb{H} = (\mathbb{R}^4 \setminus \{x; x \text{ ranges over elements of } \mathbb{R}^4: x(2) = 0 \wedge x(3) = 0\}) \cup \mathbb{C}$.

Let x be a number. We say that x is quaternion if and only if:

(Def. 2) $x \in \mathbb{H}$.

Let us observe that \mathbb{H} is non empty.

Let us consider x, y, w, z, a, b, c, d . The functor $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ yields a set and is defined as follows:

(Def. 3) $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = [x \mapsto a, y \mapsto b] + [w \mapsto c, z \mapsto d]$.

Let us consider x, y, w, z, a, b, c, d . Note that $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ is function-like and relation-like.

Next we state several propositions:

- (1) $\text{dom}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = \{x, y, w, z\}$.
- (2) $\text{rng}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] \subseteq \{a, b, c, d\}$.
- (3) Suppose x, y, w, z are mutually different. Then $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](x) = a$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](y) = b$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](w) = c$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](z) = d$.
- (4) If x, y, w, z are mutually different, then $\text{rng}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = \{a, b, c, d\}$.
- (5) $\{x_1, x_2, x_3, x_4\} \subseteq X$ iff $x_1 \in X$ and $x_2 \in X$ and $x_3 \in X$ and $x_4 \in X$.

Let us consider A, x, y, w, z and let a, b, c, d be elements of A . Then $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ is a function from $\{x, y, w, z\}$ into A .

The functor j is defined by:

(Def. 4) $j = [0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 1, 3 \mapsto 0]$.

The functor k is defined by:

(Def. 5) $k = [0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 0, 3 \mapsto 1]$.

One can check the following observations:

- * i is quaternion,
- * j is quaternion, and
- * k is quaternion.

Let us observe that there exists a number which is quaternion.

Let us mention that every element of \mathbb{H} is quaternion.

Let x, y, w, z be elements of \mathbb{R} . The functor $\langle x, y, w, z \rangle_{\mathbb{H}}$ yields an element of \mathbb{H} and is defined as follows:

(Def. 6) $\langle x, y, w, z \rangle_{\mathbb{H}} = \begin{cases} x + yi, & \text{if } w = 0 \text{ and } z = 0, \\ [0 \mapsto x, 1 \mapsto y, 2 \mapsto w, 3 \mapsto z], & \text{otherwise.} \end{cases}$

Next we state three propositions:

- (6) Let a, b, c, d, e, i, j, k be sets and g be a function. Suppose $a \neq b$ and $c \neq d$ and $\text{dom } g = \{a, b, c, d\}$ and $g(a) = e$ and $g(b) = i$ and $g(c) = j$ and $g(d) = k$. Then $g = [a \mapsto e, b \mapsto i, c \mapsto j, d \mapsto k]$.
- (7) For every element g of \mathbb{H} there exist elements r, s, t, u of \mathbb{R} such that $g = \langle r, s, t, u \rangle_{\mathbb{H}}$.
- (8) If a, c, x, w are mutually different, then $[a \mapsto b, c \mapsto d, x \mapsto y, w \mapsto z] = \{\langle a, b \rangle, \langle c, d \rangle, \langle x, y \rangle, \langle w, z \rangle\}$.

We adopt the following convention: a, b, c, d are elements of \mathbb{R} and r, s, t are elements of \mathbb{Q}_+ .

One can prove the following four propositions:

- (9) Let A be a subset of \mathbb{Q}_+ . Suppose there exists t such that $t \in A$ and $t \neq \emptyset$ and for all r, s such that $r \in A$ and $s \leq r$ holds $s \in A$. Then there exist elements r_1, r_2, r_3, r_4, r_5 of \mathbb{Q}_+ such that

$r_1 \in A$ and $r_2 \in A$ and $r_3 \in A$ and $r_4 \in A$ and $r_5 \in A$ and $r_1 \neq r_2$ and $r_1 \neq r_3$ and $r_1 \neq r_4$ and $r_1 \neq r_5$ and $r_2 \neq r_3$ and $r_2 \neq r_4$ and $r_2 \neq r_5$ and $r_3 \neq r_4$ and $r_3 \neq r_5$ and $r_4 \neq r_5$.

