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# Recognizing Chordal Graphs: Lex BFS and MCS[1]

Broderick Arneson
University of Alberta
Edmonton, Canada

Piotr Rudnicki
University of Alberta
Edmonton, Canada

**Summary.** We are formalizing the algorithm for recognizing chordal graphs by lexicographic breadth-first search as presented in [13, Section 3 of Chapter 4, pp. 81–84]. Then we follow with a formalization of another algorithm serving the same end but based on maximum cardinality search as presented by Tarjan and Yannakakis [25].

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The notation and terminology used in this paper are introduced in the following articles: [28], [11], [26], [32], [33], [35], [30], [10], [7], [8], [20], [29], [4], [2], [14], [23], [12], [3], [6], [9], [18], [15], [19], [16], [17], [24], [21], [1], [5], [31], [27], [22], and [34].

## 1. Preliminaries

The following propositions are true:

(1) Let $A$, $B$ be elements of $\mathbb{N}$, $X$ be a non empty set, and $F$ be a function from $\mathbb{N}$ into $X$. If $F$ is one-to-one, then $\overline{\overline{\{F(w); w \text{ ranges over elements of } \mathbb{N}\colon A \leq w \land w \leq A+B\}}} = B+1$.

(2) For all natural numbers $n$, $m$, $k$ such that $m \leq k$ and $n < m$ holds $k -' m < k -' n$.

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(3) For all natural numbers $n$, $k$ such that $n < k$ holds $(k -' (n + 1)) + 1 = k -' n$.

(4) For all natural numbers $n$, $m$, $k$ such that $k \neq 0$ holds $(n + m \cdot k) \div k = (n \div k) + m$.

Let $S$ be a set. We say that $S$ has finite elements if and only if:

(Def. 1) Every element of $S$ is finite.

Let us note that there exists a set which is non empty and has finite elements and there exists a subset of $2^{\mathbb{N}}$ which is non empty and finite and has finite elements.

Let $S$ be a set with finite elements. One can check that every element of $S$ is finite.

Let $f$, $g$ be functions. The functor $f[\cup]g$ yielding a function is defined by:

(Def. 2) $\mathrm{dom}(f[\cup]g) = \mathrm{dom}\, f \cup \mathrm{dom}\, g$ and for every set $x$ such that $x \in \mathrm{dom}\, f \cup \mathrm{dom}\, g$ holds $(f[\cup]g)(x) = f(x) \cup g(x)$.

The following three propositions are true:

(5) For all natural numbers $m$, $n$, $k$ holds $m \in \mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' n)$ iff $k -' n < m$ and $m \leq k$.

(6) For all natural numbers $n$, $k$, $m$ such that $n \leq m$ holds $\mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' n) \subseteq \mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' m)$.

(7) For all natural numbers $n$, $k$ such that $n < k$ holds $(\mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' n)) \cup \{k -' n\} = \mathrm{Seg}\, k \setminus \mathrm{Seg}(k -' (n + 1))$.

Let $f$ be a binary relation. We say that $f$ is natsubset yielding if and only if:

(Def. 3) $\mathrm{rng}\, f \subseteq 2^{\mathbb{N}}$.

Let us mention that there exists a function which is finite-yielding and natsubset yielding.

Let $f$ be a finite-yielding natsubset yielding function and let $x$ be a set. Then $f(x)$ is a finite subset of $\mathbb{N}$.

One can prove the following proposition

(8) For every ordinal number $X$ and for all finite subsets $a$, $b$ of $X$ such that $a \neq b$ holds $(a, 1)$-bag $\neq (b, 1)$-bag.

Let $F$ be a natural-yielding function, let $S$ be a set, and let $k$ be a natural number. The functor $F$.incSubset$(S, k)$ yielding a natural-yielding function is defined by the conditions (Def. 4).

(Def. 4)(i) $\mathrm{dom}(F.\mathrm{incSubset}(S, k)) = \mathrm{dom}\, F$, and

(ii) for every set $y$ holds if $y \in S$ and $y \in \mathrm{dom}\, F$, then $(F.\mathrm{incSubset}(S, k))(y) = F(y) + k$ and if $y \notin S$, then $(F.\mathrm{incSubset}(S, k))(y) = F(y)$.

Let $n$ be an ordinal number, let $T$ be a connected term order of $n$, and let $B$ be a non empty finite subset of Bags $n$. The functor $\max(B,T)$ yields a bag of $n$ and is defined as follows:

(Def. 5) $\max(B,T) \in B$ and for every bag $x$ of $n$ such that $x \in B$ holds $x \leq_T \max(B,T)$.

Let $O$ be an ordinal number. Observe that InvLexOrder $O$ is connected.

## 2. Miscellany on Graphs

Let $G$ be a graph. Note that there exists a vertex sequence of $G$ which is non empty and one-to-one.

Let $G$ be a graph and let $V$ be a non empty vertex sequence of $G$. A walk of $G$ is called a walk of $V$ if:

(Def. 6) It.vertexSeq() $= V$.

Let $G$ be a graph and let $V$ be a non empty one-to-one vertex sequence of $G$. One can check that every walk of $V$ is path-like.

We now state two propositions:

(9) For every graph $G$ and for all walks $W_1$, $W_2$ of $G$ such that $W_1$ is trivial and $W_1$.last() $= W_2$.first() holds $W_1$.append($W_2$) $= W_2$.

(10) Let $G$, $H$ be graphs, $A$, $B$, $C$ be sets, $G_1$ be a subgraph of $G$ induced by $A$, $H_1$ be a subgraph of $H$ induced by $B$, $G_2$ be a subgraph of $G_1$ induced by $C$, and $H_2$ be a subgraph of $H_1$ induced by $C$. Suppose $G =_G H$ and $A \subseteq B$ and $C \subseteq A$ and $C$ is a non empty subset of the vertices of $G$. Then $G_2 =_G H_2$.

Let $G$ be a v-graph. We say that $G$ is natural v-labeled if and only if:

(Def. 7) The vlabel of $G$ is natural-yielding.

## 3. Graphs with Two Vertex Labels

The natural number V2-LabelSelector is defined by:

(Def. 8) V2-LabelSelector $= 8$.

Let $G$ be a graph structure. We say that $G$ is v2-labeled if and only if:

(Def. 9) V2-LabelSelector $\in \operatorname{dom} G$ and there exists a function $f$ such that $G$(V2-LabelSelector) $= f$ and $\operatorname{dom} f \subseteq$ the vertices of $G$.

Let us note that there exists a graph structure which is graph-like, weighted, elabeled, vlabeled, and v2-labeled.

A v2-graph is a v2-labeled graph. A vv-graph is a vlabeled v2-labeled graph.

Let $G$ be a v2-graph. The v2-label of $G$ yields a function and is defined as follows:

(Def. 10) The v2-label of $G = G(\text{V2-LabelSelector})$.

