

Model Checking. Part I

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Summary. This text includes definitions of the Kripke structure, CTL (Computation Tree Logic), and verification of the basic algorithm for Model Checking based on CTL in [10].

MML identifier: MODEL_C_1, version: 7.8.03 4.75.958

The articles [21], [20], [16], [9], [18], [14], [6], [7], [4], [3], [5], [11], [2], [8], [13], [12], [17], [15], [1], and [19] provide the notation and terminology for this paper.

Let x, S be sets and let a be an element of S . The functor $\text{k.id}(x, S, a)$ yields an element of S and is defined by:

$$\text{(Def. 1)} \quad \text{k.id}(x, S, a) = \begin{cases} x, & \text{if } x \in S, \\ a, & \text{otherwise.} \end{cases}$$

Let x be a set. The functor $\text{k.nat } x$ yields an element of \mathbb{N} and is defined by:

$$\text{(Def. 2)} \quad \text{k.nat } x = \begin{cases} x, & \text{if } x \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Let f be a function and let x, a be sets. The functor $\text{UnivF}(x, f, a)$ yielding a set is defined by:

$$\text{(Def. 3)} \quad \text{UnivF}(x, f, a) = \begin{cases} f(x), & \text{if } x \in \text{dom } f, \\ a, & \text{otherwise.} \end{cases}$$

Let a be a set. The functor $\text{Castboolean } a$ yields a boolean set and is defined by:

$$\text{(Def. 4)} \quad \text{Castboolean } a = \begin{cases} a, & \text{if } a \text{ is a boolean set,} \\ \text{false}, & \text{otherwise.} \end{cases}$$

Let X, a be sets. The functor $\text{CastBool}(a, X)$ yielding a subset of X is defined as follows:

$$\text{(Def. 5)} \quad \text{CastBool}(a, X) = \begin{cases} a, & \text{if } a \subseteq X, \\ \emptyset, & \text{otherwise.} \end{cases}$$

For simplicity, we adopt the following rules: n denotes an element of \mathbb{N} , a denotes a set, D denotes a non empty set, and p, q denote finite sequences of elements of \mathbb{N} .

Let x be a variable. Then $\langle x \rangle$ is a finite sequence of elements of \mathbb{N} .

Let us consider n . The functor $\text{atom.}n$ yields a finite sequence of elements of \mathbb{N} and is defined by:

$$\text{(Def. 6)} \quad \text{atom.}n = \langle 5 + n \rangle.$$

Let us consider p . The functor $\neg p$ yielding a finite sequence of elements of \mathbb{N} is defined by:

$$\text{(Def. 7)} \quad \neg p = \langle 0 \rangle \hat{\ } p.$$

Let us consider q . The functor $p \wedge q$ yielding a finite sequence of elements of \mathbb{N} is defined by:

$$\text{(Def. 8)} \quad p \wedge q = \langle 1 \rangle \hat{\ } p \hat{\ } q.$$

Let us consider p . The functor $\text{EX}p$ yielding a finite sequence of elements of \mathbb{N} is defined as follows:

$$\text{(Def. 9)} \quad \text{EX}p = \langle 2 \rangle \hat{\ } p.$$

The functor $\text{EG}p$ yielding a finite sequence of elements of \mathbb{N} is defined by:

$$\text{(Def. 10)} \quad \text{EG}p = \langle 3 \rangle \hat{\ } p.$$

Let us consider q . The functor $p \text{EU} q$ yields a finite sequence of elements of \mathbb{N} and is defined as follows:

$$\text{(Def. 11)} \quad p \text{EU} q = \langle 4 \rangle \hat{\ } p \hat{\ } q.$$

The non empty set CTL-WFF is defined by the conditions (Def. 12).

$$\text{(Def. 12)} \quad \text{For every } a \text{ such that } a \in \text{CTL-WFF} \text{ holds } a \text{ is a finite sequence of elements of } \mathbb{N} \text{ and for every } n \text{ holds } \text{atom.}n \in \text{CTL-WFF} \text{ and for every } p \text{ such that } p \in \text{CTL-WFF} \text{ holds } \neg p \in \text{CTL-WFF} \text{ and for all } p, q \text{ such that } p \in \text{CTL-WFF} \text{ and } q \in \text{CTL-WFF} \text{ holds } p \wedge q \in \text{CTL-WFF} \text{ and for every } p \text{ such that } p \in \text{CTL-WFF} \text{ holds } \text{EX}p \in \text{CTL-WFF} \text{ and for every } p \text{ such that } p \in \text{CTL-WFF} \text{ holds } \text{EG}p \in \text{CTL-WFF} \text{ and for all } p, q \text{ such that } p \in \text{CTL-WFF} \text{ and } q \in \text{CTL-WFF} \text{ holds } p \text{EU} q \in \text{CTL-WFF} \text{ and for every } D \text{ such that for every } a \text{ such that } a \in D \text{ holds } a \text{ is a finite sequence of elements of } \mathbb{N} \text{ and for every } n \text{ holds } \text{atom.}n \in D \text{ and for every } p \text{ such that } p \in D \text{ holds } \neg p \in D \text{ and for all } p, q \text{ such that } p \in D \text{ and } q \in D \text{ holds } p \wedge q \in D \text{ and for every } p \text{ such that } p \in D \text{ holds } \text{EX}p \in D \text{ and for every } p \text{ such that } p \in D \text{ holds } \text{EG}p \in D \text{ and for all } p, q \text{ such that } p \in D \text{ and } q \in D \text{ holds } p \text{EU} q \in D \text{ holds } \text{CTL-WFF} \subseteq D.$$

Let I_1 be a finite sequence of elements of \mathbb{N} . We say that I_1 is CTL-formula-like if and only if:

$$\text{(Def. 13)} \quad I_1 \text{ is an element of CTL-WFF.}$$

Let us mention that there exists a finite sequence of elements of \mathbb{N} which is CTL-formula-like.

A CTL-formula is a CTL-formula-like finite sequence of elements of \mathbb{N} .