- (10) $[0 \mapsto a, 1 \mapsto b, 2 \mapsto c, 3 \mapsto d] \notin \mathbb{C}$.
- (11) Let $a, b, c, d, x, y, z, w, x', y', z', w'$ be sets. Suppose a, b, c, d are mutually different and $[a \mapsto x, b \mapsto y, c \mapsto z, d \mapsto w] = [a \mapsto x', b \mapsto y', c \mapsto z', d \mapsto w']$. Then $x = x'$ and $y = y'$ and $z = z'$ and $w = w'$.
- (12) For all elements $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ of \mathbb{R} such that $\langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}} = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ holds $x_1 = y_1$ and $x_2 = y_2$ and $x_3 = y_3$ and $x_4 = y_4$.

Let x, y be quaternion numbers. The functor $x + y$ is defined by:

- (Def. 7) There exist elements $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ of \mathbb{R} such that $x = \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}}$ and $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x + y = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 \rangle_{\mathbb{H}}$.

Let us observe that the functor $x + y$ is commutative.

Let z be a quaternion number. The functor $-z$ yields a quaternion number and is defined by:

- (Def. 8) $z + -z = 0$.

Let us observe that the functor $-z$ is involutive.

Let x, y be quaternion numbers. The functor $x - y$ is defined as follows:

- (Def. 9) $x - y = x + -y$.

Let x, y be quaternion numbers. The functor $x \cdot y$ is defined by the condition

(Def. 10).

- (Def. 10) There exist elements $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ of \mathbb{R} such that $x = \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}}$ and $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x \cdot y = \langle x_1 \cdot y_1 - x_2 \cdot y_2 - x_3 \cdot y_3 - x_4 \cdot y_4, (x_1 \cdot y_2 + x_2 \cdot y_1 + x_3 \cdot y_4) - x_4 \cdot y_3, (x_1 \cdot y_3 + y_1 \cdot x_3 + y_2 \cdot x_4) - y_4 \cdot x_2, (x_1 \cdot y_4 + x_4 \cdot y_1 + x_2 \cdot y_3) - x_3 \cdot y_2 \rangle_{\mathbb{H}}$.

Let z, z' be quaternion numbers. One can verify the following observations:

- * $z + z'$ is quaternion,
- * $z \cdot z'$ is quaternion, and
- * $z - z'$ is quaternion.

j is an element of \mathbb{H} and it can be characterized by the condition:

- (Def. 11) $j = \langle 0, 0, 1, 0 \rangle_{\mathbb{H}}$.

Then k is an element of \mathbb{H} and it can be characterized by the condition:

- (Def. 12) $k = \langle 0, 0, 0, 1 \rangle_{\mathbb{H}}$.

One can prove the following propositions:

- (13) $i \cdot i = -1$.
- (14) $j \cdot j = -1$.

- (15) $k \cdot k = -1$.
- (16) $i \cdot j = k$.
- (17) $j \cdot k = i$.
- (18) $k \cdot i = j$.
- (19) $i \cdot j = -j \cdot i$.
- (20) $j \cdot k = -k \cdot j$.
- (21) $k \cdot i = -i \cdot k$.

Let z be a quaternion number. The functor $\Re(z)$ is defined as follows:

- (Def. 13)(i) There exists a complex number z' such that $z = z'$ and $\Re(z) = \Re(z')$ if $z \in \mathbb{C}$,
- (ii) there exists a function f from 4 into \mathbb{R} such that $z = f$ and $\Re(z) = f(0)$, otherwise.

