

Recognizing Chordal Graphs: Lex BFS and MCS¹

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Summary. We are formalizing the algorithm for recognizing chordal graphs by lexicographic breadth-first search as presented in [13, Section 3 of Chapter 4, pp. 81–84]. Then we follow with a formalization of another algorithm serving the same end but based on maximum cardinality search as presented by Tarjan and Yannakakis [25].

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The notation and terminology used in this paper are introduced in the following articles: [28], [11], [26], [32], [33], [35], [30], [10], [7], [8], [20], [29], [4], [2], [14], [23], [12], [3], [6], [9], [18], [15], [19], [16], [17], [24], [21], [1], [5], [31], [27], [22], and [34].

1. PRELIMINARIES

The following propositions are true:

- (1) Let A, B be elements of \mathbb{N} , X be a non empty set, and F be a function from \mathbb{N} into X . If F is one-to-one, then $\overline{\{F(w); w \text{ ranges over elements of } \mathbb{N}: A \leq w \wedge w \leq A + B\}} = B + 1$.
- (2) For all natural numbers n, m, k such that $m \leq k$ and $n < m$ holds $k -' m < k -' n$.

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- (3) For all natural numbers n, k such that $n < k$ holds $(k -' (n + 1)) + 1 = k -' n$.
- (4) For all natural numbers n, m, k such that $k \neq 0$ holds $(n + m \cdot k) \div k = (n \div k) + m$.

Let S be a set. We say that S has finite elements if and only if:

(Def. 1) Every element of S is finite.

Let us note that there exists a set which is non empty and has finite elements and there exists a subset of $2^{\mathbb{N}}$ which is non empty and finite and has finite elements.

Let S be a set with finite elements. One can check that every element of S is finite.

Let f, g be functions. The functor $f[\cup]g$ yielding a function is defined by:

(Def. 2) $\text{dom}(f[\cup]g) = \text{dom } f \cup \text{dom } g$ and for every set x such that $x \in \text{dom } f \cup \text{dom } g$ holds $(f[\cup]g)(x) = f(x) \cup g(x)$.

The following three propositions are true:

- (5) For all natural numbers m, n, k holds $m \in \text{Seg } k \setminus \text{Seg}(k -' n)$ iff $k -' n < m$ and $m \leq k$.
- (6) For all natural numbers n, k, m such that $n \leq m$ holds $\text{Seg } k \setminus \text{Seg}(k -' n) \subseteq \text{Seg } k \setminus \text{Seg}(k -' m)$.
- (7) For all natural numbers n, k such that $n < k$ holds $(\text{Seg } k \setminus \text{Seg}(k -' n)) \cup \{k -' n\} = \text{Seg } k \setminus \text{Seg}(k -' (n + 1))$.

Let f be a binary relation. We say that f is natsubset yielding if and only if:

(Def. 3) $\text{rng } f \subseteq 2^{\mathbb{N}}$.

Let us mention that there exists a function which is finite-yielding and natsubset yielding.

Let f be a finite-yielding natsubset yielding function and let x be a set. Then $f(x)$ is a finite subset of \mathbb{N} .

One can prove the following proposition

- (8) For every ordinal number X and for all finite subsets a, b of X such that $a \neq b$ holds $(a, 1)\text{-bag} \neq (b, 1)\text{-bag}$.

Let F be a natural-yielding function, let S be a set, and let k be a natural number. The functor $F.\text{incSubset}(S, k)$ yielding a natural-yielding function is defined by the conditions (Def. 4).

- (Def. 4)(i) $\text{dom}(F.\text{incSubset}(S, k)) = \text{dom } F$, and
- (ii) for every set y holds if $y \in S$ and $y \in \text{dom } F$, then $(F.\text{incSubset}(S, k))(y) = F(y) + k$ and if $y \notin S$, then $(F.\text{incSubset}(S, k))(y) = F(y)$.

Let n be an ordinal number, let T be a connected term order of n , and let B be a non empty finite subset of Bags n . The functor $\max(B, T)$ yields a bag of n and is defined as follows:

- (Def. 5) $\max(B, T) \in B$ and for every bag x of n such that $x \in B$ holds $x \leq_T \max(B, T)$.

Let O be an ordinal number. Observe that $\text{InvLexOrder } O$ is connected.

2. MISCELLANY ON GRAPHS

Let G be a graph. Note that there exists a vertex sequence of G which is non empty and one-to-one.

Let G be a graph and let V be a non empty vertex sequence of G . A walk of G is called a walk of V if:

- (Def. 6) $\text{It.vertexSeq}() = V$.

Let G be a graph and let V be a non empty one-to-one vertex sequence of G . One can check that every walk of V is path-like.

We now state two propositions:

- (9) For every graph G and for all walks W_1, W_2 of G such that W_1 is trivial and $W_1.\text{last}() = W_2.\text{first}()$ holds $W_1.\text{append}(W_2) = W_2$.
- (10) Let G, H be graphs, A, B, C be sets, G_1 be a subgraph of G induced by A , H_1 be a subgraph of H induced by B , G_2 be a subgraph of G_1 induced by C , and H_2 be a subgraph of H_1 induced by C . Suppose $G =_G H$ and $A \subseteq B$ and $C \subseteq A$ and C is a non empty subset of the vertices of G . Then $G_2 =_G H_2$.

Let G be a v-graph. We say that G is natural v-labeled if and only if:

- (Def. 7) The vlabel of G is natural-yielding.

3. GRAPHS WITH TWO VERTEX LABELS

The natural number V2-LabelSelector is defined by:

- (Def. 8) $\text{V2-LabelSelector} = 8$.

Let G be a graph structure. We say that G is v2-labeled if and only if:

- (Def. 9) $\text{V2-LabelSelector} \in \text{dom } G$ and there exists a function f such that $G(\text{V2-LabelSelector}) = f$ and $\text{dom } f \subseteq \text{the vertices of } G$.

Let us note that there exists a graph structure which is graph-like, weighted, elabeled, vlabeled, and v2-labeled.

A v2-graph is a v2-labeled graph. A vv-graph is a vlabeled v2-labeled graph.

Let G be a v2-graph. The v2-label of G yields a function and is defined as follows:

(Def. 10) The v2-label of $G = G(\text{V2-LabelSelector})$.

Next we state the proposition

(11) For every v2-graph G holds $\text{dom}(\text{the v2-label of } G) \subseteq \text{the vertices of } G$.

Let G be a graph and let X be a set. Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is graph-like.

We now state the proposition

(12) For every graph G and for every set X holds

$$G.\text{set}(\text{V2-LabelSelector}, X) =_G G.$$

Let G be a finite graph and let X be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is finite.

Let G be a loopless graph and let X be a set.

Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is loopless.

Let G be a trivial graph and let X be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is trivial.

Let G be a non trivial graph and let X be a set. One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non trivial.

Let G be a non-multi graph and let X be a set. One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non-multi.

Let G be a non-directed-multi graph and let X be a set. One can verify that $G.\text{set}(\text{V2-LabelSelector}, X)$ is non-directed-multi.

Let G be a connected graph and let X be a set.

Note that $G.\text{set}(\text{V2-LabelSelector}, X)$ is connected.

Let G be an acyclic graph and let X be a set.

One can verify that $G.\text{set}(\text{V2-LabelSelector}, X)$ is acyclic.

Let G be a v-graph and let X be a set.

One can check that $G.\text{set}(\text{V2-LabelSelector}, X)$ is v-labeled.

Let G be a e-graph and let X be a set. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is e-labeled.

Let G be a w-graph and let X be a set. Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is weighted.

Let G be a v2-graph and let X be a set.

One can verify that $G.\text{set}(\text{VLabelSelector}, X)$ is v2-labeled.

Let G be a graph, let Y be a set, and let X be a partial function from the vertices of G to Y . Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is v2-labeled.

Let G be a graph and let X be a many sorted set indexed by the vertices of G . Observe that $G.\text{set}(\text{V2-LabelSelector}, X)$ is v2-labeled.

Let G be a graph. One can verify that $G.\text{set}(\text{V2-LabelSelector}, \emptyset)$ is v2-labeled.

Let G be a v2-graph. We say that G is natural v2-labeled if and only if:

(Def. 11) The v2-label of G is natural-yielding.

We say that G is finite v2-labeled if and only if:

(Def. 12) The v_2 -label of G is finite-yielding.

We say that G is natsubset v_2 -labeled if and only if:

(Def. 13) The v_2 -label of G is natsubset yielding.

One can check that there exists a weighted elabeled vlabeled v_2 -labeled graph which is finite, natural v -labeled, finite v_2 -labeled, natsubset v_2 -labeled, and chordal and there exists a weighted elabeled vlabeled v_2 -labeled graph which is finite, natural v -labeled, natural v_2 -labeled, and chordal.

Let G be a natural v -labeled v -graph. Observe that the v label of G is natural-yielding.

Let G be a natural v_2 -labeled v_2 -graph. Observe that the v_2 -label of G is natural-yielding.

Let G be a finite v_2 -labeled v_2 -graph. Observe that the v_2 -label of G is finite-yielding.

Let G be a natsubset v_2 -labeled v_2 -graph. One can verify that the v_2 -label of G is natsubset yielding.

Let G be a vv -graph and let v, x be sets. One can check that $G.\text{labelVertex}(v, x)$ is v_2 -labeled.

Next we state the proposition

(13) For every vv -graph G and for all sets v, x holds the v_2 -label of $G =$ the v_2 -label of $G.\text{labelVertex}(v, x)$.

Let G be a natural v -labeled vv -graph, let v be a set, and let x be a natural number. Observe that $G.\text{labelVertex}(v, x)$ is natural v -labeled.

Let G be a natural v_2 -labeled vv -graph, let v be a set, and let x be a natural number. Observe that $G.\text{labelVertex}(v, x)$ is natural v_2 -labeled.

Let G be a finite v_2 -labeled vv -graph, let v be a set, and let x be a natural number. Note that $G.\text{labelVertex}(v, x)$ is finite v_2 -labeled.

Let G be a natsubset v_2 -labeled vv -graph, let v be a set, and let x be a natural number. One can check that $G.\text{labelVertex}(v, x)$ is natsubset v_2 -labeled.

Let G be a graph. Note that there exists a subgraph of G which is vlabeled and v_2 -labeled.

Let G be a v_2 -graph and let G_2 be a v_2 -labeled subgraph of G . We say that G_2 inherits v_2 -label if and only if:

(Def. 14) The v_2 -label of $G_2 =$ (the v_2 -label of G)|(the vertices of G_2).

Let G be a v_2 -graph. Note that there exists a v_2 -labeled subgraph of G which inherits v_2 -label.

Let G be a v_2 -graph. A v_2 -subgraph of G is a v_2 -labeled subgraph of G inheriting v_2 -label.

Let G be a vv -graph. Note that there exists a vlabeled v_2 -labeled subgraph of G which inherits vlabeled and v_2 -label.

Let G be a vv -graph. A vv -subgraph of G is a vlabeled v_2 -labeled subgraph of G inheriting vlabeled and v_2 -label.

Let G be a natural v -labeled v -graph. Note that every v -subgraph of G is natural v -labeled.

Let G be a graph and let V, E be sets. Observe that there exists a subgraph of G induced by V and E which is weighted, elabeled, vlabeled, and $v2$ -labeled.

Let G be a vv -graph and let V, E be sets. Observe that there exists a vlabeled $v2$ -labeled subgraph of G induced by V and E which inherits vlabeled and $v2$ -label.

Let G be a vv -graph and let V, E be sets. A (V, E) -induced vv -subgraph of G is a vlabeled $v2$ -labeled subgraph of G induced by V and E inheriting vlabeled and $v2$ -label.

Let G be a vv -graph and let V be a set. A V -induced vv -subgraph of G is a $(V, G.edgesBetween(V))$ -induced vv -subgraph of G .

4. MORE ON GRAPH SEQUENCES

Let s be a many sorted set indexed by \mathbb{N} . We say that s is iterative if and only if:

(Def. 15) For all natural numbers k, n such that $s(k) = s(n)$ holds $s(k + 1) = s(n + 1)$.

Let G_3 be a many sorted set indexed by \mathbb{N} . We say that G_3 is eventually constant if and only if:

(Def. 16) There exists a natural number n such that for every natural number m such that $n \leq m$ holds $G_3(n) = G_3(m)$.

Let us observe that there exists a many sorted set indexed by \mathbb{N} which is halting, iterative, and eventually constant.

The following proposition is true

(14) For every many sorted set G_4 indexed by \mathbb{N} such that G_4 is halting and iterative holds G_4 is eventually constant.

One can check that every many sorted set indexed by \mathbb{N} which is halting and iterative is also eventually constant.

The following proposition is true

(15) For every many sorted set G_4 indexed by \mathbb{N} such that G_4 is eventually constant holds G_4 is halting.

Let us mention that every many sorted set indexed by \mathbb{N} which is eventually constant is also halting.

One can prove the following two propositions:

(16) Let G_4 be an iterative eventually constant many sorted set indexed by \mathbb{N} and n be a natural number. If $G_4.Lifespan() \leq n$, then $G_4(G_4.Lifespan()) = G_4(n)$.