Next we state the proposition

(11) For every v2-graph $G$ holds dom (the v2-label of $G$) $\subseteq$ the vertices of $G$.

Let $G$ be a graph and let $X$ be a set. Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is graph-like.

We now state the proposition

(12) For every graph $G$ and for every set $X$ holds $G.\text{set}(\text{V2-LabelSelector}, X) =_G G$.

Let $G$ be a finite graph and let $X$ be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is finite.

Let $G$ be a loopless graph and let $X$ be a set.

Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is loopless.

Let $G$ be a trivial graph and let $X$ be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is trivial.

Let $G$ be a non trivial graph and let $X$ be a set. One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non trivial.

Let $G$ be a non-multi graph and let $X$ be a set. One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non-multi.

Let $G$ be a non-directed-multi graph and let $X$ be a set. One can verify that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non-directed-multi.

Let $G$ be a connected graph and let $X$ be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is connected.

Let $G$ be an acyclic graph and let $X$ be a set.

One can verify that $G.\text{set}(\text{V2-LabelSelector}, X)$ is acyclic.

Let $G$ be a v-graph and let $X$ be a set.

One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is vlabeled.

Let $G$ be a e-graph and let $X$ be a set. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is elabeled.

Let $G$ be a w-graph and let $X$ be a set. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is weighted.

Let $G$ be a v2-graph and let $X$ be a set.

One can verify that $G.\text{set}(\text{VLabelSelector}, X)$ is v2-labeled.

Let $G$ be a graph, let $Y$ be a set, and let $X$ be a partial function from the vertices of $G$ to $Y$. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is v2-labeled.

Let $G$ be a graph and let $X$ be a many sorted set indexed by the vertices of $G$. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is v2-labeled.

Let $G$ be a graph. One can verify that $G.\text{set}(\text{V2-LabelSelector}, \emptyset)$ is v2-labeled.

Let $G$ be a v2-graph. We say that $G$ is natural v2-labeled if and only if:

(Def. 11) The v2-label of $G$ is natural-yielding.

We say that $G$ is finite v2-labeled if and only if:

(Def. 12) The v2-label of $G$ is finite-yielding.

We say that $G$ is natsubset v2-labeled if and only if:

(Def. 13) The v2-label of $G$ is natsubset yielding.

One can check that there exists a weighted elabeled vlabeled v2-labeled graph which is finite, natural v-labeled, finite v2-labeled, natsubset v2-labeled, and chordal and there exists a weighted elabeled vlabeled v2-labeled graph which is finite, natural v-labeled, natural v2-labeled, and chordal.

Let $G$ be a natural v-labeled v-graph. Observe that the vlabel of $G$ is natural-yielding.

Let $G$ be a natural v2-labeled v2-graph. Observe that the v2-label of $G$ is natural-yielding.

Let $G$ be a finite v2-labeled v2-graph. Observe that the v2-label of $G$ is finite-yielding.

Let $G$ be a natsubset v2-labeled v2-graph. One can verify that the v2-label of $G$ is natsubset yielding.

Let $G$ be a vv-graph and let $v$, $x$ be sets. One can check that $G.\mathrm{labelVertex}(v, x)$ is v2-labeled.

Next we state the proposition

(13) For every vv-graph $G$ and for all sets $v$, $x$ holds the v2-label of $G =$ the v2-label of $G.\mathrm{labelVertex}(v, x)$.

Let $G$ be a natural v-labeled vv-graph, let $v$ be a set, and let $x$ be a natural number. Observe that $G.\mathrm{labelVertex}(v, x)$ is natural v-labeled.

Let $G$ be a natural v2-labeled vv-graph, let $v$ be a set, and let $x$ be a natural number. Observe that $G.\mathrm{labelVertex}(v, x)$ is natural v2-labeled.

Let $G$ be a finite v2-labeled vv-graph, let $v$ be a set, and let $x$ be a natural number. Note that $G.\mathrm{labelVertex}(v, x)$ is finite v2-labeled.

Let $G$ be a natsubset v2-labeled vv-graph, let $v$ be a set, and let $x$ be a natural number. One can check that $G.\mathrm{labelVertex}(v, x)$ is natsubset v2-labeled.

Let $G$ be a graph. Note that there exists a subgraph of $G$ which is vlabeled and v2-labeled.

Let $G$ be a v2-graph and let $G_2$ be a v2-labeled subgraph of $G$. We say that $G_2$ inherits v2-label if and only if:

(Def. 14) The v2-label of $G_2 =$ (the v2-label of $G$)$\restriction$(the vertices of $G_2$).

Let $G$ be a v2-graph. Note that there exists a v2-labeled subgraph of $G$ which inherits v2-label.

Let $G$ be a v2-graph. A v2-subgraph of $G$ is a v2-labeled subgraph of $G$ inheriting v2-label.

Let $G$ be a vv-graph. Note that there exists a vlabeled v2-labeled subgraph of $G$ which inherits vlabel and v2-label.

Let $G$ be a vv-graph. A vv-subgraph of $G$ is a vlabeled v2-labeled subgraph of $G$ inheriting vlabel and v2-label.

Let $G$ be a natural v-labeled v-graph. Note that every v-subgraph of $G$ is natural v-labeled.

Let $G$ be a graph and let $V$, $E$ be sets. Observe that there exists a subgraph of $G$ induced by $V$ and $E$ which is weighted, elabeled, vlabeled, and v2-labeled.

Let $G$ be a vv-graph and let $V$, $E$ be sets. Observe that there exists a vlabeled v2-labeled subgraph of $G$ induced by $V$ and $E$ which inherits vlabel and v2-label.

Let $G$ be a vv-graph and let $V$, $E$ be sets. A $(V, E)$-induced vv-subgraph of $G$ is a vlabeled v2-labeled subgraph of $G$ induced by $V$ and $E$ inheriting vlabel and v2-label.

Let $G$ be a vv-graph and let $V$ be a set. A $V$-induced vv-subgraph of $G$ is a $(V, G.\mathrm{edgesBetween}(V))$-induced vv-subgraph of $G$.

## 4. More on Graph Sequences

Let $s$ be a many sorted set indexed by $\mathbb{N}$. We say that $s$ is iterative if and only if:

(Def. 15) For all natural numbers $k$, $n$ such that $s(k) = s(n)$ holds $s(k+1) = s(n+1)$.

Let $G_3$ be a many sorted set indexed by $\mathbb{N}$. We say that $G_3$ is eventually constant if and only if:

(Def. 16) There exists a natural number $n$ such that for every natural number $m$ such that $n \leq m$ holds $G_3(n) = G_3(m)$.

Let us observe that there exists a many sorted set indexed by $\mathbb{N}$ which is halting, iterative, and eventually constant.