One can prove the following proposition

- (1) a is a CTL-formula iff $a \in \text{CTL-WFF}$.

In the sequel F, G, H, H_1, H_2 denote CTL-formulae.

Let us consider n . One can verify that $\text{atom. } n$ is CTL-formula-like.

Let us consider H . One can verify the following observations:

- * $\neg H$ is CTL-formula-like,
- * $\text{EX } H$ is CTL-formula-like, and
- * $\text{EG } H$ is CTL-formula-like.

Let us consider G . One can verify that $H \wedge G$ is CTL-formula-like and $H \text{ EU } G$ is CTL-formula-like.

Let us consider H . We say that H is atomic if and only if:

- (Def. 14) There exists n such that $H = \text{atom. } n$.

We say that H is negative if and only if:

- (Def. 15) There exists H_1 such that $H = \neg H_1$.

We say that H is conjunctive if and only if:

- (Def. 16) There exist F, G such that $H = F \wedge G$.

We say that H is exist-next-formula if and only if:

- (Def. 17) There exists H_1 such that $H = \text{EX } H_1$.

We say that H is exist-global-formula if and only if:

- (Def. 18) There exists H_1 such that $H = \text{EG } H_1$.

We say that H is exist-until-formula if and only if:

- (Def. 19) There exist F, G such that $H = F \text{ EU } G$.

Let us consider F, G . The functor $F \vee G$ yielding a CTL-formula is defined by:

- (Def. 20) $F \vee G = \neg(\neg F \wedge \neg G)$.

One can prove the following proposition

- (2) H is atomic, or negative, or conjunctive, or exist-next-formula, or exist-global-formula, or exist-until-formula.

Let us consider H . Let us assume that H is negative, or exist-next-formula, or exist-global-formula. The functor $\text{Arg}(H)$ yielding a CTL-formula is defined as follows:

- (Def. 21)(i) $\neg \text{Arg}(H) = H$ if H is negative,
(ii) $\text{EX Arg}(H) = H$ if H is exist-next-formula,
(iii) $\text{EG Arg}(H) = H$, otherwise.

Let us consider H . Let us assume that H is conjunctive or exist-until-formula. The functor $\text{LeftArg}(H)$ yields a CTL-formula and is defined as follows:

- (Def. 22)(i) There exists H_1 such that $\text{LeftArg}(H) \wedge H_1 = H$ if H is conjunctive,
(ii) there exists H_1 such that $\text{LeftArg}(H) \text{EU } H_1 = H$, otherwise.

The functor $\text{RightArg}(H)$ yielding a CTL-formula is defined by:

- (Def. 23)(i) There exists H_1 such that $H_1 \wedge \text{RightArg}(H) = H$ if H is conjunctive,
(ii) there exists H_1 such that $H_1 \text{EU } \text{RightArg}(H) = H$, otherwise.

Let x be a set. The functor $\text{CastCTLformula } x$ yields a CTL-formula and is defined by:

- (Def. 24) $\text{CastCTLformula } x = \begin{cases} x, & \text{if } x \in \text{CTL-WFF}, \\ \text{atom. } 0, & \text{otherwise.} \end{cases}$

Let P_1 be a set. We consider Kripke structures over P_1 as systems
 $\langle \text{worlds, starts, possibilities, a label} \rangle$,

where the worlds constitute a set, the starts constitute a subset of the worlds, the possibilities constitute a total relation between the worlds and the worlds, and the label is a function from the worlds into 2^{P_1} .

We introduce CTL model structures which are systems

$\langle \text{assignments, basic assignments, a conjunction, a negation, a next-operation, a global-operation, an until-operation} \rangle$,

where the assignments constitute a non empty set, the basic assignments constitute a non empty subset of the assignments, the conjunction is a binary operation on the assignments, the negation is a unary operation on the assignments, the next-operation is a unary operation on the assignments, the global-operation is a unary operation on the assignments, and the until-operation is a binary operation on the assignments.

Let V be a CTL model structure. An assignment of V is an element of the assignments of V .

The subset the atomic WFF of CTL-WFF is defined by:

- (Def. 25) The atomic WFF = $\{x; x \text{ ranges over CTL-formulae: } x \text{ is atomic}\}$.

Let V be a CTL model structure, let K_1 be a function from the atomic WFF into the basic assignments of V , and let f be a function from CTL-WFF into the assignments of V . We say that f is an evaluation for K_1 if and only if the condition (Def. 26) is satisfied.

- (Def. 26) Let H be a CTL-formula. Then
(i) if H is atomic, then $f(H) = K_1(H)$,
(ii) if H is negative, then $f(H) = (\text{the negation of } V)(f(\text{Arg}(H)))$,
(iii) if H is conjunctive, then $f(H) = (\text{the conjunction of } V)(f(\text{LeftArg}(H)), f(\text{RightArg}(H)))$,
(iv) if H is exist-next-formula, then $f(H) = (\text{the next-operation of } V)(f(\text{Arg}(H)))$,

- (v) if H is exist-global-formula, then $f(H) =$ (the global-operation of $V)(f(\text{Arg}(H)))$, and
- (vi) if H is exist-until-formula, then $f(H) =$ (the until-operation of $V)(f(\text{LeftArg}(H)), f(\text{RightArg}(H)))$.

Let V be a CTL model structure, let K_1 be a function from the atomic WFF into the basic assignments of V , let f be a function from CTL-WFF into the assignments of V , and let n be an element of \mathbb{N} . We say that f is a n -pre-evaluation for K_1 if and only if the condition (Def. 27) is satisfied.

- (Def. 27) Let H be a CTL-formula such that $\text{len } H \leq n$. Then
- (i) if H is atomic, then $f(H) = K_1(H)$,
 - (ii) if H is negative, then $f(H) =$ (the negation of $V)(f(\text{Arg}(H)))$,
 - (iii) if H is conjunctive, then $f(H) =$ (the conjunction of $V)(f(\text{LeftArg}(H)), f(\text{RightArg}(H)))$,
 - (iv) if H is exist-next-formula, then $f(H) =$ (the next-operation of $V)(f(\text{Arg}(H)))$,
 - (v) if H is exist-global-formula, then $f(H) =$ (the global-operation of $V)(f(\text{Arg}(H)))$, and
 - (vi) if H is exist-until-formula, then $f(H) =$ (the until-operation of $V)(f(\text{LeftArg}(H)), f(\text{RightArg}(H)))$.