The functor $\Im_1(z)$ is defined by:

- (Def. 14)(i) There exists a complex number z' such that $z = z'$ and $\Im_1(z) = \Im_1(z')$ if $z \in \mathbb{C}$,
- (ii) there exists a function f from 4 into \mathbb{R} such that $z = f$ and $\Im_1(z) = f(1)$, otherwise.

The functor $\Im_2(z)$ is defined as follows:

- (Def. 15)(i) $\Im_2(z) = 0$ if $z \in \mathbb{C}$,
- (ii) there exists a function f from 4 into \mathbb{R} such that $z = f$ and $\Im_2(z) = f(2)$, otherwise.

The functor $\Im_3(z)$ is defined by:

- (Def. 16)(i) $\Im_3(z) = 0$ if $z \in \mathbb{C}$,
- (ii) there exists a function f from 4 into \mathbb{R} such that $z = f$ and $\Im_3(z) = f(3)$, otherwise.

Let z be a quaternion number. One can check the following observations:

- * $\Re(z)$ is real,
- * $\Im_1(z)$ is real,
- * $\Im_2(z)$ is real, and
- * $\Im_3(z)$ is real.

Let z be a quaternion number. Then $\Re(z)$ is a real number. Then $\Im_1(z)$ is a real number. Then $\Im_2(z)$ is a real number. Then $\Im_3(z)$ is a real number.

One can prove the following two propositions:

- (22) For every function f from 4 into \mathbb{R} there exist a, b, c, d such that $f = [0 \mapsto a, 1 \mapsto b, 2 \mapsto c, 3 \mapsto d]$.
- (23) $\Re(\langle a, b, c, d \rangle_{\mathbb{H}}) = a$ and $\Im_1(\langle a, b, c, d \rangle_{\mathbb{H}}) = b$ and $\Im_2(\langle a, b, c, d \rangle_{\mathbb{H}}) = c$ and $\Im_3(\langle a, b, c, d \rangle_{\mathbb{H}}) = d$.

In the sequel z, z_1, z_2, z_3, z_4 denote quaternion numbers.

Next we state two propositions:

$$(24) \quad z = \langle \Re(z), \Im_1(z), \Im_2(z), \Im_3(z) \rangle_{\mathbb{H}}.$$

$$(25) \quad \text{If } \Re(z_1) = \Re(z_2) \text{ and } \Im_1(z_1) = \Im_1(z_2) \text{ and } \Im_2(z_1) = \Im_2(z_2) \text{ and } \Im_3(z_1) = \Im_3(z_2), \text{ then } z_1 = z_2.$$

The quaternion number $0_{\mathbb{H}}$ is defined as follows:

$$(\text{Def. 17}) \quad 0_{\mathbb{H}} = 0.$$

The quaternion number $1_{\mathbb{H}}$ is defined as follows:

$$(\text{Def. 18}) \quad 1_{\mathbb{H}} = 1.$$

One can prove the following propositions:

$$(26) \quad \text{If } \Re(z) = 0 \text{ and } \Im_1(z) = 0 \text{ and } \Im_2(z) = 0 \text{ and } \Im_3(z) = 0, \text{ then } z = 0_{\mathbb{H}}.$$

$$(27) \quad \text{If } z = 0, \text{ then } (\Re(z))^2 + (\Im_1(z))^2 + (\Im_2(z))^2 + (\Im_3(z))^2 = 0.$$

$$(28) \quad \text{If } (\Re(z))^2 + (\Im_1(z))^2 + (\Im_2(z))^2 + (\Im_3(z))^2 = 0, \text{ then } z = 0_{\mathbb{H}}.$$

$$(29) \quad \Re(1_{\mathbb{H}}) = 1 \text{ and } \Im_1(1_{\mathbb{H}}) = 0 \text{ and } \Im_2(1_{\mathbb{H}}) = 0 \text{ and } \Im_3(1_{\mathbb{H}}) = 0.$$

$$(30) \quad \Re(i) = 0 \text{ and } \Im_1(i) = 1 \text{ and } \Im_2(i) = 0 \text{ and } \Im_3(i) = 0.$$