The following proposition is true

(14) For every many sorted set $G_4$ indexed by $\mathbb{N}$ such that $G_4$ is halting and iterative holds $G_4$ is eventually constant.

One can check that every many sorted set indexed by $\mathbb{N}$ which is halting and iterative is also eventually constant.

The following proposition is true

(15) For every many sorted set $G_4$ indexed by $\mathbb{N}$ such that $G_4$ is eventually constant holds $G_4$ is halting.

Let us mention that every many sorted set indexed by $\mathbb{N}$ which is eventually constant is also halting.

One can prove the following two propositions:

(16) Let $G_4$ be an iterative eventually constant many sorted set indexed by $\mathbb{N}$ and $n$ be a natural number. If $G_4.\mathrm{Lifespan}() \leq n$, then $G_4(G_4.\mathrm{Lifespan}()) = G_4(n)$.

(17) Let $G_4$ be an iterative eventually constant many sorted set indexed by $\mathbb{N}$ and $n$, $m$ be natural numbers. If $G_4$.Lifespan() $\leq n$ and $n \leq m$, then $G_4(m) = G_4(n)$.

Let $G_3$ be a v-graph sequence. We say that $G_3$ is natural v-labeled if and only if:

(Def. 17) For every natural number $x$ holds $G_3(x)$ is natural v-labeled.

Let $G_3$ be a graph sequence. We say that $G_3$ is chordal if and only if:

(Def. 18) For every natural number $x$ holds $G_3(x)$ is chordal.

We say that $G_3$ has fixed vertices if and only if:

(Def. 19) For all natural numbers $n$, $m$ holds the vertices of $G_3(n)$ = the vertices of $G_3(m)$.

We say that $G_3$ is v2-labeled if and only if:

(Def. 20) For every natural number $x$ holds $G_3(x)$ is v2-labeled.

Let us observe that there exists a graph sequence which is weighted, elabeled, vlabeled, and v2-labeled.

A v2-graph sequence is a v2-labeled graph sequence. A vv-graph sequence is a vlabeled v2-labeled graph sequence.

Let $G_5$ be a v2-graph sequence and let $x$ be a natural number. Note that $G_5(x)$ is v2-labeled.

Let $G_5$ be a v2-graph sequence. We say that $G_5$ is natural v2-labeled if and only if:

(Def. 21) For every natural number $x$ holds $G_5(x)$ is natural v2-labeled.

We say that $G_5$ is finite v2-labeled if and only if:

(Def. 22) For every natural number $x$ holds $G_5(x)$ is finite v2-labeled.

We say that $G_5$ is natsubset v2-labeled if and only if:

(Def. 23) For every natural number $x$ holds $G_5(x)$ is natsubset v2-labeled.

Let us mention that there exists a weighted elabeled vlabeled v2-labeled graph sequence which is finite, natural v-labeled, finite v2-labeled, natsubset v2-labeled, and chordal and there exists a weighted elabeled vlabeled v2-labeled graph sequence which is finite, natural v-labeled, natural v2-labeled, and chordal.

Let $G_4$ be a v-graph sequence and let $x$ be a natural number. Then $G_4(x)$ is a v-graph.

Let $G_5$ be a natural v-labeled v-graph sequence and let $x$ be a natural number. Observe that $G_5(x)$ is natural v-labeled.

Let $G_5$ be a natural v2-labeled v2-graph sequence and let $x$ be a natural number. One can check that $G_5(x)$ is natural v2-labeled.

Let $G_5$ be a finite v2-labeled v2-graph sequence and let $x$ be a natural number. One can verify that $G_5(x)$ is finite v2-labeled.

Let $G_5$ be a natsubset v2-labeled v2-graph sequence and let $x$ be a natural number. Note that $G_5(x)$ is natsubset v2-labeled.

Let $G_5$ be a chordal graph sequence and let $x$ be a natural number. One can check that $G_5(x)$ is chordal.

Let $G_4$ be a v-graph sequence and let $n$ be a natural number. Then $G_4(n)$ is a v-graph.

Let $G_4$ be a finite v-graph sequence and let $n$ be a natural number. One can check that $G_4(n)$ is finite.

Let $G_4$ be a vv-graph sequence and let $n$ be a natural number. Then $G_4(n)$ is a vv-graph.

Let $G_4$ be a finite vv-graph sequence and let $n$ be a natural number. One can verify that $G_4(n)$ is finite.

Let $G_4$ be a chordal vv-graph sequence and let $n$ be a natural number. Note that $G_4(n)$ is chordal.

Let $G_4$ be a natural v-labeled vv-graph sequence and let $n$ be a natural number. One can check that $G_4(n)$ is natural v-labeled.

Let $G_4$ be a finite v2-labeled vv-graph sequence and let $n$ be a natural number. Note that $G_4(n)$ is finite v2-labeled.

Let $G_4$ be a natsubset v2-labeled vv-graph sequence and let $n$ be a natural number. One can check that $G_4(n)$ is natsubset v2-labeled.

Let $G_4$ be a natural v2-labeled vv-graph sequence and let $n$ be a natural number. Observe that $G_4(n)$ is natural v2-labeled.

## 5. Vertices Numbering Sequences

Let $G_3$ be a v-graph sequence. We say that $G_3$ has initially empty v-label if and only if:

(Def. 24) The vlabel of $G_3(0) = \emptyset$.

We say that $G_3$ is adding one at a step if and only if the condition (Def. 25) is satisfied.

(Def. 25) Let $n$ be a natural number. Suppose $n < G_3.\mathrm{Lifespan}()$. Then there exists a set $w$ such that $w \notin \mathrm{dom}\,(\text{the vlabel of } G_3(n))$ and the vlabel of $G_3(n+1) = (\text{the vlabel of } G_3(n)) {+\cdot} (w \dot{\longmapsto} (G_3.\mathrm{Lifespan}() -' n))$.

Let $G_3$ be a v-graph sequence. We say that $G_3$ is v-label numbering if and only if the condition (Def. 26) is satisfied.

(Def. 26) $G_3$ is iterative, eventually constant, finite, natural v-labeled, and adding one at a step and has fixed vertices and initially empty v-label.

One can check that there exists a v-graph sequence which is iterative, eventually constant, finite, natural v-labeled, and adding one at a step and has fixed vertices and initially empty v-label.

Let us observe that there exists a v-graph sequence which is v-label numbering.

One can check the following observations:

* every v-graph sequence which is v-label numbering is also iterative,
* every v-graph sequence which is v-label numbering is also eventually constant,
* every v-graph sequence which is v-label numbering is also finite,
* every v-graph sequence which is v-label numbering has also fixed vertices,
* every v-graph sequence which is v-label numbering is also natural v-labeled,
* every v-graph sequence which is v-label numbering has also initially empty v-label, and
* every v-graph sequence which is v-label numbering is also adding one at a step.