Let V be a CTL model structure, let K_1 be a function from the atomic WFF into the basic assignments of V , let f, h be functions from CTL-WFF into the assignments of V , let n be an element of \mathbb{N} , and let H be a CTL-formula. The functor $\text{GraftEval}(V, K_1, f, h, n, H)$ yields a set and is defined as follows:

- (Def. 28) $\text{GraftEval}(V, K_1, f, h, n, H) =$
- $$\left\{ \begin{array}{l} f(H), \text{ if } \text{len } H > n + 1, \\ K_1(H), \text{ if } \text{len } H = n + 1 \text{ and } H \text{ is atomic,} \\ \text{(the negation of } V)(h(\text{Arg}(H))), \text{ if } \text{len } H = n + 1 \text{ and } H \text{ is negative,} \\ \text{(the conjunction of } V)(h(\text{LeftArg}(H)), h(\text{RightArg}(H))), \\ \quad \text{if } \text{len } H = n + 1 \text{ and } H \text{ is conjunctive,} \\ \text{(the next-operation of } V)(h(\text{Arg}(H))), \text{ if } \text{len } H = n + 1 \text{ and } H \text{ is} \\ \quad \text{exist-next-formula,} \\ \text{(the global-operation of } V)(h(\text{Arg}(H))), \text{ if } \text{len } H = n + 1 \text{ and } H \text{ is} \\ \quad \text{exist-global-formula,} \\ \text{(the until-operation of } V)(h(\text{LeftArg}(H)), h(\text{RightArg}(H))), \\ \quad \text{if } \text{len } H = n + 1 \text{ and } H \text{ is exist-until-formula,} \\ h(H), \text{ if } \text{len } H < n + 1, \\ \emptyset, \text{ otherwise.} \end{array} \right.$$

We follow the rules: V is a CTL model structure, K_1 is a function from the atomic WFF into the basic assignments of V , and f, f_1, f_2 are functions from CTL-WFF into the assignments of V .

Let V be a CTL model structure, let K_1 be a function from the atomic

WFF into the basic assignments of V , and let n be an element of \mathbb{N} . The functor $\text{EvalSet}(V, K_1, n)$ yields a non empty set and is defined by:

(Def. 29) $\text{EvalSet}(V, K_1, n) = \{h; h \text{ ranges over functions from CTL-WFF into the assignments of } V: h \text{ is a } n\text{-pre-evaluation for } K_1\}$.

Let V be a CTL model structure, let v_0 be an element of the assignments of V , and let x be a set. The functor $\text{CastEval}(V, x, v_0)$ yielding a function from CTL-WFF into the assignments of V is defined by:

(Def. 30) $\text{CastEval}(V, x, v_0) = \begin{cases} x, & \text{if } x \in (\text{the assignments of } V)^{\text{CTL-WFF}}, \\ \text{CTL-WFF} \longmapsto v_0, & \text{otherwise.} \end{cases}$

Let V be a CTL model structure and let K_1 be a function from the atomic WFF into the basic assignments of V . The functor $\text{EvalFamily}(V, K_1)$ yielding a non empty set is defined by the condition (Def. 31).

(Def. 31) Let p be a set. Then $p \in \text{EvalFamily}(V, K_1)$ if and only if the following conditions are satisfied:

- (i) $p \in 2^{(\text{the assignments of } V)^{\text{CTL-WFF}}}$, and
- (ii) there exists an element n of \mathbb{N} such that $p = \text{EvalSet}(V, K_1, n)$.

We now state two propositions:

- (3) There exists f which is an evaluation for K_1 .
- (4) If f_1 is an evaluation for K_1 and f_2 is an evaluation for K_1 , then $f_1 = f_2$.

Let V be a CTL model structure, let K_1 be a function from the atomic WFF into the basic assignments of V , and let H be a CTL-formula. The functor $\text{Evaluate}(H, K_1)$ yields an assignment of V and is defined by:

(Def. 32) There exists a function f from CTL-WFF into the assignments of V such that f is an evaluation for K_1 and $\text{Evaluate}(H, K_1) = f(H)$.

Let V be a CTL model structure and let f be an assignment of V . The functor $\neg f$ yields an assignment of V and is defined as follows:

(Def. 33) $\neg f = (\text{the negation of } V)(f)$.

Let V be a CTL model structure and let f, g be assignments of V . The functor $f \wedge g$ yielding an assignment of V is defined by:

(Def. 34) $f \wedge g = (\text{the conjunction of } V)(f, g)$.

Let V be a CTL model structure and let f be an assignment of V . The functor $\text{EX } f$ yields an assignment of V and is defined by:

(Def. 35) $\text{EX } f = (\text{the next-operation of } V)(f)$.

The functor $\text{EG } f$ yielding an assignment of V is defined as follows:

(Def. 36) $\text{EG } f = (\text{the global-operation of } V)(f)$.

Let V be a CTL model structure and let f, g be assignments of V . The functor $f \text{ EU } g$ yields an assignment of V and is defined as follows:

(Def. 37) $f \text{ EU } g = (\text{the until-operation of } V)(f, g)$.

The functor $f \vee g$ yielding an assignment of V is defined as follows:

(Def. 38) $f \vee g = \neg(\neg f \wedge \neg g)$.