$$(31) \quad \Re(j) = 0 \text{ and } \Im_1(j) = 0 \text{ and } \Im_2(j) = 1 \text{ and } \Im_3(j) = 0 \text{ and } \Re(k) = 0 \text{ and } \Im_1(k) = 0 \text{ and } \Im_2(k) = 0 \text{ and } \Im_3(k) = 1.$$

$$(32) \quad \Re(z_1 + z_2 + z_3 + z_4) = \Re(z_1) + \Re(z_2) + \Re(z_3) + \Re(z_4) \text{ and } \Im_1(z_1 + z_2 + z_3 + z_4) = \Im_1(z_1) + \Im_1(z_2) + \Im_1(z_3) + \Im_1(z_4) \text{ and } \Im_2(z_1 + z_2 + z_3 + z_4) = \Im_2(z_1) + \Im_2(z_2) + \Im_2(z_3) + \Im_2(z_4) \text{ and } \Im_3(z_1 + z_2 + z_3 + z_4) = \Im_3(z_1) + \Im_3(z_2) + \Im_3(z_3) + \Im_3(z_4).$$

In the sequel x denotes a real number.

We now state three propositions:

$$(33) \quad \text{If } z_1 = x, \text{ then } \Re(z_1 \cdot i) = 0 \text{ and } \Im_1(z_1 \cdot i) = x \text{ and } \Im_2(z_1 \cdot i) = 0 \text{ and } \Im_3(z_1 \cdot i) = 0.$$

$$(34) \quad \text{If } z_1 = x, \text{ then } \Re(z_1 \cdot j) = 0 \text{ and } \Im_1(z_1 \cdot j) = 0 \text{ and } \Im_2(z_1 \cdot j) = x \text{ and } \Im_3(z_1 \cdot j) = 0.$$

$$(35) \quad \text{If } z_1 = x, \text{ then } \Re(z_1 \cdot k) = 0 \text{ and } \Im_1(z_1 \cdot k) = 0 \text{ and } \Im_2(z_1 \cdot k) = 0 \text{ and } \Im_3(z_1 \cdot k) = x.$$

Let x be a real number and let y be a quaternion number. The functor $x + y$ is defined as follows:

$$(\text{Def. 19}) \quad \text{There exist elements } y_1, y_2, y_3, y_4 \text{ of } \mathbb{R} \text{ such that } y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}} \text{ and } x + y = \langle x + y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}.$$

Let x be a real number and let y be a quaternion number. The functor $x - y$ is defined by:

$$(\text{Def. 20}) \quad x - y = x + -y.$$

Let x be a real number and let y be a quaternion number. The functor $x \cdot y$ is defined as follows:

(Def. 21) There exist elements y_1, y_2, y_3, y_4 of \mathbb{R} such that $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x \cdot y = \langle x \cdot y_1, x \cdot y_2, x \cdot y_3, x \cdot y_4 \rangle_{\mathbb{H}}$.

Let x be a real number and let z' be a quaternion number. One can verify the following observations:

- * $x + z'$ is quaternion,
- * $x \cdot z'$ is quaternion, and
- * $x - z'$ is quaternion.

Let z_1, z_2 be quaternion numbers. Then $z_1 + z_2$ is an element of \mathbb{H} and it can be characterized by the condition:

(Def. 22) $z_1 + z_2 = \Re(z_1) + \Re(z_2) + (\Im_1(z_1) + \Im_1(z_2)) \cdot i + (\Im_2(z_1) + \Im_2(z_2)) \cdot j + (\Im_3(z_1) + \Im_3(z_2)) \cdot k$.

The following proposition is true

(36) $\Re(z_1 + z_2) = \Re(z_1) + \Re(z_2)$ and $\Im_1(z_1 + z_2) = \Im_1(z_1) + \Im_1(z_2)$ and $\Im_2(z_1 + z_2) = \Im_2(z_1) + \Im_2(z_2)$ and $\Im_3(z_1 + z_2) = \Im_3(z_1) + \Im_3(z_2)$.