A v-label numbering sequence is a v-label numbering v-graph sequence.

Let $G_3$ be a v-label numbering sequence and let $n$ be a natural number. The functor $G_3\,.\mathrm{PickedAt}\,n$ yields a set and is defined by:

(Def. 27)(i) $G_3\,.\mathrm{PickedAt}\,n = \mathrm{choose}(\text{the vertices of } G_3(0))$ if $n \geq G_3.\mathrm{Lifespan}()$,

(ii) $G_3\,.\mathrm{PickedAt}\,n \notin \mathrm{dom}\,(\text{the vlabel of } G_3(n))$ and the vlabel of $G_3(n+1) = (\text{the vlabel of } G_3(n)) +\cdot ((G_3\,.\mathrm{PickedAt}\,n) \dot{\longmapsto} (G_3.\mathrm{Lifespan}() -' n))$, otherwise.

The following propositions are true:

(18) Let $G_3$ be a v-label numbering sequence and $n$ be a natural number. If $n < G_3.\mathrm{Lifespan}()$, then $G_3\,.\mathrm{PickedAt}\,n \in G_3(n+1).\mathrm{labeledV}()$ and $G_3(n+1).\mathrm{labeledV}() = G_3(n).\mathrm{labeledV}() \cup \{G_3\,.\mathrm{PickedAt}\,n\}$.

(19) Let $G_3$ be a v-label numbering sequence and $n$ be a natural number. If $n < G_3.\mathrm{Lifespan}()$, then $(\text{the vlabel of } G_3(n+1))(G_3\,.\mathrm{PickedAt}\,n) = G_3.\mathrm{Lifespan}() -' n$.

(20) For every v-label numbering sequence $G_3$ and for every natural number $n$ such that $n \leq G_3.\mathrm{Lifespan}()$ holds $\mathrm{card}(G_3(n).\mathrm{labeledV}()) = n$.

(21) For every v-label numbering sequence $G_3$ and for every natural number $n$ holds $\mathrm{rng}\,(\text{the vlabel of } G_3(n)) = \mathrm{Seg}(G_3.\mathrm{Lifespan}()) \setminus \mathrm{Seg}(G_3.\mathrm{Lifespan}() -' n)$.

(22) Let $G_3$ be a v-label numbering sequence, $n$ be a natural number, and $x$ be a set. Then $(\text{the vlabel of } G_3(n))(x) \leq G_3.\mathrm{Lifespan}()$ and if $x \in G_3(n).\mathrm{labeledV}()$, then $1 \leq (\text{the vlabel of } G_3(n))(x)$.

(23) Let $G_3$ be a v-label numbering sequence and $n$, $m$ be natural numbers. Suppose $G_3.\mathrm{Lifespan}() -' n < m$ and $m \leq G_3.\mathrm{Lifespan}()$. Then there exists a vertex $v$ of $G_3(n)$ such that $v \in G_3(n).\mathrm{labeledV}()$ and (the vlabel

of $G_3(n))(v) = m$.

(24) Let $G_3$ be a v-label numbering sequence and $m$, $n$ be natural numbers. If $m \leq n$, then the vlabel of $G_3(m) \subseteq$ the vlabel of $G_3(n)$.

(25) For every v-label numbering sequence $G_3$ and for every natural number $n$ holds the vlabel of $G_3(n)$ is one-to-one.

(26) Let $G_3$ be a v-label numbering sequence, $m$, $n$ be natural numbers, and $v$ be a set. Suppose $v \in G_3(m)$.labeledV() and $v \in G_3(n)$.labeledV(). Then (the vlabel of $G_3(m))(v) =$ (the vlabel of $G_3(n))(v)$.

(27) Let $G_3$ be a v-label numbering sequence, $v$ be a set, and $m$, $n$ be natural numbers. If $v \in G_3(m)$.labeledV() and (the vlabel of $G_3(m))(v) = n$, then $G_3$ .PickedAt($G_3$.Lifespan() $-'$ $n$) $= v$.

(28) Let $G_3$ be a v-label numbering sequence and $m$, $n$ be natural numbers. If $n < G_3$.Lifespan() and $n < m$, then $G_3$ .PickedAt $n \in G_3(m)$.labeledV() and (the vlabel of $G_3(m))(G_3$ .PickedAt $n) = G_3$.Lifespan() $-'$ $n$.

(29) Let $G_3$ be a v-label numbering sequence, $m$ be a natural number, and $v$ be a set. Suppose $v \in G_3(m)$.labeledV(). Then $G_3$.Lifespan() $-'$ (the vlabel of $G_3(m))(v) < m$ and $G_3$.Lifespan() $-'$ $m <$ (the vlabel of $G_3(m))(v)$.

(30) Let $G_3$ be a v-label numbering sequence, $i$ be a natural number, and $a$, $b$ be sets. Suppose $a \in G_3(i)$.labeledV() and $b \in G_3(i)$.labeledV() and (the vlabel of $G_3(i))(a) <$ (the vlabel of $G_3(i))(b)$. Then $b \in G_3(G_3$.Lifespan() $-'$ (the vlabel of $G_3(i))(a))$.labeledV().

(31) Let $G_3$ be a v-label numbering sequence, $i$ be a natural number, and $a$, $b$ be sets. Suppose $a \in G_3(i)$.labeledV() and $b \in G_3(i)$.labeledV() and (the vlabel of $G_3(i))(a) <$ (the vlabel of $G_3(i))(b)$. Then $a \notin G_3(G_3$.Lifespan() $-'$ (the vlabel of $G_3(i))(b))$.labeledV().

## 6. Lexicographical Breadth-First Search

Let $G$ be a graph. The functor LexBFS:Init $G$ yields a natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph and is defined as follows:

(Def. 28) LexBFS:Init $G$ $=$ $G$.set(VLabelSelector, $\emptyset$).set(V2-LabelSelector, (the vertices of $G) \longmapsto \emptyset$).

Let $G$ be a finite graph. Then LexBFS:Init $G$ is a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph.

Let $G$ be a finite finite v2-labeled natsubset v2-labeled vv-graph. Let us assume that dom (the v2-label of $G$) $=$ the vertices of $G$. The functor LexBFS:PickUnnumbered $G$ yields a vertex of $G$ and is defined by:

(Def. 29)(i) LexBFS:PickUnnumbered $G$ $=$ choose(the vertices of $G$) if dom (the vlabel of $G$) $=$ the vertices of $G$,

(ii) there exists a non empty finite subset $S$ of $2^{\mathbb{N}}$ and there exists a non empty finite subset $B$ of Bags $\mathbb{N}$ and there exists a function $F$ such that $S = \operatorname{rng} F$ and $F =$ (the v2-label of $G$)$\restriction$((the vertices of $G$) $\setminus$ dom (the vlabel of $G$)) and for every finite subset $x$ of $\mathbb{N}$ such that $x \in S$ holds $(x, 1)$-bag $\in B$ and for every set $x$ such that $x \in B$ there exists a finite subset $y$ of $\mathbb{N}$ such that $y \in S$ and $x = (y, 1)$-bag and LexBFS:PickUnnumbered $G =$ choose$(F^{-1}(\{\text{support} \max(B, \text{InvLexOrder}\,\mathbb{N})\}))$, otherwise.