Next we state several propositions:

- (5) $\text{Evaluate}(\neg H, K_1) = \neg \text{Evaluate}(H, K_1)$.
- (6) $\text{Evaluate}(H_1 \wedge H_2, K_1) = \text{Evaluate}(H_1, K_1) \wedge \text{Evaluate}(H_2, K_1)$.
- (7) $\text{Evaluate}(\text{EX } H, K_1) = \text{EX } \text{Evaluate}(H, K_1)$.
- (8) $\text{Evaluate}(\text{EG } H, K_1) = \text{EG } \text{Evaluate}(H, K_1)$.
- (9) $\text{Evaluate}(H_1 \text{ EU } H_2, K_1) = \text{Evaluate}(H_1, K_1) \text{ EU } \text{Evaluate}(H_2, K_1)$.
- (10) $\text{Evaluate}(H_1 \vee H_2, K_1) = \text{Evaluate}(H_1, K_1) \vee \text{Evaluate}(H_2, K_1)$.

Let f be a function and let n be an element of \mathbb{N} . We introduce f^n as a synonym of f^n .

Let S be a set, let f be a function from S into S , and let n be an element of \mathbb{N} . Then f^n is a function from S into S .

We use the following convention: S is a non empty set, R is a total relation between S and S , and s, s_0, s_1 are elements of S .

The scheme *ExistPath* deals with a non empty set \mathcal{A} , a total relation \mathcal{B} between \mathcal{A} and \mathcal{A} , an element \mathcal{C} of \mathcal{A} , and a unary functor \mathcal{F} yielding a set, and states that:

There exists a function f from \mathbb{N} into \mathcal{A} such that $f(0) = \mathcal{C}$
and for every element n of \mathbb{N} holds $\langle f(n), f(n+1) \rangle \in \mathcal{B}$ and
 $f(n+1) \in \mathcal{F}(f(n))$

provided the following requirement is met:

- For every element s of \mathcal{A} holds $\mathcal{B}^\circ\{s\} \cap \mathcal{F}(s)$ is a non empty subset of \mathcal{A} .

Let S be a non empty set and let R be a total relation between S and S . A function from \mathbb{N} into S is said to be an infinity path of R if:

(Def. 39) For every element n of \mathbb{N} holds $\langle \text{it}(n), \text{it}(n+1) \rangle \in R$.

Let S be a non empty set. The functor $\text{ModelSP } S$ yields a non empty set and is defined by:

(Def. 40) $\text{ModelSP } S = \text{Boolean}^S$.

Let S be a non empty set. Observe that $\text{ModelSP } S$ is non empty.

Let S be a non empty set and let f be a set. The functor $\text{Fid}(f, S)$ yielding a function from S into Boolean is defined by:

(Def. 41) $\text{Fid}(f, S) = \begin{cases} f, & \text{if } f \in \text{ModelSP } S, \\ S \mapsto \text{false}, & \text{otherwise.} \end{cases}$

Now we present several schemes. The scheme *Func1EX* deals with a non empty set \mathcal{A} , a function \mathcal{B} from \mathcal{A} into Boolean , and a binary functor \mathcal{F} yielding a boolean set, and states that:

There exists a set g such that $g \in \text{ModelSP } \mathcal{A}$ and for every set s
such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}) = \text{true}$ iff $(\text{Fid}(g, \mathcal{A}))(s) = \text{true}$
for all values of the parameters.

The scheme *Func1Unique* deals with a non empty set \mathcal{A} , a function \mathcal{B} from \mathcal{A} into *Boolean*, and a binary functor \mathcal{F} yielding a boolean set, and states that:

Let g_1, g_2 be sets. Suppose that

- (i) $g_1 \in \text{ModelSP } \mathcal{A}$,
- (ii) for every set s such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}) = \text{true}$ iff $(\text{Fid}(g_1, \mathcal{A}))(s) = \text{true}$,
- (iii) $g_2 \in \text{ModelSP } \mathcal{A}$, and
- (iv) for every set s such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}) = \text{true}$ iff $(\text{Fid}(g_2, \mathcal{A}))(s) = \text{true}$.

Then $g_1 = g_2$

for all values of the parameters.

The scheme *UnOpEX* deals with a non empty set \mathcal{A} and a unary functor \mathcal{F} yielding an element of \mathcal{A} , and states that:

There exists a unary operation o on \mathcal{A} such that for every set f such that $f \in \mathcal{A}$ holds $o(f) = \mathcal{F}(f)$

for all values of the parameters.

The scheme *UnOpUnique* deals with a non empty set \mathcal{A} , a non empty set \mathcal{B} , and a unary functor \mathcal{F} yielding an element of \mathcal{B} , and states that:

Let o_1, o_2 be unary operations on \mathcal{B} . Suppose for every set f such that $f \in \mathcal{B}$ holds $o_1(f) = \mathcal{F}(f)$ and for every set f such that $f \in \mathcal{B}$ holds $o_2(f) = \mathcal{F}(f)$. Then $o_1 = o_2$

for all values of the parameters.

The scheme *Func2EX* deals with a non empty set \mathcal{A} , a function \mathcal{B} from \mathcal{A} into *Boolean*, a function \mathcal{C} from \mathcal{A} into *Boolean*, and a ternary functor \mathcal{F} yielding a boolean set, and states that:

There exists a set h such that $h \in \text{ModelSP } \mathcal{A}$ and for every set s such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}, \mathcal{C}) = \text{true}$ iff $(\text{Fid}(h, \mathcal{A}))(s) = \text{true}$

for all values of the parameters.

The scheme *Func2Unique* deals with a non empty set \mathcal{A} , a function \mathcal{B} from \mathcal{A} into *Boolean*, a function \mathcal{C} from \mathcal{A} into *Boolean*, and a ternary functor \mathcal{F} yielding a boolean set, and states that:

Let h_1, h_2 be sets. Suppose that

- (i) $h_1 \in \text{ModelSP } \mathcal{A}$,
- (ii) for every set s such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}, \mathcal{C}) = \text{true}$ iff $(\text{Fid}(h_1, \mathcal{A}))(s) = \text{true}$,
- (iii) $h_2 \in \text{ModelSP } \mathcal{A}$, and
- (iv) for every set s such that $s \in \mathcal{A}$ holds $\mathcal{F}(s, \mathcal{B}, \mathcal{C}) = \text{true}$ iff $(\text{Fid}(h_2, \mathcal{A}))(s) = \text{true}$.