(17) Let G_4 be an iterative eventually constant many sorted set indexed by \mathbb{N} and n, m be natural numbers. If $G_4.\text{Lifespan}() \leq n$ and $n \leq m$, then $G_4(m) = G_4(n)$.

Let G_3 be a v-graph sequence. We say that G_3 is natural v-labeled if and only if:

(Def. 17) For every natural number x holds $G_3(x)$ is natural v-labeled.

Let G_3 be a graph sequence. We say that G_3 is chordal if and only if:

(Def. 18) For every natural number x holds $G_3(x)$ is chordal.

We say that G_3 has fixed vertices if and only if:

(Def. 19) For all natural numbers n, m holds the vertices of $G_3(n) =$ the vertices of $G_3(m)$.

We say that G_3 is v2-labeled if and only if:

(Def. 20) For every natural number x holds $G_3(x)$ is v2-labeled.

Let us observe that there exists a graph sequence which is weighted, elabeled, vlabeled, and v2-labeled.

A v2-graph sequence is a v2-labeled graph sequence. A vv-graph sequence is a vlabeled v2-labeled graph sequence.

Let G_5 be a v2-graph sequence and let x be a natural number. Note that $G_5(x)$ is v2-labeled.

Let G_5 be a v2-graph sequence. We say that G_5 is natural v2-labeled if and only if:

(Def. 21) For every natural number x holds $G_5(x)$ is natural v2-labeled.

We say that G_5 is finite v2-labeled if and only if:

(Def. 22) For every natural number x holds $G_5(x)$ is finite v2-labeled.

We say that G_5 is natsubset v2-labeled if and only if:

(Def. 23) For every natural number x holds $G_5(x)$ is natsubset v2-labeled.

Let us mention that there exists a weighted elabeled vlabeled v2-labeled graph sequence which is finite, natural v-labeled, finite v2-labeled, natsubset v2-labeled, and chordal and there exists a weighted elabeled vlabeled v2-labeled graph sequence which is finite, natural v-labeled, natural v2-labeled, and chordal.

Let G_4 be a v-graph sequence and let x be a natural number. Then $G_4(x)$ is a v-graph.

Let G_5 be a natural v-labeled v-graph sequence and let x be a natural number. Observe that $G_5(x)$ is natural v-labeled.

Let G_5 be a natural v2-labeled v2-graph sequence and let x be a natural number. One can check that $G_5(x)$ is natural v2-labeled.

Let G_5 be a finite v2-labeled v2-graph sequence and let x be a natural number. One can verify that $G_5(x)$ is finite v2-labeled.

Let G_5 be a natsubset v2-labeled v2-graph sequence and let x be a natural number. Note that $G_5(x)$ is natsubset v2-labeled.

Let G_5 be a chordal graph sequence and let x be a natural number. One can check that $G_5(x)$ is chordal.

Let G_4 be a v-graph sequence and let n be a natural number. Then $G_4(n)$ is a v-graph.

Let G_4 be a finite v-graph sequence and let n be a natural number. One can check that $G_4(n)$ is finite.

Let G_4 be a vv-graph sequence and let n be a natural number. Then $G_4(n)$ is a vv-graph.

Let G_4 be a finite vv-graph sequence and let n be a natural number. One can verify that $G_4(n)$ is finite.

Let G_4 be a chordal vv-graph sequence and let n be a natural number. Note that $G_4(n)$ is chordal.

Let G_4 be a natural v-labeled vv-graph sequence and let n be a natural number. One can check that $G_4(n)$ is natural v-labeled.

Let G_4 be a finite v2-labeled vv-graph sequence and let n be a natural number. Note that $G_4(n)$ is finite v2-labeled.

Let G_4 be a natsubset v2-labeled vv-graph sequence and let n be a natural number. One can check that $G_4(n)$ is natsubset v2-labeled.

Let G_4 be a natural v2-labeled vv-graph sequence and let n be a natural number. Observe that $G_4(n)$ is natural v2-labeled.

5. VERTICES NUMBERING SEQUENCES

Let G_3 be a v-graph sequence. We say that G_3 has initially empty v-label if and only if:

(Def. 24) The vlabel of $G_3(0) = \emptyset$.

We say that G_3 is adding one at a step if and only if the condition (Def. 25) is satisfied.

(Def. 25) Let n be a natural number. Suppose $n < G_3.\text{Lifespan}()$. Then there exists a set w such that $w \notin \text{dom}(\text{the vlabel of } G_3(n))$ and the vlabel of $G_3(n+1) = (\text{the vlabel of } G_3(n)) + \cdot (w \mapsto (G_3.\text{Lifespan}() -' n))$.

Let G_3 be a v-graph sequence. We say that G_3 is v-label numbering if and only if the condition (Def. 26) is satisfied.

(Def. 26) G_3 is iterative, eventually constant, finite, natural v-labeled, and adding one at a step and has fixed vertices and initially empty v-label.

One can check that there exists a v-graph sequence which is iterative, eventually constant, finite, natural v-labeled, and adding one at a step and has fixed vertices and initially empty v-label.

Let us observe that there exists a v-graph sequence which is v-label numbering.

One can check the following observations:

- * every v-graph sequence which is v-label numbering is also iterative,
- * every v-graph sequence which is v-label numbering is also eventually constant,
- * every v-graph sequence which is v-label numbering is also finite,
- * every v-graph sequence which is v-label numbering has also fixed vertices,
- * every v-graph sequence which is v-label numbering is also natural v-labeled,
- * every v-graph sequence which is v-label numbering has also initially empty v-label, and
- * every v-graph sequence which is v-label numbering is also adding one at a step.

A v-label numbering sequence is a v-label numbering v-graph sequence.

Let G_3 be a v-label numbering sequence and let n be a natural number. The functor $G_3.PickedAt\ n$ yields a set and is defined by:

- (Def. 27)(i) $G_3.PickedAt\ n = choose(\text{the vertices of } G_3(0))$ if $n \geq G_3.Lifespan()$,
(ii) $G_3.PickedAt\ n \notin \text{dom}(\text{the vlabel of } G_3(n))$ and the vlabel of $G_3(n+1) = (\text{the vlabel of } G_3(n)) + ((G_3.PickedAt\ n) \mapsto (G_3.Lifespan() -' n))$, otherwise.