Let $G$ be a vv-graph, let $v$ be a set, and let $k$ be a natural number. The functor LexBFS:LabelAdjacent$(G, v, k)$ yielding a vv-graph is defined as follows:

(Def. 30) LexBFS:LabelAdjacent$(G, v, k) = G$.set(V2-LabelSelector, (the v2-label of $G$)$[\cup]$(($G$.adjacentSet($\{v\}$)) $\setminus$ dom (the vlabel of $G$) $\longmapsto \{k\}$)).

Next we state four propositions:

(32) Let $G$ be a vv-graph, $v$, $x$ be sets, and $k$ be a natural number. If $x \notin G$.adjacentSet($\{v\}$), then (the v2-label of $G$)$(x)$ = (the v2-label of LexBFS:LabelAdjacent$(G, v, k)$)$(x)$.

(33) Let $G$ be a vv-graph, $v$, $x$ be sets, and $k$ be a natural number. Suppose $x \in$ dom (the vlabel of $G$). Then (the v2-label of $G$)$(x)$ = (the v2-label of LexBFS:LabelAdjacent$(G, v, k)$)$(x)$.

(34) Let $G$ be a vv-graph, $v$, $x$ be sets, and $k$ be a natural number. Suppose $x \in G$.adjacentSet($\{v\}$) and $x \notin$ dom (the vlabel of $G$). Then (the v2-label of LexBFS:LabelAdjacent$(G, v, k)$)$(x)$ = (the v2-label of $G$)$(x) \cup \{k\}$.

(35) Let $G$ be a vv-graph, $v$ be a set, and $k$ be a natural number. Suppose dom (the v2-label of $G$) = the vertices of $G$. Then dom (the v2-label of LexBFS:LabelAdjacent$(G, v, k)$) = the vertices of $G$.

Let $G$ be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph, let $v$ be a vertex of $G$, and let $k$ be a natural number. Then LexBFS:LabelAdjacent$(G, v, k)$ is a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph.

Let $G$ be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph, let $v$ be a vertex of $G$, and let $n$ be a natural number. The functor LexBFS:Update$(G, v, n)$ yielding a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph is defined by:

(Def. 31) LexBFS:Update$(G, v, n) =$
LexBFS:LabelAdjacent$(G.\text{labelVertex}(v, G.\text{order}() -' n), v, G.\text{order}() -' n)$.

Let $G$ be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph. The functor LexBFS:Step $G$ yields a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph and is defined as follows:

(Def. 32) $$\text{LexBFS:Step}\,G = \begin{cases} G, & \text{if } G.\text{order}() \leq \text{card dom (the vlabel of } G), \\ \text{LexBFS:Update}(G, \text{LexBFS:PickUnnumbered}\,G, & \\ \quad \text{card dom (the vlabel of } G)), & \text{otherwise.} \end{cases}$$

Let $G$ be a finite graph. The functor LexBFS:CSeq $G$ yields a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph sequence and is defined by:

(Def. 33) (LexBFS:CSeq $G$)(0) = LexBFS:Init $G$ and for every natural number $n$ holds (LexBFS:CSeq $G$)($n+1$) = LexBFS:Step(LexBFS:CSeq $G$)($n$).

We now state the proposition

(36) For every finite graph $G$ holds LexBFS:CSeq $G$ is iterative.

Let $G$ be a finite graph. Observe that LexBFS:CSeq $G$ is iterative.

Next we state a number of propositions:

(37) For every graph $G$ holds the vlabel of LexBFS:Init $G = \emptyset$.

(38) Let $G$ be a graph and $v$ be a set. Then dom (the v2-label of LexBFS:Init $G$) = the vertices of $G$ and (the v2-label of LexBFS:Init $G$)($v$) $= \emptyset$.

(39) For every graph $G$ holds $G =_G$ LexBFS:Init $G$.

(40) Let $G$ be a finite finite v2-labeled natsubset v2-labeled vv-graph and $x$ be a set. Suppose that

(i) $x \notin$ dom (the vlabel of $G$),

(ii) dom (the v2-label of $G$) = the vertices of $G$, and

(iii) dom (the vlabel of $G$) $\neq$ the vertices of $G$.

Then ((the v2-label of $G$)($x$), 1) -bag $\leq_{\text{InvLexOrder}\,\mathbb{N}}$ ((the v2-label of $G$)(LexBFS:PickUnnumbered $G$), 1) -bag .

(41) Let $G$ be a finite finite v2-labeled natsubset v2-labeled vv-graph. Suppose dom (the v2-label of $G$) = the vertices of $G$ and dom (the vlabel of $G$) $\neq$ the vertices of $G$. Then LexBFS:PickUnnumbered $G \notin$ dom (the vlabel of $G$).

(42) For every finite graph $G$ and for every natural number $n$ holds (LexBFS:CSeq $G$)($n$) $=_G G$.

(43) For every finite graph $G$ and for all natural numbers $m$, $n$ holds (LexBFS:CSeq $G$)($m$) $=_G$ (LexBFS:CSeq $G$)($n$).

(44) Let $G$ be a finite graph and $n$ be a natural number. Suppose card dom (the vlabel of (LexBFS:CSeq $G$)($n$)) $<$ $G$.order(). Then the vlabel of (LexBFS:CSeq $G$)($n+1$) = (the vlabel of (LexBFS:CSeq $G$)($n$))$+\cdot$(LexBFS:PickUnnumbered(LexBFS:CSeq $G$)($n$) $\dot{\longmapsto}$($G$.order() $-'$ card dom (the vlabel of (LexBFS:CSeq $G$)($n$)))).

(45) For every finite graph $G$ and for every natural number $n$ holds dom (the v2-label of (LexBFS:CSeq $G$)($n$)) = the vertices of (LexBFS:CSeq $G$)($n$).

(46) For every finite graph $G$ and for every natural number $n$ such that $n \leq$ $G$.order() holds card dom (the vlabel of (LexBFS:CSeq $G$)($n$)) $= n$.

(47) For every finite graph $G$ and for every natural number $n$ such that $G$.order() $\leq n$ holds (LexBFS:CSeq $G$)($G$.order()) =

(LexBFS:CSeq $G$)($n$).