Then $h_1 = h_2$

for all values of the parameters.

Let S be a non empty set and let f be a set. The functor $\text{Not}_0(f, S)$ yielding an element of $\text{ModelSP } S$ is defined as follows:

(Def. 42) For every set s such that $s \in S$ holds $\neg \text{Castboolean}(\text{Fid}(f, S))(s) = \text{true}$
iff $(\text{Fid}(\text{Not}_0(f, S), S))(s) = \text{true}$.

Let S be a non empty set. The functor $\text{Not } S$ yields a unary operation on $\text{ModelSP } S$ and is defined by:

(Def. 43) For every set f such that $f \in \text{ModelSP } S$ holds $(\text{Not } S)(f) = \text{Not}_0(f, S)$.

Let S be a non empty set, let R be a total relation between S and S , let f be a function from S into Boolean , and let x be a set. The functor $\text{EneXt}_{\text{univ}}(x, f, R)$ yielding an element of Boolean is defined by:

(Def. 44)
$$\text{EneXt}_{\text{univ}}(x, f, R) = \begin{cases} \text{true}, & \\ \text{if } x \in S \text{ and there exists an infinity path } p_1 & \\ \text{of } R \text{ such that } p_1(0) = x \text{ and } f(p_1(1)) = \text{true}, & \\ \text{false, otherwise.} & \end{cases}$$

Let S be a non empty set, let R be a total relation between S and S , and let f be a set. The functor $\text{EneXt}_0(f, R)$ yielding an element of $\text{ModelSP } S$ is defined as follows:

(Def. 45) For every set s such that $s \in S$ holds $\text{EneXt}_{\text{univ}}(s, \text{Fid}(f, S), R) = \text{true}$
iff $(\text{Fid}(\text{EneXt}_0(f, R), S))(s) = \text{true}$.

Let S be a non empty set and let R be a total relation between S and S . The functor $\text{EneXt } R$ yields a unary operation on $\text{ModelSP } S$ and is defined by:

(Def. 46) For every set f such that $f \in \text{ModelSP } S$ holds $(\text{EneXt } R)(f) = \text{EneXt}_0(f, R)$.

Let S be a non empty set, let R be a total relation between S and S , let f be a function from S into Boolean , and let x be a set. The functor $\text{EGlobal}_{\text{univ}}(x, f, R)$ yielding an element of Boolean is defined by:

(Def. 47)
$$\text{EGlobal}_{\text{univ}}(x, f, R) = \begin{cases} \text{true}, & \\ \text{if } x \in S \text{ and there exists an infinity path } & \\ p_1 \text{ of } R \text{ such that } p_1(0) = x \text{ and for every} & \\ \text{element } n \text{ of } \mathbb{N} \text{ holds } f(p_1(n)) = \text{true}, & \\ \text{false, otherwise.} & \end{cases}$$

Let S be a non empty set, let R be a total relation between S and S , and let f be a set. The functor $\text{EGlobal}_0(f, R)$ yielding an element of $\text{ModelSP } S$ is defined as follows:

(Def. 48) For every set s such that $s \in S$ holds $\text{EGlobal}_{\text{univ}}(s, \text{Fid}(f, S), R) = \text{true}$
iff $(\text{Fid}(\text{EGlobal}_0(f, R), S))(s) = \text{true}$.

Let S be a non empty set and let R be a total relation between S and S . The functor $\text{EGlobal } R$ yields a unary operation on $\text{ModelSP } S$ and is defined as follows:

(Def. 49) For every set f such that $f \in \text{ModelSP } S$ holds $(\text{EGlobal } R)(f) = \text{EGlobal}_0(f, R)$.

Let S be a non empty set and let f, g be sets. The functor $\text{And}_0(f, g, S)$ yields an element of $\text{ModelSP } S$ and is defined as follows:

(Def. 50) For every set s such that $s \in S$ holds $\text{Castboolean}(\text{Fid}(f, S))(s) \wedge \text{Castboolean}(\text{Fid}(g, S))(s) = \text{true}$ iff $(\text{Fid}(\text{And}_0(f, g, S), S))(s) = \text{true}$.

Let S be a non empty set. The $\text{and } S$ yielding a binary operation on $\text{ModelSP } S$ is defined by:

(Def. 51) For all sets f, g such that $f \in \text{ModelSP } S$ and $g \in \text{ModelSP } S$ holds (the $\text{and } S$)(f, g) = $\text{And}_0(f, g, S)$.

Let S be a non empty set, let R be a total relation between S and S , let f, g be functions from S into Boolean , and let x be a set. The functor $\text{EUntill}_{\text{univ}}(x, f, g, R)$ yielding an element of Boolean is defined as follows:

(Def. 52)
$$\text{EUntill}_{\text{univ}}(x, f, g, R) = \begin{cases} \text{true, if } x \in S \text{ and there exists an infinity path} \\ \quad p_1 \text{ of } R \text{ such that } p_1(0) = x \text{ and there exists} \\ \quad \text{an element } m \text{ of } \mathbb{N} \text{ such that for every} \\ \quad \text{element } j \text{ of } \mathbb{N} \text{ such that } j < m \text{ holds} \\ \quad f(p_1(j)) = \text{true} \text{ and } g(p_1(m)) = \text{true,} \\ \text{false, otherwise.} \end{cases}$$

Let S be a non empty set, let R be a total relation between S and S , and let f, g be sets. The functor $\text{EUntill}_0(f, g, R)$ yields an element of $\text{ModelSP } S$ and is defined by:

(Def. 53) For every set s such that $s \in S$ holds $\text{EUntill}_{\text{univ}}(s, \text{Fid}(f, S), \text{Fid}(g, S), R) = \text{true}$ iff $(\text{Fid}(\text{EUntill}_0(f, g, R), S))(s) = \text{true}$.