The following propositions are true:

- (18) Let G_3 be a v-label numbering sequence and n be a natural number. If $n < G_3.Lifespan()$, then $G_3.PickedAt\ n \in G_3(n+1).labeledV()$ and $G_3(n+1).labeledV() = G_3(n).labeledV() \cup \{G_3.PickedAt\ n\}$.
- (19) Let G_3 be a v-label numbering sequence and n be a natural number. If $n < G_3.Lifespan()$, then $(\text{the vlabel of } G_3(n+1))(G_3.PickedAt\ n) = G_3.Lifespan() -' n$.
- (20) For every v-label numbering sequence G_3 and for every natural number n such that $n \leq G_3.Lifespan()$ holds $\text{card}(G_3(n).labeledV()) = n$.
- (21) For every v-label numbering sequence G_3 and for every natural number n holds $\text{rng}(\text{the vlabel of } G_3(n)) = \text{Seg}(G_3.Lifespan()) \setminus \text{Seg}(G_3.Lifespan() -' n)$.
- (22) Let G_3 be a v-label numbering sequence, n be a natural number, and x be a set. Then $(\text{the vlabel of } G_3(n))(x) \leq G_3.Lifespan()$ and if $x \in G_3(n).labeledV()$, then $1 \leq (\text{the vlabel of } G_3(n))(x)$.
- (23) Let G_3 be a v-label numbering sequence and n, m be natural numbers. Suppose $G_3.Lifespan() -' n < m$ and $m \leq G_3.Lifespan()$. Then there exists a vertex v of $G_3(n)$ such that $v \in G_3(n).labeledV()$ and $(\text{the vlabel of } G_3(n))(v) = m$.

of $G_3(n)(v) = m$.

- (24) Let G_3 be a v-label numbering sequence and m, n be natural numbers. If $m \leq n$, then the vlabel of $G_3(m) \subseteq$ the vlabel of $G_3(n)$.
- (25) For every v-label numbering sequence G_3 and for every natural number n holds the vlabel of $G_3(n)$ is one-to-one.
- (26) Let G_3 be a v-label numbering sequence, m, n be natural numbers, and v be a set. Suppose $v \in G_3(m).\text{labeledV}()$ and $v \in G_3(n).\text{labeledV}()$. Then $(\text{the vlabel of } G_3(m))(v) = (\text{the vlabel of } G_3(n))(v)$.
- (27) Let G_3 be a v-label numbering sequence, v be a set, and m, n be natural numbers. If $v \in G_3(m).\text{labeledV}()$ and $(\text{the vlabel of } G_3(m))(v) = n$, then $G_3.\text{PickedAt}(G_3.\text{Lifespan}() -' n) = v$.
- (28) Let G_3 be a v-label numbering sequence and m, n be natural numbers. If $n < G_3.\text{Lifespan}()$ and $n < m$, then $G_3.\text{PickedAt } n \in G_3(m).\text{labeledV}()$ and $(\text{the vlabel of } G_3(m))(G_3.\text{PickedAt } n) = G_3.\text{Lifespan}() -' n$.
- (29) Let G_3 be a v-label numbering sequence, m be a natural number, and v be a set. Suppose $v \in G_3(m).\text{labeledV}()$. Then $G_3.\text{Lifespan}() -' (\text{the vlabel of } G_3(m))(v) < m$ and $G_3.\text{Lifespan}() -' m < (\text{the vlabel of } G_3(m))(v)$.
- (30) Let G_3 be a v-label numbering sequence, i be a natural number, and a, b be sets. Suppose $a \in G_3(i).\text{labeledV}()$ and $b \in G_3(i).\text{labeledV}()$ and $(\text{the vlabel of } G_3(i))(a) < (\text{the vlabel of } G_3(i))(b)$. Then $b \in G_3(G_3.\text{Lifespan}() -' (\text{the vlabel of } G_3(i))(a)).\text{labeledV}()$.
- (31) Let G_3 be a v-label numbering sequence, i be a natural number, and a, b be sets. Suppose $a \in G_3(i).\text{labeledV}()$ and $b \in G_3(i).\text{labeledV}()$ and $(\text{the vlabel of } G_3(i))(a) < (\text{the vlabel of } G_3(i))(b)$. Then $a \notin G_3(G_3.\text{Lifespan}() -' (\text{the vlabel of } G_3(i))(b)).\text{labeledV}()$.

6. LEXICOGRAPHICAL BREADTH-FIRST SEARCH

Let G be a graph. The functor $\text{LexBFS:Init } G$ yields a natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph and is defined as follows:

- (Def. 28) $\text{LexBFS:Init } G = G.\text{set}(\text{VLabelSelector}, \emptyset).\text{set}(\text{V2-LabelSelector}, (\text{the vertices of } G) \mapsto \emptyset)$.

Let G be a finite graph. Then $\text{LexBFS:Init } G$ is a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph.

Let G be a finite finite v2-labeled natsubset v2-labeled vv-graph. Let us assume that $\text{dom}(\text{the v2-label of } G) = \text{the vertices of } G$. The functor $\text{LexBFS:PickUnnumbered } G$ yields a vertex of G and is defined by:

- (Def. 29)(i) $\text{LexBFS:PickUnnumbered } G = \text{choose}(\text{the vertices of } G)$ if $\text{dom}(\text{the vlabel of } G) = \text{the vertices of } G$,

- (ii) there exists a non empty finite subset S of $2^{\mathbb{N}}$ and there exists a non empty finite subset B of Bags \mathbb{N} and there exists a function F such that $S = \text{rng } F$ and $F = (\text{the v2-label of } G) \upharpoonright ((\text{the vertices of } G) \setminus \text{dom}(\text{the vlabel of } G))$ and for every finite subset x of \mathbb{N} such that $x \in S$ holds $(x, 1)\text{-bag} \in B$ and for every set x such that $x \in B$ there exists a finite subset y of \mathbb{N} such that $y \in S$ and $x = (y, 1)\text{-bag}$ and $\text{LexBFS:PickUnnumbered } G = \text{choose}(F^{-1}(\{\text{support max}(B, \text{InvLexOrder } \mathbb{N})\}))$, otherwise.

Let G be a vv-graph, let v be a set, and let k be a natural number. The functor $\text{LexBFS:LabelAdjacent}(G, v, k)$ yielding a vv-graph is defined as follows:

(Def. 30) $\text{LexBFS:LabelAdjacent}(G, v, k) = G.\text{set}(\text{V2-LabelSelector}, (\text{the v2-label of } G) \upharpoonright ((G.\text{adjacentSet}(\{v\}) \setminus \text{dom}(\text{the vlabel of } G)) \mapsto \{k\}))$.