(48) For every finite graph $G$ and for all natural numbers $m$, $n$ such that $G$.order() $\leq m$ and $m \leq n$ holds (LexBFS:CSeq $G$)($m$) $=$ (LexBFS:CSeq $G$)($n$).

(49) For every finite graph $G$ holds LexBFS:CSeq $G$ is eventually constant.

Let $G$ be a finite graph. Note that LexBFS:CSeq $G$ is eventually constant.

We now state two propositions:

(50) Let $G$ be a finite graph and $n$ be a natural number. Then dom (the vlabel of (LexBFS:CSeq $G$)($n$)) $=$ the vertices of (LexBFS:CSeq $G$)($n$) if and only if $G$.order() $\leq n$.

(51) For every finite graph $G$ holds (LexBFS:CSeq $G$).Lifespan() $=$ $G$.order().

Let $G$ be a finite chordal graph and let $i$ be a natural number. One can check that (LexBFS:CSeq $G$)($i$) is chordal.

Let $G$ be a finite chordal graph. One can check that LexBFS:CSeq $G$ is chordal.

One can prove the following proposition

(52) For every finite graph $G$ holds LexBFS:CSeq $G$ is v-label numbering.

Let $G$ be a finite graph. Note that LexBFS:CSeq $G$ is v-label numbering.

We now state several propositions:

(53) For every finite graph $G$ and for every natural number $n$ such that $n <$ $G$.order() holds LexBFS:CSeq $G$ .PickedAt $n$ $=$ LexBFS:PickUnnumbered(LexBFS:CSeq $G$)($n$).

(54) Let $G$ be a finite graph and $n$ be a natural number. Suppose $n <$ $G$.order(). Then there exists a vertex $w$ of (LexBFS:CSeq $G$)($n$) such that

  (i) $w =$ LexBFS:PickUnnumbered(LexBFS:CSeq $G$)($n$), and

  (ii) for every set $v$ holds if $v \in G$.adjacentSet($\{w\}$) and $v \notin$ dom (the vlabel of (LexBFS:CSeq $G$)($n$)), then (the v2-label of (LexBFS:CSeq $G$)($n+1$))($v$) $=$ (the v2-label of (LexBFS:CSeq $G$)($n$))($v$)$\cup\{G$.order()$-'n\}$ and if $v \notin G$.adjacentSet($\{w\}$) or $v \in$ dom (the vlabel of (LexBFS:CSeq $G$)($n$)), then (the v2-label of (LexBFS:CSeq $G$)($n+1$))($v$) $=$ (the v2-label of (LexBFS:CSeq $G$)($n$))($v$).

(55) Let $G$ be a finite graph, $i$ be a natural number, and $v$ be a set. Then (the v2-label of (LexBFS:CSeq $G$)($i$))($v$) $\subseteq$ Seg($G$.order())$\setminus$Seg($G$.order()$-'i$).

(56) Let $G$ be a finite graph, $x$ be a set, and $i$, $j$ be natural numbers. If $i \leq j$, then (the v2-label of (LexBFS:CSeq $G$)($i$))($x$) $\subseteq$ (the v2-label of (LexBFS:CSeq $G$)($j$))($x$).

(57) Let $G$ be a finite graph, $m$, $n$ be natural numbers, and $x$, $y$ be sets. Suppose $n <$ $G$.order() and $n < m$ and $y =$ LexBFS:PickUnnumbered(LexBFS:CSeq $G$)($n$) and $x \notin$ dom (the vlabel of

(LexBFS:CSeq $G$)($n$)) and $x \in G$.adjacentSet($\{y\}$). Then $G$.order() $-'$ $n \in$ (the v2-label of (LexBFS:CSeq $G$)($m$))($x$).

(58) Let $G$ be a finite graph and $m$, $n$ be natural numbers. Suppose $m < n$. Let $x$ be a set. Suppose $G$.order() $-'$ $m \notin$ (the v2-label of (LexBFS:CSeq $G$)($m+1$))($x$). Then $G$.order() $-'$ $m \notin$ (the v2-label of (LexBFS:CSeq $G$)($n$))($x$).

(59) Let $G$ be a finite graph and $m$, $n$, $k$ be natural numbers. Suppose $k < n$ and $n \leq m$. Let $x$ be a set. Suppose $G$.order() $-'$ $k \notin$ (the v2-label of (LexBFS:CSeq $G$)($n$))($x$). Then $G$.order() $-'$ $k \notin$ (the v2-label of (LexBFS:CSeq $G$)($m$))($x$).

(60) Let $G$ be a finite graph, $m$, $n$ be natural numbers, and $x$ be a vertex of (LexBFS:CSeq $G$)($m$). Suppose $n \in$ (the v2-label of (LexBFS:CSeq $G$)($m$))($x$). Then there exists a vertex $y$ of (LexBFS:CSeq $G$)($m$) such that LexBFS:PickUnnumbered(LexBFS:CSeq $G$) ($G$.order()$-'n$) $= y$ and $y \notin$ dom (the vlabel of (LexBFS:CSeq $G$)($G$.order()$-'$ $n$)) and $x \in G$.adjacentSet($\{y\}$).

Let $G_4$ be a finite natural v-labeled vv-graph sequence. Then $G_4$.Result() is a finite natural v-labeled vv-graph.

The following four propositions are true:

(61) For every finite graph $G$ holds (LexBFS:CSeq $G$).Result().labeledV() = the vertices of $G$.

(62) For every finite graph $G$ holds (the vlabel of (LexBFS:CSeq $G$).Result()$)^{-1}$ is a vertex scheme of $G$.

(63) Let $G$ be a finite graph, $i$, $j$ be natural numbers, and $a$, $b$ be vertices of (LexBFS:CSeq $G$)($i$). Suppose that

(i) $a \in$ dom (the vlabel of (LexBFS:CSeq $G$)($i$)),

(ii) $b \in$ dom (the vlabel of (LexBFS:CSeq $G$)($i$)),

(iii) (the vlabel of (LexBFS:CSeq $G$)($i$))($a$) $<$ (the vlabel of (LexBFS:CSeq $G$)($i$))($b$), and

(iv) $j = G$.order() $-'$ (the vlabel of (LexBFS:CSeq $G$)($i$))($b$).

Then ((the v2-label of (LexBFS:CSeq $G$)($j$))($a$), 1)-bag $\leq_{\mathrm{InvLexOrder}\,\mathbb{N}}$ ((the v2-label of (LexBFS:CSeq $G$)($j$))($b$), 1)-bag.