Let S be a non empty set and let R be a total relation between S and S . The functor $\text{EUntill } R$ yields a binary operation on $\text{ModelSP } S$ and is defined as follows:

(Def. 54) For all sets f, g such that $f \in \text{ModelSP } S$ and $g \in \text{ModelSP } S$ holds $(\text{EUntill } R)(f, g) = \text{EUntill}_0(f, g, R)$.

Let S be a non empty set, let X be a non empty subset of $\text{ModelSP } S$, and let s be a set. The functor $\text{F-LABEL}(s, X)$ yields a subset of X and is defined as follows:

(Def. 55) For every set x holds $x \in \text{F-LABEL}(s, X)$ iff $x \in X$ and there exists a function f from S into Boolean such that $f = x$ and $f(s) = \text{true}$.

Let S be a non empty set and let X be a non empty subset of $\text{ModelSP } S$. The functor $\text{Label } X$ yields a function from S into 2^X and is defined by:

(Def. 56) For every set x such that $x \in S$ holds $(\text{Label } X)(x) = \text{F-LABEL}(x, X)$.

Let S be a non empty set, let S_0 be a subset of S , let R be a total relation between S and S , and let P_1 be a non empty subset of $\text{ModelSP } S$. The functor $\text{KModel}(R, S_0, P_1)$ yields a Kripke structure over P_1 and is defined as follows:

(Def. 57) $\text{KModel}(R, S_0, P_1) = \langle S, S_0, R, \text{Label } P_1 \rangle$.

Let S be a non empty set, let S_0 be a subset of S , let R be a total relation between S and S , and let P_1 be a non empty subset of $\text{ModelSP } S$. One can check that the worlds of $\text{KModel}(R, S_0, P_1)$ is non empty.

Let S be a non empty set, let S_0 be a subset of S , let R be a total relation between S and S , and let P_1 be a non empty subset of $\text{ModelSP } S$. One can verify that ModelSP (the worlds of $\text{KModel}(R, S_0, P_1)$) is non empty.

Let S be a non empty set, let R be a total relation between S and S , and let B_1 be a non empty subset of $\text{ModelSP } S$. The functor $\text{CTLModel}(R, B_1)$ yielding a CTL model structure is defined as follows:

(Def. 58) $\text{CTLModel}(R, B_1) = \langle \text{ModelSP } S, B_1, \text{ the and } S, \text{Not } S, \text{EneXt } R, \text{EGlobal } R, \text{EUntill } R \rangle$.

In the sequel B_1 is a non empty subset of $\text{ModelSP } S$ and k_1 is a function from the atomic WFF into the basic assignments of $\text{CTLModel}(R, B_1)$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, let s be an element of S , and let f be an assignment of $\text{CTLModel}(R, B_1)$. The predicate $s \models f$ is defined by:

(Def. 59) $(\text{Fid}(f, S))(s) = \text{true}$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, let s be an element of S , and let f be an assignment of $\text{CTLModel}(R, B_1)$. We introduce $s \not\models f$ as an antonym of $s \models f$.

Next we state several propositions:

- (11) For every assignment a of $\text{CTLModel}(R, B_1)$ such that $a \in B_1$ holds $s \models a$ iff $a \in (\text{Label } B_1)(s)$.
- (12) For every assignment f of $\text{CTLModel}(R, B_1)$ holds $s \models \neg f$ iff $s \not\models f$.
- (13) For all assignments f, g of $\text{CTLModel}(R, B_1)$ holds $s \models f \wedge g$ iff $s \models f$ and $s \models g$.
- (14) For every assignment f of $\text{CTLModel}(R, B_1)$ holds $s \models \text{EX } f$ iff there exists an infinity path p_1 of R such that $p_1(0) = s$ and $p_1(1) \models f$.
- (15) Let f be an assignment of $\text{CTLModel}(R, B_1)$. Then $s \models \text{EG } f$ if and only if there exists an infinity path p_1 of R such that $p_1(0) = s$ and for every element n of \mathbb{N} holds $p_1(n) \models f$.
- (16) Let f, g be assignments of $\text{CTLModel}(R, B_1)$. Then $s \models f \text{EU } g$ if and only if there exists an infinity path p_1 of R such that $p_1(0) = s$ and there exists an element m of \mathbb{N} such that for every element j of \mathbb{N} such that $j < m$ holds $p_1(j) \models f$ and $p_1(m) \models g$.
- (17) For all assignments f, g of $\text{CTLModel}(R, B_1)$ holds $s \models f \vee g$ iff $s \models f$ or $s \models g$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, let k_1 be a function from the atomic

WFF into the basic assignments of $\text{CTLModel}(R, B_1)$, let s be an element of S , and let H be a CTL-formula. The predicate $s \models_{k_1} H$ is defined by:

(Def. 60) $s \models \text{Evaluate}(H, k_1)$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, let k_1 be a function from the atomic WFF into the basic assignments of $\text{CTLModel}(R, B_1)$, let s be an element of S , and let H be a CTL-formula. We introduce $s \not\models_{k_1} H$ as an antonym of $s \models_{k_1} H$.

The following propositions are true:

- (18) If H is atomic, then $s \models_{k_1} H$ iff $k_1(H) \in (\text{Label } B_1)(s)$.
- (19) $s \models_{k_1} \neg H$ iff $s \not\models_{k_1} H$.
- (20) $s \models_{k_1} H_1 \wedge H_2$ iff $s \models_{k_1} H_1$ and $s \models_{k_1} H_2$.
- (21) $s \models_{k_1} H_1 \vee H_2$ iff $s \models_{k_1} H_1$ or $s \models_{k_1} H_2$.
- (22) $s \models_{k_1} \text{EX } H$ iff there exists an infinity path p_1 of R such that $p_1(0) = s$ and $p_1(1) \models_{k_1} H$.
- (23) $s \models_{k_1} \text{EG } H$ iff there exists an infinity path p_1 of R such that $p_1(0) = s$ and for every element n of \mathbb{N} holds $p_1(n) \models_{k_1} H$.
- (24) $s \models_{k_1} H_1 \text{EU } H_2$ if and only if there exists an infinity path p_1 of R such that $p_1(0) = s$ and there exists an element m of \mathbb{N} such that for every element j of \mathbb{N} such that $j < m$ holds $p_1(j) \models_{k_1} H_1$ and $p_1(m) \models_{k_1} H_2$.
- (25) For every s_0 there exists an infinity path p_1 of R such that $p_1(0) = s_0$.
- (26) Let R be a relation between S and S . Then R is total if and only if for every set x such that $x \in S$ there exists a set y such that $y \in S$ and $\langle x, y \rangle \in R$.