Next we state four propositions:

- (32) Let G be a vv-graph, v, x be sets, and k be a natural number. If $x \notin G.\text{adjacentSet}(\{v\})$, then $(\text{the v2-label of } G)(x) = (\text{the v2-label of } \text{LexBFS:LabelAdjacent}(G, v, k))(x)$.
- (33) Let G be a vv-graph, v, x be sets, and k be a natural number. Suppose $x \in \text{dom}(\text{the vlabel of } G)$. Then $(\text{the v2-label of } G)(x) = (\text{the v2-label of } \text{LexBFS:LabelAdjacent}(G, v, k))(x)$.
- (34) Let G be a vv-graph, v, x be sets, and k be a natural number. Suppose $x \in G.\text{adjacentSet}(\{v\})$ and $x \notin \text{dom}(\text{the vlabel of } G)$. Then $(\text{the v2-label of } \text{LexBFS:LabelAdjacent}(G, v, k))(x) = (\text{the v2-label of } G)(x) \cup \{k\}$.
- (35) Let G be a vv-graph, v be a set, and k be a natural number. Suppose $\text{dom}(\text{the v2-label of } G) = \text{the vertices of } G$. Then $\text{dom}(\text{the v2-label of } \text{LexBFS:LabelAdjacent}(G, v, k)) = \text{the vertices of } G$.

Let G be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph, let v be a vertex of G , and let k be a natural number. Then $\text{LexBFS:LabelAdjacent}(G, v, k)$ is a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph.

Let G be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph, let v be a vertex of G , and let n be a natural number. The functor $\text{LexBFS:Update}(G, v, n)$ yielding a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph is defined by:

(Def. 31) $\text{LexBFS:Update}(G, v, n) = \text{LexBFS:LabelAdjacent}(G.\text{labelVertex}(v, G.\text{order}() - 'n), v, G.\text{order}() - 'n)$.

Let G be a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph. The functor $\text{LexBFS:Step } G$ yields a finite natural v-labeled finite v2-labeled natsubset v2-labeled vv-graph and is defined as follows:

(Def. 32) $\text{LexBFS:Step } G = \begin{cases} G, & \text{if } G.\text{order}() \leq \text{card dom}(\text{the vlabel of } G), \\ \text{LexBFS:Update}(G, \text{LexBFS:PickUnnumbered } G, & \\ \quad \text{card dom}(\text{the vlabel of } G)), & \text{otherwise.} \end{cases}$

Let G be a finite graph. The functor $\text{LexBFS:CSeq } G$ yields a finite natural v -labeled finite v^2 -labeled natsubset v^2 -labeled vv -graph sequence and is defined by:

- (Def. 33) $(\text{LexBFS:CSeq } G)(0) = \text{LexBFS:Init } G$ and for every natural number n holds $(\text{LexBFS:CSeq } G)(n+1) = \text{LexBFS:Step}(\text{LexBFS:CSeq } G)(n)$.

We now state the proposition

- (36) For every finite graph G holds $\text{LexBFS:CSeq } G$ is iterative.

Let G be a finite graph. Observe that $\text{LexBFS:CSeq } G$ is iterative.

Next we state a number of propositions:

- (37) For every graph G holds the v label of $\text{LexBFS:Init } G = \emptyset$.

- (38) Let G be a graph and v be a set. Then $\text{dom}(\text{the } v^2\text{-label of } \text{LexBFS:Init } G) = \text{the vertices of } G$ and $(\text{the } v^2\text{-label of } \text{LexBFS:Init } G)(v) = \emptyset$.

- (39) For every graph G holds $G =_G \text{LexBFS:Init } G$.

- (40) Let G be a finite finite v^2 -labeled natsubset v^2 -labeled vv -graph and x be a set. Suppose that

- (i) $x \notin \text{dom}(\text{the } v\text{label of } G)$,
- (ii) $\text{dom}(\text{the } v^2\text{-label of } G) = \text{the vertices of } G$, and
- (iii) $\text{dom}(\text{the } v\text{label of } G) \neq \text{the vertices of } G$.

Then $((\text{the } v^2\text{-label of } G)(x), 1)\text{-bag} \leq_{\text{InvLexOrder } \mathbb{N}} ((\text{the } v^2\text{-label of } G)(\text{LexBFS:PickUnnumbered } G), 1)\text{-bag}$.

- (41) Let G be a finite finite v^2 -labeled natsubset v^2 -labeled vv -graph. Suppose $\text{dom}(\text{the } v^2\text{-label of } G) = \text{the vertices of } G$ and $\text{dom}(\text{the } v\text{label of } G) \neq \text{the vertices of } G$. Then $\text{LexBFS:PickUnnumbered } G \notin \text{dom}(\text{the } v\text{label of } G)$.

- (42) For every finite graph G and for every natural number n holds $(\text{LexBFS:CSeq } G)(n) =_G G$.

- (43) For every finite graph G and for all natural numbers m, n holds $(\text{LexBFS:CSeq } G)(m) =_G (\text{LexBFS:CSeq } G)(n)$.

- (44) Let G be a finite graph and n be a natural number. Suppose $\text{card } \text{dom}(\text{the } v\text{label of } (\text{LexBFS:CSeq } G)(n)) < G.\text{order}()$. Then the v label of $(\text{LexBFS:CSeq } G)(n+1) = (\text{the } v\text{label of } (\text{LexBFS:CSeq } G)(n) + (\text{LexBFS:PickUnnumbered}(\text{LexBFS:CSeq } G)(n) \mapsto (G.\text{order}() - \text{card } \text{dom}(\text{the } v\text{label of } (\text{LexBFS:CSeq } G)(n))))$.

- (45) For every finite graph G and for every natural number n holds $\text{dom}(\text{the } v^2\text{-label of } (\text{LexBFS:CSeq } G)(n)) = \text{the vertices of } (\text{LexBFS:CSeq } G)(n)$.

- (46) For every finite graph G and for every natural number n such that $n \leq G.\text{order}()$ holds $\text{card } \text{dom}(\text{the } v\text{label of } (\text{LexBFS:CSeq } G)(n)) = n$.

- (47) For every finite graph G and for every natural number n such that $G.\text{order}() \leq n$ holds $(\text{LexBFS:CSeq } G)(G.\text{order}()) =$

- (LexBFS:CSeq G)(n).
- (48) For every finite graph G and for all natural numbers m, n such that $G.order() \leq m$ and $m \leq n$ holds $(LexBFS:CSeq G)(m) = (LexBFS:CSeq G)(n)$.
- (49) For every finite graph G holds LexBFS:CSeq G is eventually constant.
- Let G be a finite graph. Note that LexBFS:CSeq G is eventually constant. We now state two propositions:
- (50) Let G be a finite graph and n be a natural number. Then dom (the vlabel of $(LexBFS:CSeq G)(n)$) = the vertices of $(LexBFS:CSeq G)(n)$ and only if $G.order() \leq n$.
- (51) For every finite graph G holds $(LexBFS:CSeq G).Lifespan() = G.order()$.

Let G be a finite chordal graph and let i be a natural number. One can check that $(LexBFS:CSeq G)(i)$ is chordal.