(64) Let $G$ be a finite graph, $i$, $j$ be natural numbers, and $v$ be a vertex of (LexBFS:CSeq $G$)($i$). Suppose $j \in$ (the v2-label of (LexBFS:CSeq $G$)($i$))($v$). Then there exists a vertex $w$ of (LexBFS:CSeq $G$)($i$) such that $w \in$ dom (the vlabel of (LexBFS:CSeq $G$)($i$)) and (the vlabel of (LexBFS:CSeq $G$)($i$))($w$) $= j$ and $v \in G$.adjacentSet($\{w\}$).

Let $G$ be a natural v-labeled v-graph. We say that $G$ has property *L3* if and only if the condition (Def. 34) is satisfied.

(Def. 34) Let $a$, $b$, $c$ be vertices of $G$. Suppose that $a \in \mathrm{dom}\,$(the vlabel of $G$) and $b \in \mathrm{dom}\,$(the vlabel of $G$) and $c \in \mathrm{dom}\,$(the vlabel of $G$) and (the vlabel of $G$)($a$) $<$ (the vlabel of $G$)($b$) and (the vlabel of $G$)($b$) $<$ (the vlabel of $G$)($c$) and $a$ and $c$ are adjacent and $b$ and $c$ are not adjacent. Then there exists a vertex $d$ of $G$ such that

(i) $d \in \mathrm{dom}\,$(the vlabel of $G$),

(ii) (the vlabel of $G$)($c$) $<$ (the vlabel of $G$)($d$),

(iii) $b$ and $d$ are adjacent,

(iv) $a$ and $d$ are not adjacent, and

(v) for every vertex $e$ of $G$ such that $e \neq d$ and $e$ and $b$ are adjacent and $e$ and $a$ are not adjacent holds (the vlabel of $G$)($e$) $<$ (the vlabel of $G$)($d$).

One can prove the following three propositions:

(65) For every finite graph $G$ and for every natural number $n$ holds (LexBFS:CSeq $G$)($n$) has property *L3*.

(66) Let $G$ be a finite chordal natural v-labeled v-graph. Suppose $G$ has property *L3* and dom (the vlabel of $G$) = the vertices of $G$. Let $V$ be a vertex scheme of $G$. If $V^{-1}$ = the vlabel of $G$, then $V$ is perfect.

(67) For every finite chordal vv-graph $G$ holds (the vlabel of (LexBFS:CSeq $G$).Result())$^{-1}$ is a perfect vertex scheme of $G$.

## 7. The Maximum Cardinality Search Algorithm

Let $G$ be a finite graph. The functor MCS:Init $G$ yields a finite natural v-labeled natural v2-labeled vv-graph and is defined by:

(Def. 35) MCS:Init $G$ = $G$.set(VLabelSelector, $\emptyset$).set(V2-LabelSelector, (the vertices of $G$) $\longmapsto 0$).

Let $G$ be a finite natural v2-labeled vv-graph. Let us assume that dom (the v2-label of $G$) = the vertices of $G$. The functor MCS:PickUnnumbered $G$ yields a vertex of $G$ and is defined by:

(Def. 36)(i) MCS:PickUnnumbered $G$ = choose(the vertices of $G$) if dom (the vlabel of $G$) = the vertices of $G$,

(ii) there exists a finite non empty natural-membered set $S$ and there exists a function $F$ such that $S = \mathrm{rng}\, F$ and $F$ = (the v2-label of $G$)$\restriction$((the vertices of $G$) $\setminus$ dom (the vlabel of $G$)) and MCS:PickUnnumbered $G$ = choose($F^{-1}(\{\max S\})$), otherwise.

Let $G$ be a finite natural v2-labeled vv-graph and let $v$ be a set. The functor MCS:LabelAdjacent($G, v$) yields a finite natural v2-labeled vv-graph and is defined by:

(Def. 37) MCS:LabelAdjacent$(G, v)$ = $G$.set(V2-LabelSelector, (the v2-label of $G$) .incSubset$((G$.adjacentSet$(\{v\})) \setminus$ dom (the vlabel of $G$), 1)).

Let $G$ be a finite natural v-labeled natural v2-labeled vv-graph and let $v$ be a vertex of $G$. Then MCS:LabelAdjacent$(G, v)$ is a finite natural v-labeled natural v2-labeled vv-graph.

Let $G$ be a finite natural v-labeled natural v2-labeled vv-graph, let $v$ be a vertex of $G$, and let $n$ be a natural number. The functor MCS:Update$(G, v, n)$ yielding a finite natural v-labeled natural v2-labeled vv-graph is defined as follows:

(Def. 38) MCS:Update$(G, v, n)$ = MCS:LabelAdjacent$(G$.labelVertex$(v, G$.order$()-' n), v)$.

Let $G$ be a finite natural v-labeled natural v2-labeled vv-graph. The functor MCS:Step $G$ yielding a finite natural v-labeled natural v2-labeled vv-graph is defined by:

(Def. 39) $\text{MCS:Step}\, G = \begin{cases} G, \text{ if } G.\text{order}() \leq \text{card dom (the vlabel of } G), \\ \text{MCS:Update}(G, \text{MCS:PickUnnumbered}\, G, \text{card dom} \\ \quad \text{(the vlabel of } G)), \text{ otherwise.} \end{cases}$

Let $G$ be a finite graph. The functor MCS:CSeq $G$ yields a finite natural v-labeled natural v2-labeled vv-graph sequence and is defined by:

(Def. 40) (MCS:CSeq $G$)(0) = MCS:Init $G$ and for every natural number $n$ holds (MCS:CSeq $G$)$(n + 1)$ = MCS:Step(MCS:CSeq $G$)$(n)$.

The following proposition is true

(68) For every finite graph $G$ holds MCS:CSeq $G$ is iterative.

Let $G$ be a finite graph. Observe that MCS:CSeq $G$ is iterative.

We now state a number of propositions:

(69) For every finite graph $G$ holds the vlabel of MCS:Init $G = \emptyset$.

(70) Let $G$ be a finite graph and $v$ be a set. Then dom (the v2-label of MCS:Init $G$) = the vertices of $G$ and (the v2-label of MCS:Init $G$)$(v) = 0$.

(71) For every finite graph $G$ holds $G =_G$ MCS:Init $G$.

(72) Let $G$ be a finite natural v2-labeled vv-graph and $x$ be a set. Suppose that
  (i) $x \notin$ dom (the vlabel of $G$),
  (ii) dom (the v2-label of $G$) = the vertices of $G$, and
  (iii) dom (the vlabel of $G$) $\neq$ the vertices of $G$.

  Then (the v2-label of $G$)$(x) \leq$ (the v2-label of $G$)(MCS:PickUnnumbered $G$).

(73) Let $G$ be a finite natural v2-labeled vv-graph. Suppose dom (the v2-label of $G$) = the vertices of $G$ and dom (the vlabel of $G$) $\neq$ the vertices of $G$. Then MCS:PickUnnumbered $G \notin$ dom (the vlabel of $G$).