Let z_1, z_2 be elements of \mathbb{H} . Then $z_1 \cdot z_2$ is an element of \mathbb{H} and it can be characterized by the condition:

(Def. 23) $z_1 \cdot z_2 = (\Re(z_1) \cdot \Re(z_2) - \Im_1(z_1) \cdot \Im_1(z_2) - \Im_2(z_1) \cdot \Im_2(z_2) - \Im_3(z_1) \cdot \Im_3(z_2)) + ((\Re(z_1) \cdot \Im_1(z_2) + \Im_1(z_1) \cdot \Re(z_2) + \Im_2(z_1) \cdot \Im_3(z_2)) - \Im_3(z_1) \cdot \Im_2(z_2)) \cdot i + ((\Re(z_1) \cdot \Im_2(z_2) + \Im_2(z_1) \cdot \Re(z_2) + \Im_3(z_1) \cdot \Im_1(z_2)) - \Im_1(z_1) \cdot \Im_3(z_2)) \cdot j + ((\Re(z_1) \cdot \Im_3(z_2) + \Im_3(z_1) \cdot \Re(z_2) + \Im_1(z_1) \cdot \Im_2(z_2)) - \Im_2(z_1) \cdot \Im_1(z_2)) \cdot k$.

We now state four propositions:

(37) $z = \Re(z) + \Im_1(z) \cdot i + \Im_2(z) \cdot j + \Im_3(z) \cdot k$.

(38) Suppose $\Im_1(z_1) = 0$ and $\Im_1(z_2) = 0$ and $\Im_2(z_1) = 0$ and $\Im_2(z_2) = 0$ and $\Im_3(z_1) = 0$ and $\Im_3(z_2) = 0$. Then $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2)$ and $\Im_1(z_1 \cdot z_2) = \Im_2(z_1) \cdot \Im_3(z_2) - \Im_3(z_1) \cdot \Im_2(z_2)$ and $\Im_2(z_1 \cdot z_2) = \Im_3(z_1) \cdot \Im_1(z_2) - \Im_1(z_1) \cdot \Im_3(z_2)$ and $\Im_3(z_1 \cdot z_2) = \Im_1(z_1) \cdot \Im_2(z_2) - \Im_2(z_1) \cdot \Im_1(z_2)$.

(39) Suppose $\Re(z_1) = 0$ and $\Re(z_2) = 0$. Then $\Re(z_1 \cdot z_2) = -\Im_1(z_1) \cdot \Im_1(z_2) - \Im_2(z_1) \cdot \Im_2(z_2) - \Im_3(z_1) \cdot \Im_3(z_2)$ and $\Im_1(z_1 \cdot z_2) = \Im_2(z_1) \cdot \Im_3(z_2) - \Im_3(z_1) \cdot \Im_2(z_2)$ and $\Im_2(z_1 \cdot z_2) = \Im_3(z_1) \cdot \Im_1(z_2) - \Im_1(z_1) \cdot \Im_3(z_2)$ and $\Im_3(z_1 \cdot z_2) = \Im_1(z_1) \cdot \Im_2(z_2) - \Im_2(z_1) \cdot \Im_1(z_2)$.

(40) $\Re(z \cdot z) = (\Re(z))^2 - (\Im_1(z))^2 - (\Im_2(z))^2 - (\Im_3(z))^2$ and $\Im_1(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_1(z))$ and $\Im_2(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_2(z))$ and $\Im_3(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_3(z))$.

Let z be a quaternion number. Then $-z$ is an element of \mathbb{H} and it can be characterized by the condition:

(Def. 24) $-z = -\Re(z) + (-\Im_1(z)) \cdot i + (-\Im_2(z)) \cdot j + (-\Im_3(z)) \cdot k$.