Let G be a finite chordal graph. One can check that LexBFS:CSeq G is chordal.

One can prove the following proposition

- (52) For every finite graph G holds LexBFS:CSeq G is v-label numbering.
- Let G be a finite graph. Note that LexBFS:CSeq G is v-label numbering. We now state several propositions:
- (53) For every finite graph G and for every natural number n such that $n < G.order()$ holds $LexBFS:CSeq G.PickedAt n = LexBFS:PickUnnumbered(LexBFS:CSeq G)(n)$.
- (54) Let G be a finite graph and n be a natural number. Suppose $n < G.order()$. Then there exists a vertex w of $(LexBFS:CSeq G)(n)$ such that
- (i) $w = LexBFS:PickUnnumbered(LexBFS:CSeq G)(n)$, and
 - (ii) for every set v holds if $v \in G.adjacentSet(\{w\})$ and $v \notin dom$ (the vlabel of $(LexBFS:CSeq G)(n)$), then (the v2-label of $(LexBFS:CSeq G)(n + 1)(v) = (the v2-label of (LexBFS:CSeq G)(n)(v) \cup \{G.order() - 'n\}$ and if $v \notin G.adjacentSet(\{w\})$ or $v \in dom$ (the vlabel of $(LexBFS:CSeq G)(n)$), then (the v2-label of $(LexBFS:CSeq G)(n + 1)(v) = (the v2-label of (LexBFS:CSeq G)(n)(v)$.

- (55) Let G be a finite graph, i be a natural number, and v be a set. Then (the v2-label of $(LexBFS:CSeq G)(i)(v) \subseteq Seg(G.order()) \setminus Seg(G.order() - 'i)$.
- (56) Let G be a finite graph, x be a set, and i, j be natural numbers. If $i \leq j$, then (the v2-label of $(LexBFS:CSeq G)(i)(x) \subseteq (the v2-label of (LexBFS:CSeq G)(j)(x)$.
- (57) Let G be a finite graph, m, n be natural numbers, and x, y be sets. Suppose $n < G.order()$ and $n < m$ and $y = LexBFS:PickUnnumbered(LexBFS:CSeq G)(n)$ and $x \notin dom$ (the vlabel of

$(\text{LexBFS:CSeq } G)(n))$ and $x \in G.\text{adjacentSet}(\{y\})$. Then $G.\text{order}() -' n \in$ (the v2-label of $(\text{LexBFS:CSeq } G)(m))(x)$.

- (58) Let G be a finite graph and m, n be natural numbers. Suppose $m < n$. Let x be a set. Suppose $G.\text{order}() -' m \notin$ (the v2-label of $(\text{LexBFS:CSeq } G)(m+1))(x)$. Then $G.\text{order}() -' m \notin$ (the v2-label of $(\text{LexBFS:CSeq } G)(n))(x)$.
- (59) Let G be a finite graph and m, n, k be natural numbers. Suppose $k < n$ and $n \leq m$. Let x be a set. Suppose $G.\text{order}() -' k \notin$ (the v2-label of $(\text{LexBFS:CSeq } G)(n))(x)$. Then $G.\text{order}() -' k \notin$ (the v2-label of $(\text{LexBFS:CSeq } G)(m))(x)$.
- (60) Let G be a finite graph, m, n be natural numbers, and x be a vertex of $(\text{LexBFS:CSeq } G)(m)$. Suppose $n \in$ (the v2-label of $(\text{LexBFS:CSeq } G)(m))(x)$. Then there exists a vertex y of $(\text{LexBFS:CSeq } G)(m)$ such that $\text{LexBFS:PickUnnumbered}(\text{LexBFS:CSeq } G)(G.\text{order}() -' n) = y$ and $y \notin \text{dom}$ (the vlabel of $(\text{LexBFS:CSeq } G)(G.\text{order}() -' n)$) and $x \in G.\text{adjacentSet}(\{y\})$.

Let G_4 be a finite natural v-labeled vv-graph sequence. Then $G_4.\text{Result}()$ is a finite natural v-labeled vv-graph.

The following four propositions are true:

- (61) For every finite graph G holds $(\text{LexBFS:CSeq } G).\text{Result}().\text{labeledV}() =$ the vertices of G .
- (62) For every finite graph G holds (the vlabel of $(\text{LexBFS:CSeq } G).\text{Result}()^{-1}$) is a vertex scheme of G .
- (63) Let G be a finite graph, i, j be natural numbers, and a, b be vertices of $(\text{LexBFS:CSeq } G)(i)$. Suppose that
- (i) $a \in \text{dom}$ (the vlabel of $(\text{LexBFS:CSeq } G)(i)$),
 - (ii) $b \in \text{dom}$ (the vlabel of $(\text{LexBFS:CSeq } G)(i)$),
 - (iii) (the vlabel of $(\text{LexBFS:CSeq } G)(i))(a) <$ (the vlabel of $(\text{LexBFS:CSeq } G)(i))(b)$, and
 - (iv) $j = G.\text{order}() -'$ (the vlabel of $(\text{LexBFS:CSeq } G)(i))(b)$.
- Then $((\text{the v2-label of } (\text{LexBFS:CSeq } G)(j))(a), 1)\text{-bag} \leq_{\text{InvLexOrder } \mathbb{N}}$ $((\text{the v2-label of } (\text{LexBFS:CSeq } G)(j))(b), 1)\text{-bag}$.
- (64) Let G be a finite graph, i, j be natural numbers, and v be a vertex of $(\text{LexBFS:CSeq } G)(i)$. Suppose $j \in$ (the v2-label of $(\text{LexBFS:CSeq } G)(i))(v)$. Then there exists a vertex w of $(\text{LexBFS:CSeq } G)(i)$ such that $w \in \text{dom}$ (the vlabel of $(\text{LexBFS:CSeq } G)(i)$) and (the vlabel of $(\text{LexBFS:CSeq } G)(i))(w) = j$ and $v \in G.\text{adjacentSet}(\{w\})$.

Let G be a natural v-labeled v-graph. We say that G has property $L3$ if and only if the condition (Def. 34) is satisfied.

(Def. 34) Let a, b, c be vertices of G . Suppose that $a \in \text{dom}(\text{the vlabel of } G)$ and $b \in \text{dom}(\text{the vlabel of } G)$ and $c \in \text{dom}(\text{the vlabel of } G)$ and $(\text{the vlabel of } G)(a) < (\text{the vlabel of } G)(b)$ and $(\text{the vlabel of } G)(b) < (\text{the vlabel of } G)(c)$ and a and c are adjacent and b and c are not adjacent. Then there exists a vertex d of G such that

- (i) $d \in \text{dom}(\text{the vlabel of } G)$,
- (ii) $(\text{the vlabel of } G)(c) < (\text{the vlabel of } G)(d)$,
- (iii) b and d are adjacent,
- (iv) a and d are not adjacent, and
- (v) for every vertex e of G such that $e \neq d$ and e and b are adjacent and e and a are not adjacent holds $(\text{the vlabel of } G)(e) < (\text{the vlabel of } G)(d)$.