(74) Let $G$ be a finite natural v2-labeled vv-graph and $v$, $x$ be sets. If $x \notin G$.adjacentSet$(\{v\})$, then (the v2-label of $G$)$(x)$ = (the v2-label of

MCS:LabelAdjacent$(G, v))(x)$.

(75) Let $G$ be a finite natural v2-labeled vv-graph and $v$, $x$ be sets. Suppose $x \in$ dom (the vlabel of $G$). Then (the v2-label of $G)(x) =$ (the v2-label of MCS:LabelAdjacent$(G, v))(x)$.

(76) Let $G$ be a finite natural v2-labeled vv-graph and $v$, $x$ be sets. Suppose $x \in$ dom (the v2-label of $G$) and $x \in G$.adjacentSet$(\{v\})$ and $x \notin$ dom (the vlabel of $G$). Then (the v2-label of MCS:LabelAdjacent$(G, v))(x) =$ (the v2-label of $G)(x) + 1$.

(77) Let $G$ be a finite natural v2-labeled vv-graph and $v$ be a set. Suppose dom (the v2-label of $G$) = the vertices of $G$. Then dom (the v2-label of MCS:LabelAdjacent$(G, v)$) = the vertices of $G$.

(78) For every finite graph $G$ and for every natural number $n$ holds (MCS:CSeq $G)(n) =_G G$.

(79) For every finite graph $G$ and for all natural numbers $m$, $n$ holds (MCS:CSeq $G)(m) =_G$ (MCS:CSeq $G)(n)$.

Let $G$ be a finite chordal graph and let $n$ be a natural number. Observe that (MCS:CSeq $G)(n)$ is chordal.

Let $G$ be a finite chordal graph. Observe that MCS:CSeq $G$ is chordal.

One can prove the following propositions:

(80) For every finite graph $G$ and for every natural number $n$ holds dom (the v2-label of (MCS:CSeq $G)(n)$) = the vertices of (MCS:CSeq $G)(n)$.

(81) Let $G$ be a finite graph and $n$ be a natural number. Suppose card dom (the vlabel of (MCS:CSeq $G)(n)$) $< G$.order(). Then the vlabel of (MCS:CSeq $G)(n+1) =$ (the vlabel of (MCS:CSeq $G)(n)$) $+\cdot$(MCS:PickUnnumbered(MCS:CSeq $G)(n) \dot{\longmapsto} (G$.order()$-'$card dom (the vlabel of (MCS:CSeq $G)(n)$))).

(82) For every finite graph $G$ and for every natural number $n$ such that $n \leq G$.order() holds card dom (the vlabel of (MCS:CSeq $G)(n)$) $= n$.

(83) For every finite graph $G$ and for every natural number $n$ such that $G$.order() $\leq n$ holds (MCS:CSeq $G)(G$.order()) = (MCS:CSeq $G)(n)$.

(84) For every finite graph $G$ and for all natural numbers $m$, $n$ such that $G$.order() $\leq m$ and $m \leq n$ holds (MCS:CSeq $G)(m) =$ (MCS:CSeq $G)(n)$.

(85) For every finite graph $G$ holds MCS:CSeq $G$ is eventually constant.

Let $G$ be a finite graph. Observe that MCS:CSeq $G$ is eventually constant.

The following propositions are true:

(86) Let $G$ be a finite graph and $n$ be a natural number. Then dom (the vlabel of (MCS:CSeq $G)(n)$) = the vertices of (MCS:CSeq $G)(n)$ if and only if $G$.order() $\leq n$.

(87) For every finite graph $G$ holds (MCS:CSeq $G$).Lifespan() $= G$.order().

(88) For every finite graph $G$ holds MCS:CSeq $G$ is v-label numbering.

Let $G$ be a finite graph. Note that MCS:CSeq $G$ is v-label numbering.

Next we state three propositions:

(89) For every finite graph $G$ and for every natural number $n$ such that $n <$ $G$.order() holds MCS:CSeq $G$ .PickedAt $n =$ MCS:PickUnnumbered(MCS:CSeq $G$)($n$).

(90) Let $G$ be a finite graph and $n$ be a natural number. Suppose $n <$ $G$.order(). Then there exists a vertex $w$ of (MCS:CSeq $G$)($n$) such that

(i) $w =$ MCS:PickUnnumbered(MCS:CSeq $G$)($n$), and

(ii) for every set $v$ holds if $v \in G$.adjacentSet($\{w\}$) and $v \notin$ dom (the vlabel of (MCS:CSeq $G$)($n$)), then (the v2-label of (MCS:CSeq $G$)($n+1$))($v$) = (the v2-label of (MCS:CSeq $G$)($n$))($v$) + 1 and if $v \notin G$.adjacentSet($\{w\}$) or $v \in$ dom (the vlabel of (MCS:CSeq $G$)($n$)), then (the v2-label of (MCS:CSeq $G$)($n+1$))($v$) = (the v2-label of (MCS:CSeq $G$)($n$))($v$).

(91) Let $G$ be a finite graph, $n$ be a natural number, and $x$ be a set. Suppose $x \notin$ dom (the vlabel of (MCS:CSeq $G$)($n$)). Then (the v2-label of (MCS:CSeq $G$)($n$))($x$) = card(($G$.adjacentSet($\{x\}$)) $\cap$ dom (the vlabel of (MCS:CSeq $G$)($n$))).

Let $G$ be a natural v-labeled v-graph. We say that $G$ has property $T$ if and only if the condition (Def. 41) is satisfied.

(Def. 41) Let $a$, $b$, $c$ be vertices of $G$. Suppose that $a \in$ dom (the vlabel of $G$) and $b \in$ dom (the vlabel of $G$) and $c \in$ dom (the vlabel of $G$) and (the vlabel of $G$)($a$) < (the vlabel of $G$)($b$) and (the vlabel of $G$)($b$) < (the vlabel of $G$)($c$) and $a$ and $c$ are adjacent and $b$ and $c$ are not adjacent. Then there exists a vertex $d$ of $G$ such that

(i) $d \in$ dom (the vlabel of $G$),

(ii) (the vlabel of $G$)($b$) < (the vlabel of $G$)($d$),

(iii) $b$ and $d$ are adjacent, and

(iv) $a$ and $d$ are not adjacent.

We now state three propositions:

(92) For every finite graph $G$ and for every natural number $n$ holds (MCS:CSeq $G$)($n$) has property $T$.

(93) For every finite graph $G$ holds (LexBFS:CSeq $G$).Result() has property $T$.

(94) Let $G$ be a finite chordal natural v-labeled v-graph. Suppose $G$ has property $T$ and dom (the vlabel of $G$) = the vertices of $G$. Let $V$ be a vertex scheme of $G$. If $V^{-1}$ = the vlabel of $G$, then $V$ is perfect.

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