Let S be a non empty set, let R be a total relation between S and S , let s_0 be an element of S , let p_1 be an infinity path of R , and let n be a set. The functor $\text{PrePath}(n, s_0, p_1)$ yielding an element of S is defined as follows:

(Def. 61) $\text{PrePath}(n, s_0, p_1) = \begin{cases} s_0, & \text{if } n = 0, \\ p_1(\text{k.nat}(\text{k.nat } n - 1)), & \text{otherwise.} \end{cases}$

The following propositions are true:

- (27) If $\langle s_0, s_1 \rangle \in R$, then there exists an infinity path p_1 of R such that $p_1(0) = s_0$ and $p_1(1) = s_1$.
- (28) For every assignment f of $\text{CTLModel}(R, B_1)$ holds $s \models \text{EX } f$ iff there exists an element s_1 of S such that $\langle s, s_1 \rangle \in R$ and $s_1 \models f$.

Let S be a non empty set, let R be a total relation between S and S , and let H be a subset of S . The functor $\text{Pred}(H, R)$ yields a subset of S and is defined by:

(Def. 62) $\text{Pred}(H, R) = \{s; s \text{ ranges over elements of } S: \bigvee_{t: \text{element of } S} (t \in H \wedge \langle s, t \rangle \in R)\}$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, and let f be an assignation of $\text{CTLModel}(R, B_1)$. The functor $\text{SIGMA } f$ yields a subset of S and is defined as follows:

(Def. 63) $\text{SIGMA } f = \{s; s \text{ ranges over elements of } S: s \models f\}$.

One can prove the following proposition

(29) For all assignations f, g of $\text{CTLModel}(R, B_1)$ such that $\text{SIGMA } f = \text{SIGMA } g$ holds $f = g$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, and let T be a subset of S . The functor $\text{Tau}(T, R, B_1)$ yielding an assignation of $\text{CTLModel}(R, B_1)$ is defined as follows:

(Def. 64) For every set s such that $s \in S$ holds $(\text{Fid}(\text{Tau}(T, R, B_1), S))(s) = \chi_{T,S}(s)$.

The following propositions are true:

(30) For all subsets T_1, T_2 of S such that $\text{Tau}(T_1, R, B_1) = \text{Tau}(T_2, R, B_1)$ holds $T_1 = T_2$.

(31) For every assignation f of $\text{CTLModel}(R, B_1)$ holds $\text{Tau}(\text{SIGMA } f, R, B_1) = f$.

(32) For every subset T of S holds $\text{SIGMA } \text{Tau}(T, R, B_1) = T$.

(33) For all assignations f, g of $\text{CTLModel}(R, B_1)$ holds $\text{SIGMA } \neg f = S \setminus \text{SIGMA } f$ and $\text{SIGMA}(f \wedge g) = \text{SIGMA } f \cap \text{SIGMA } g$ and $\text{SIGMA}(f \vee g) = \text{SIGMA } f \cup \text{SIGMA } g$.

(34) For all subsets G_1, G_2 of S such that $G_1 \subseteq G_2$ and for every element s of S such that $s \models \text{Tau}(G_1, R, B_1)$ holds $s \models \text{Tau}(G_2, R, B_1)$.

(35) For all assignations f_1, f_2 of $\text{CTLModel}(R, B_1)$ such that for every element s of S such that $s \models f_1$ holds $s \models f_2$ holds $\text{SIGMA } f_1 \subseteq \text{SIGMA } f_2$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, and let f, g be assignations of $\text{CTLModel}(R, B_1)$. The functor $\text{Fax}(f, g)$ yielding an assignation of

$\text{CTLModel}(R, B_1)$ is defined by:

(Def. 65) $\text{Fax}(f, g) = f \wedge \text{EX } g$.

Next we state the proposition

(36) Let f, g_1, g_2 be assignations of $\text{CTLModel}(R, B_1)$. Suppose that for every element s of S such that $s \models g_1$ holds $s \models g_2$. Let s be an element of S . If $s \models \text{Fax}(f, g_1)$, then $s \models \text{Fax}(f, g_2)$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, let f be an assignation of $\text{CTLModel}(R, B_1)$, and let G be a subset of S . The functor $\text{SigFaxTau}(f, G, R, B_1)$ yielding a subset of S is defined by:

(Def. 66) $\text{SigFaxTau}(f, G, R, B_1) = \text{SIGMA Fax}(f, \text{Tau}(G, R, B_1))$.

One can prove the following proposition

(37) For every assignation f of $\text{CTLModel}(R, B_1)$ and for all subsets G_1, G_2 of S such that $G_1 \subseteq G_2$ holds $\text{SigFaxTau}(f, G_1, R, B_1) \subseteq \text{SigFaxTau}(f, G_2, R, B_1)$.

Let S be a non empty set, let R be a total relation between S and S , let p_1 be an infinity path of R , and let k be an element of \mathbb{N} . The functor $\text{PathShift}(p_1, k)$ yielding an infinity path of R is defined as follows:

(Def. 67) For every element n of \mathbb{N} holds $(\text{PathShift}(p_1, k))(n) = p_1(n + k)$.