One can prove the following three propositions:

- (65) For every finite graph G and for every natural number n holds $(\text{LexBFS:CSeq } G)(n)$ has property $L3$.
- (66) Let G be a finite chordal natural v-labeled v-graph. Suppose G has property $L3$ and $\text{dom}(\text{the vlabel of } G) = \text{the vertices of } G$. Let V be a vertex scheme of G . If $V^{-1} = \text{the vlabel of } G$, then V is perfect.
- (67) For every finite chordal vv-graph G holds $(\text{the vlabel of } (\text{LexBFS:CSeq } G).\text{Result}())^{-1}$ is a perfect vertex scheme of G .

7. THE MAXIMUM CARDINALITY SEARCH ALGORITHM

Let G be a finite graph. The functor $\text{MCS:Init } G$ yields a finite natural v-labeled natural v2-labeled vv-graph and is defined by:

(Def. 35) $\text{MCS:Init } G = G.\text{set}(\text{VLabelSelector}, \emptyset).\text{set}(\text{V2-LabelSelector}, (\text{the vertices of } G) \mapsto 0)$.

Let G be a finite natural v2-labeled vv-graph. Let us assume that $\text{dom}(\text{the v2-label of } G) = \text{the vertices of } G$. The functor $\text{MCS:PickUnnumbered } G$ yields a vertex of G and is defined by:

- (Def. 36)(i) $\text{MCS:PickUnnumbered } G = \text{choose}(\text{the vertices of } G)$ if $\text{dom}(\text{the vlabel of } G) = \text{the vertices of } G$,
- (ii) there exists a finite non empty natural-membered set S and there exists a function F such that $S = \text{rng } F$ and $F = (\text{the v2-label of } G) \upharpoonright ((\text{the vertices of } G) \setminus \text{dom}(\text{the vlabel of } G))$ and $\text{MCS:PickUnnumbered } G = \text{choose}(F^{-1}(\{\max S\}))$, otherwise.

Let G be a finite natural v2-labeled vv-graph and let v be a set. The functor $\text{MCS:LabelAdjacent}(G, v)$ yields a finite natural v2-labeled vv-graph and is defined by:

(Def. 37) $\text{MCS:LabelAdjacent}(G, v) = G.\text{set}(\text{V2-LabelSelector}, (\text{the v2-label of } G).\text{incSubset}((G.\text{adjacentSet}(\{v\})) \setminus \text{dom}(\text{the vlabel of } G), 1))$.

Let G be a finite natural v -labeled natural $v2$ -labeled vv -graph and let v be a vertex of G . Then $\text{MCS:LabelAdjacent}(G, v)$ is a finite natural v -labeled natural $v2$ -labeled vv -graph.

Let G be a finite natural v -labeled natural $v2$ -labeled vv -graph, let v be a vertex of G , and let n be a natural number. The functor $\text{MCS:Update}(G, v, n)$ yielding a finite natural v -labeled natural $v2$ -labeled vv -graph is defined as follows:

(Def. 38) $\text{MCS:Update}(G, v, n) = \text{MCS:LabelAdjacent}(G.\text{labelVertex}(v, G.\text{order}() - n), v)$.

Let G be a finite natural v -labeled natural $v2$ -labeled vv -graph. The functor $\text{MCS:Step } G$ yielding a finite natural v -labeled natural $v2$ -labeled vv -graph is defined by:

(Def. 39) $\text{MCS:Step } G = \begin{cases} G, & \text{if } G.\text{order}() \leq \text{card dom}(\text{the vlabel of } G), \\ \text{MCS:Update}(G, \text{MCS:PickUnnumbered } G, \text{card dom}(\text{the vlabel of } G)), & \text{otherwise.} \end{cases}$

Let G be a finite graph. The functor $\text{MCS:CSeq } G$ yields a finite natural v -labeled natural $v2$ -labeled vv -graph sequence and is defined by:

(Def. 40) $(\text{MCS:CSeq } G)(0) = \text{MCS:Init } G$ and for every natural number n holds $(\text{MCS:CSeq } G)(n + 1) = \text{MCS:Step}(\text{MCS:CSeq } G)(n)$.

The following proposition is true

(68) For every finite graph G holds $\text{MCS:CSeq } G$ is iterative.

Let G be a finite graph. Observe that $\text{MCS:CSeq } G$ is iterative.

We now state a number of propositions:

(69) For every finite graph G holds the v label of $\text{MCS:Init } G = \emptyset$.

(70) Let G be a finite graph and v be a set. Then $\text{dom}(\text{the } v2\text{-label of } \text{MCS:Init } G) = \text{the vertices of } G$ and $(\text{the } v2\text{-label of } \text{MCS:Init } G)(v) = 0$.

(71) For every finite graph G holds $G =_G \text{MCS:Init } G$.

(72) Let G be a finite natural $v2$ -labeled vv -graph and x be a set. Suppose that

- (i) $x \notin \text{dom}(\text{the vlabel of } G)$,
- (ii) $\text{dom}(\text{the } v2\text{-label of } G) = \text{the vertices of } G$, and
- (iii) $\text{dom}(\text{the vlabel of } G) \neq \text{the vertices of } G$.

Then $(\text{the } v2\text{-label of } G)(x) \leq (\text{the } v2\text{-label of } G)(\text{MCS:PickUnnumbered } G)$.

(73) Let G be a finite natural $v2$ -labeled vv -graph. Suppose $\text{dom}(\text{the } v2\text{-label of } G) = \text{the vertices of } G$ and $\text{dom}(\text{the vlabel of } G) \neq \text{the vertices of } G$. Then $\text{MCS:PickUnnumbered } G \notin \text{dom}(\text{the vlabel of } G)$.

(74) Let G be a finite natural $v2$ -labeled vv -graph and v, x be sets. If $x \notin G.\text{adjacentSet}(\{v\})$, then $(\text{the } v2\text{-label of } G)(x) = (\text{the } v2\text{-label of } G)(\text{MCS:PickUnnumbered } G)$.

$\text{MCS:LabelAdjacent}(G, v)(x)$.