The following proposition is true

(41) $\Re(-z) = -\Re(z)$ and $\Im_1(-z) = -\Im_1(z)$ and $\Im_2(-z) = -\Im_2(z)$ and $\Im_3(-z) = -\Im_3(z)$.

Let z_1, z_2 be quaternion numbers. Then $z_1 - z_2$ is an element of \mathbb{H} and it can be characterized by the condition:

$$(Def. 25) \quad z_1 - z_2 = (\Re(z_1) - \Re(z_2)) + (\Im_1(z_1) - \Im_1(z_2)) \cdot i + (\Im_2(z_1) - \Im_2(z_2)) \cdot j + (\Im_3(z_1) - \Im_3(z_2)) \cdot k.$$

One can prove the following proposition

$$(42) \quad \Re(z_1 - z_2) = \Re(z_1) - \Re(z_2) \text{ and } \Im_1(z_1 - z_2) = \Im_1(z_1) - \Im_1(z_2) \text{ and } \Im_2(z_1 - z_2) = \Im_2(z_1) - \Im_2(z_2) \text{ and } \Im_3(z_1 - z_2) = \Im_3(z_1) - \Im_3(z_2).$$

Let z be a quaternion number. The functor \bar{z} yielding a quaternion number is defined by:

$$(Def. 26) \quad \bar{z} = \Re(z) + (-\Im_1(z)) \cdot i + (-\Im_2(z)) \cdot j + (-\Im_3(z)) \cdot k.$$

Let z be a quaternion number. Then \bar{z} is an element of \mathbb{H} .

We now state a number of propositions:

$$(43) \quad \bar{z} = \langle \Re(z), -\Im_1(z), -\Im_2(z), -\Im_3(z) \rangle_{\mathbb{H}}.$$

$$(44) \quad \Re(\bar{z}) = \Re(z) \text{ and } \Im_1(\bar{z}) = -\Im_1(z) \text{ and } \Im_2(\bar{z}) = -\Im_2(z) \text{ and } \Im_3(\bar{z}) = -\Im_3(z).$$

$$(45) \quad \text{If } z = 0, \text{ then } \bar{z} = 0.$$

$$(46) \quad \text{If } \bar{z} = 0, \text{ then } z = 0.$$

$$(47) \quad \overline{1_{\mathbb{H}}} = 1_{\mathbb{H}}.$$

$$(48) \quad \Re(\bar{i}) = 0 \text{ and } \Im_1(\bar{i}) = -1 \text{ and } \Im_2(\bar{i}) = 0 \text{ and } \Im_3(\bar{i}) = 0.$$

$$(49) \quad \Re(\bar{j}) = 0 \text{ and } \Im_1(\bar{j}) = 0 \text{ and } \Im_2(\bar{j}) = -1 \text{ and } \Im_3(\bar{j}) = 0.$$

$$(50) \quad \Re(\bar{k}) = 0 \text{ and } \Im_1(\bar{k}) = 0 \text{ and } \Im_2(\bar{k}) = 0 \text{ and } \Im_3(\bar{k}) = -1.$$

$$(51) \quad \bar{\bar{i}} = -i.$$

$$(52) \quad \bar{\bar{j}} = -j.$$

$$(53) \quad \bar{\bar{k}} = -k.$$

$$(54) \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2.$$

$$(55) \quad \overline{-z} = -\bar{z}.$$

$$(56) \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2.$$

$$(57) \quad \text{If } \Im_2(z_1) \cdot \Im_3(z_2) \neq \Im_3(z_1) \cdot \Im_2(z_2), \text{ then } \overline{z_1 \cdot z_2} \neq \bar{z}_1 \cdot \bar{z}_2.$$

$$(58) \quad \text{If } \Im_1(z) = 0 \text{ and } \Im_2(z) = 0 \text{ and } \Im_3(z) = 0, \text{ then } \bar{z} = z.$$