Let S be a non empty set, let R be a total relation between S and S , let p_2, p_3 be infinity paths of R , and let n, k be elements of \mathbb{N} . The functor $\text{PathChange}(p_2, p_3, k, n)$ yielding a set is defined by:

(Def. 68) $\text{PathChange}(p_2, p_3, k, n) = \begin{cases} p_2(n), & \text{if } n < k, \\ p_3(n - k), & \text{otherwise.} \end{cases}$

Let S be a non empty set, let R be a total relation between S and S , let p_2, p_3 be infinity paths of R , and let k be an element of \mathbb{N} . The functor $\text{PathConc}(p_2, p_3, k)$ yielding a function from \mathbb{N} into S is defined as follows:

(Def. 69) For every element n of \mathbb{N} holds $(\text{PathConc}(p_2, p_3, k))(n) = \text{PathChange}(p_2, p_3, k, n)$.

We now state four propositions:

(38) Let p_2, p_3 be infinity paths of R and k be an element of \mathbb{N} . If $p_2(k) = p_3(0)$, then $\text{PathConc}(p_2, p_3, k)$ is an infinity path of R .

(39) For every assignation f of $\text{CTLModel}(R, B_1)$ and for every element s of S holds $s \models \text{EG } f$ iff $s \models \text{Fax}(f, \text{EG } f)$.

(40) Let g be an assignation of $\text{CTLModel}(R, B_1)$ and s_0 be an element of S . Suppose $s_0 \models g$. Suppose that for every element s of S such that $s \models g$ holds $s \models \text{EX } g$. Then there exists an infinity path p_1 of R such that $p_1(0) = s_0$ and for every element n of \mathbb{N} holds $p_1(n) \models g$.

(41) Let f, g be assignations of $\text{CTLModel}(R, B_1)$. Suppose that for every element s of S holds $s \models g$ iff $s \models \text{Fax}(f, g)$. Let s be an element of S . If $s \models g$, then $s \models \text{EG } f$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, and let f be an assignation of $\text{CTLModel}(R, B_1)$. The functor $\text{TransEG } f$ yielding a \subseteq -monotone function from 2^S into 2^S is defined as follows:

(Def. 70) For every subset G of S holds $(\text{TransEG } f)(G) = \text{SigFaxTau}(f, G, R, B_1)$.

One can prove the following two propositions:

(42) Let f, g be assignations of $\text{CTLModel}(R, B_1)$. Then for every element s of S holds $s \models g$ iff $s \models \text{Fax}(f, g)$ if and only if $\text{SIGMA } g$ is a fixpoint of

TransEG f .

- (43) For every assignation f of $\text{CTLModel}(R, B_1)$ holds $\text{SIGMA EG } f = \text{gfp}(S, \text{TransEG } f)$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, and let f, g, h be assignations of $\text{CTLModel}(R, B_1)$. The functor $\text{Foax}(g, f, h)$ yields an assignation of

$\text{CTLModel}(R, B_1)$ and is defined as follows:

- (Def. 71) $\text{Foax}(g, f, h) = g \vee \text{Fax}(f, h)$.

We now state the proposition

- (44) Let f, g, h_1, h_2 be assignations of $\text{CTLModel}(R, B_1)$. Suppose that for every element s of S such that $s \models h_1$ holds $s \models h_2$. Let s be an element of S . If $s \models \text{Foax}(g, f, h_1)$, then $s \models \text{Foax}(g, f, h_2)$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, let f, g be assignations of $\text{CTLModel}(R, B_1)$, and let H be a subset of S . The functor $\text{SigFoaxTau}(g, f, H, R, B_1)$ yields a subset of S and is defined as follows:

- (Def. 72) $\text{SigFoaxTau}(g, f, H, R, B_1) = \text{SIGMA Foax}(g, f, \text{Tau}(H, R, B_1))$.

Next we state three propositions:

- (45) For all assignations f, g of $\text{CTLModel}(R, B_1)$ and for all subsets H_1, H_2 of S such that $H_1 \subseteq H_2$ holds $\text{SigFoaxTau}(g, f, H_1, R, B_1) \subseteq \text{SigFoaxTau}(g, f, H_2, R, B_1)$.
- (46) For all assignations f, g of $\text{CTLModel}(R, B_1)$ and for every element s of S holds $s \models f \text{ EU } g$ iff $s \models \text{Foax}(g, f, f \text{ EU } g)$.
- (47) Let f, g, h be assignations of $\text{CTLModel}(R, B_1)$. Suppose that for every element s of S holds $s \models h$ iff $s \models \text{Foax}(g, f, h)$. Let s be an element of S . If $s \models f \text{ EU } g$, then $s \models h$.

Let S be a non empty set, let R be a total relation between S and S , let B_1 be a non empty subset of $\text{ModelSP } S$, and let f, g be assignations of $\text{CTLModel}(R, B_1)$. The functor $\text{TransEU}(f, g)$ yields a \subseteq -monotone function from 2^S into 2^S and is defined by:

- (Def. 73) For every subset H of S holds $(\text{TransEU}(f, g))(H) = \text{SigFoaxTau}(g, f, H, R, B_1)$.

One can prove the following propositions:

- (48) Let f, g, h be assignations of $\text{CTLModel}(R, B_1)$. Then for every element s of S holds $s \models h$ iff $s \models \text{Foax}(g, f, h)$ if and only if $\text{SIGMA } h$ is a fixpoint of $\text{TransEU}(f, g)$.
- (49) For all assignations f, g of $\text{CTLModel}(R, B_1)$ holds $\text{SIGMA}(f \text{ EU } g) = \text{lfp}(S, \text{TransEU}(f, g))$.

- (50) For every assignation f of $\text{CTLModel}(R, B_1)$ holds $\text{SIGMA EX } f = \text{Pred}(\text{SIGMA } f, R)$.
- (51) For every assignation f of $\text{CTLModel}(R, B_1)$ and for every subset X of S holds $(\text{TransEG } f)(X) = \text{SIGMA } f \cap \text{Pred}(X, R)$.
- (52) For all assignations f, g of $\text{CTLModel}(R, B_1)$ and for every subset X of S holds $(\text{TransEU}(f, g))(X) = \text{SIGMA } g \cup \text{SIGMA } f \cap \text{Pred}(X, R)$.

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Received November 14, 2006