- (75) Let G be a finite natural v2-labeled vv-graph and v, x be sets. Suppose $x \in \text{dom}(\text{the vlabel of } G)$. Then $(\text{the v2-label of } G)(x) = (\text{the v2-label of } \text{MCS:LabelAdjacent}(G, v))(x)$.
- (76) Let G be a finite natural v2-labeled vv-graph and v, x be sets. Suppose $x \in \text{dom}(\text{the v2-label of } G)$ and $x \in G.\text{adjacentSet}(\{v\})$ and $x \notin \text{dom}(\text{the vlabel of } G)$. Then $(\text{the v2-label of } \text{MCS:LabelAdjacent}(G, v))(x) = (\text{the v2-label of } G)(x) + 1$.
- (77) Let G be a finite natural v2-labeled vv-graph and v be a set. Suppose $\text{dom}(\text{the v2-label of } G) = \text{the vertices of } G$. Then $\text{dom}(\text{the v2-label of } \text{MCS:LabelAdjacent}(G, v)) = \text{the vertices of } G$.
- (78) For every finite graph G and for every natural number n holds $(\text{MCS:CSeq } G)(n) =_G G$.
- (79) For every finite graph G and for all natural numbers m, n holds $(\text{MCS:CSeq } G)(m) =_G (\text{MCS:CSeq } G)(n)$.

Let G be a finite chordal graph and let n be a natural number. Observe that $(\text{MCS:CSeq } G)(n)$ is chordal.

Let G be a finite chordal graph. Observe that $\text{MCS:CSeq } G$ is chordal.

One can prove the following propositions:

- (80) For every finite graph G and for every natural number n holds $\text{dom}(\text{the v2-label of } (\text{MCS:CSeq } G)(n)) = \text{the vertices of } (\text{MCS:CSeq } G)(n)$.
- (81) Let G be a finite graph and n be a natural number. Suppose $\text{card } \text{dom}(\text{the vlabel of } (\text{MCS:CSeq } G)(n)) < G.\text{order}()$. Then the vlabel of $(\text{MCS:CSeq } G)(n + 1) = (\text{the vlabel of } (\text{MCS:CSeq } G)(n) + (\text{MCS:PickUnnumbered}(\text{MCS:CSeq } G)(n) \dashrightarrow (G.\text{order}() - \text{card } \text{dom}(\text{the vlabel of } (\text{MCS:CSeq } G)(n))))$.
- (82) For every finite graph G and for every natural number n such that $n \leq G.\text{order}()$ holds $\text{card } \text{dom}(\text{the vlabel of } (\text{MCS:CSeq } G)(n)) = n$.
- (83) For every finite graph G and for every natural number n such that $G.\text{order}() \leq n$ holds $(\text{MCS:CSeq } G)(G.\text{order}()) = (\text{MCS:CSeq } G)(n)$.
- (84) For every finite graph G and for all natural numbers m, n such that $G.\text{order}() \leq m$ and $m \leq n$ holds $(\text{MCS:CSeq } G)(m) = (\text{MCS:CSeq } G)(n)$.
- (85) For every finite graph G holds $\text{MCS:CSeq } G$ is eventually constant.

Let G be a finite graph. Observe that $\text{MCS:CSeq } G$ is eventually constant.

The following propositions are true:

- (86) Let G be a finite graph and n be a natural number. Then $\text{dom}(\text{the vlabel of } (\text{MCS:CSeq } G)(n)) = \text{the vertices of } (\text{MCS:CSeq } G)(n)$ if and only if $G.\text{order}() \leq n$.
- (87) For every finite graph G holds $(\text{MCS:CSeq } G).\text{Lifespan}() = G.\text{order}()$.

- (88) For every finite graph G holds $\text{MCS:CSeq } G$ is v-label numbering.
 Let G be a finite graph. Note that $\text{MCS:CSeq } G$ is v-label numbering.
 Next we state three propositions:
- (89) For every finite graph G and for every natural number n such that $n < G.\text{order}()$ holds $\text{MCS:CSeq } G.\text{PickedAt } n = \text{MCS:PickUnnumbered}(\text{MCS:CSeq } G)(n)$.
- (90) Let G be a finite graph and n be a natural number. Suppose $n < G.\text{order}()$. Then there exists a vertex w of $(\text{MCS:CSeq } G)(n)$ such that
- (i) $w = \text{MCS:PickUnnumbered}(\text{MCS:CSeq } G)(n)$, and
 - (ii) for every set v holds if $v \in G.\text{adjacentSet}(\{w\})$ and $v \notin \text{dom}(\text{the vlabel of } (\text{MCS:CSeq } G)(n))$, then $(\text{the v2-label of } (\text{MCS:CSeq } G)(n+1))(v) = (\text{the v2-label of } (\text{MCS:CSeq } G)(n))(v) + 1$ and if $v \notin G.\text{adjacentSet}(\{w\})$ or $v \in \text{dom}(\text{the vlabel of } (\text{MCS:CSeq } G)(n))$, then $(\text{the v2-label of } (\text{MCS:CSeq } G)(n+1))(v) = (\text{the v2-label of } (\text{MCS:CSeq } G)(n))(v)$.
- (91) Let G be a finite graph, n be a natural number, and x be a set. Suppose $x \notin \text{dom}(\text{the vlabel of } (\text{MCS:CSeq } G)(n))$. Then $(\text{the v2-label of } (\text{MCS:CSeq } G)(n))(x) = \text{card}((G.\text{adjacentSet}(\{x\})) \cap \text{dom}(\text{the vlabel of } (\text{MCS:CSeq } G)(n)))$.

Let G be a natural v-labeled v-graph. We say that G has property T if and only if the condition (Def. 41) is satisfied.

- (Def. 41) Let a, b, c be vertices of G . Suppose that $a \in \text{dom}(\text{the vlabel of } G)$ and $b \in \text{dom}(\text{the vlabel of } G)$ and $c \in \text{dom}(\text{the vlabel of } G)$ and $(\text{the vlabel of } G)(a) < (\text{the vlabel of } G)(b)$ and $(\text{the vlabel of } G)(b) < (\text{the vlabel of } G)(c)$ and a and c are adjacent and b and c are not adjacent. Then there exists a vertex d of G such that
- (i) $d \in \text{dom}(\text{the vlabel of } G)$,
 - (ii) $(\text{the vlabel of } G)(b) < (\text{the vlabel of } G)(d)$,
 - (iii) b and d are adjacent, and
 - (iv) a and d are not adjacent.

We now state three propositions:

- (92) For every finite graph G and for every natural number n holds $(\text{MCS:CSeq } G)(n)$ has property T .
- (93) For every finite graph G holds $(\text{LexBFS:CSeq } G).\text{Result}()$ has property T .
- (94) Let G be a finite chordal natural v-labeled v-graph. Suppose G has property T and $\text{dom}(\text{the vlabel of } G) = \text{the vertices of } G$. Let V be a vertex scheme of G . If $V^{-1} = \text{the vlabel of } G$, then V is perfect.

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