$$(59) \quad \text{If } \Re(z) = 0, \text{ then } \bar{z} = -z.$$

$$(60) \quad \Re(z \cdot \bar{z}) = (\Re(z))^2 + (\Im_1(z))^2 + (\Im_2(z))^2 + (\Im_3(z))^2 \text{ and } \Im_1(z \cdot \bar{z}) = 0 \text{ and } \Im_2(z \cdot \bar{z}) = 0 \text{ and } \Im_3(z \cdot \bar{z}) = 0.$$

$$(61) \quad \Re(z + \bar{z}) = 2 \cdot \Re(z) \text{ and } \Im_1(z + \bar{z}) = 0 \text{ and } \Im_2(z + \bar{z}) = 0 \text{ and } \Im_3(z + \bar{z}) = 0.$$

$$(62) \quad -z = \langle -\Re(z), -\Im_1(z), -\Im_2(z), -\Im_3(z) \rangle_{\mathbb{H}}.$$

$$(63) \quad z_1 - z_2 = \langle \Re(z_1) - \Re(z_2), \Im_1(z_1) - \Im_1(z_2), \Im_2(z_1) - \Im_2(z_2), \Im_3(z_1) - \Im_3(z_2) \rangle_{\mathbb{H}}.$$

$$(64) \quad \Re(z - \bar{z}) = 0 \text{ and } \Im_1(z - \bar{z}) = 2 \cdot \Im_1(z) \text{ and } \Im_2(z - \bar{z}) = 2 \cdot \Im_2(z) \text{ and } \Im_3(z - \bar{z}) = 2 \cdot \Im_3(z).$$

Let us consider z . The functor $|z|$ yielding a real number is defined by:

$$(Def. 27) \quad |z| = \sqrt{(\Re(z))^2 + (\Im_1(z))^2 + (\Im_2(z))^2 + (\Im_3(z))^2}.$$

We now state a number of propositions:

$$(65) \quad |0_{\mathbb{H}}| = 0.$$

$$(66) \quad \text{If } |z| = 0, \text{ then } z = 0.$$

$$(67) \quad 0 \leq |z|.$$

$$(68) \quad |1_{\mathbb{H}}| = 1.$$

$$(69) \quad |i| = 1.$$

$$(70) \quad |j| = 1.$$

$$(71) \quad |k| = 1.$$

$$(72) \quad |-z| = |z|.$$

$$(73) \quad |\bar{z}| = |z|.$$

$$(74) \quad 0 \leq (\Re(z))^2 + (\Im_1(z))^2 + (\Im_2(z))^2 + (\Im_3(z))^2.$$

$$(75) \quad \Re(z) \leq |z|.$$

$$(76) \quad \Im_1(z) \leq |z|.$$

$$(77) \quad \Im_2(z) \leq |z|.$$

$$(78) \quad \Im_3(z) \leq |z|.$$

$$(79) \quad |z_1 + z_2| \leq |z_1| + |z_2|.$$

$$(80) \quad |z_1 - z_2| \leq |z_1| + |z_2|.$$

$$(81) \quad |z_1| - |z_2| \leq |z_1 + z_2|.$$

$$(82) \quad |z_1| - |z_2| \leq |z_1 - z_2|.$$

$$(83) \quad |z_1 - z_2| = |z_2 - z_1|.$$

$$(84) \quad |z_1 - z_2| = 0 \text{ iff } z_1 = z_2.$$

$$(85) \quad |z_1 - z_2| \leq |z_1 - z| + |z - z_2|.$$

$$(86) \quad ||z_1| - |z_2|| \leq |z_1 - z_2|.$$

$$(87) \quad |z_1 \cdot z_2| = |z_1| \cdot |z_2|.$$

$$(88) \quad |z \cdot z| = (\Re(z))^2 + (\Im_1(z))^2 + (\Im_2(z))^2 + (\Im_3(z))^2.$$

$$(89) \quad |z \cdot z| = |z \cdot \bar{z}|.$$

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Received November 14, 2006
