

# Integrability and the Integral of Partial Functions from $\mathbb{R}$ into $\mathbb{R}^1$

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**Summary.** In this paper, we showed the linearity of the indefinite integral  $\int_a^b f dx$ , the form of which was introduced in [11]. In addition, we proved some theorems about the integral calculus on the subinterval of  $[a, b]$ . As a result, we described the fundamental theorem of calculus, that we developed in [11], by a more general expression.

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The articles [23], [25], [26], [2], [22], [4], [14], [1], [24], [5], [27], [7], [6], [21], [9], [3], [17], [16], [15], [18], [20], [8], [10], [13], [19], [12], and [11] provide the notation and terminology for this paper.

## 1. PRELIMINARIES

We use the following convention:  $a, b, c, d, e, x$  are real numbers,  $A$  is a closed-interval subset of  $\mathbb{R}$ , and  $f, g$  are partial functions from  $\mathbb{R}$  to  $\mathbb{R}$ .

We now state several propositions:

- (1) If  $a \leq b$  and  $c \leq d$  and  $a + c = b + d$ , then  $a = b$  and  $c = d$ .
- (2) If  $a \leq b$ , then  $]x - a, x + a[ \subseteq ]x - b, x + b[$ .

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- (3) For every binary relation  $R$  and for all sets  $A, B, C$  such that  $A \subseteq B$  and  $A \subseteq C$  holds  $R \upharpoonright B \upharpoonright A = R \upharpoonright C \upharpoonright A$ .
- (4) For all sets  $A, B, C$  such that  $A \subseteq B$  and  $A \subseteq C$  holds  $\chi_{B,B} \upharpoonright A = \chi_{C,C} \upharpoonright A$ .
- (5) If  $a \leq b$ , then  $\text{vol}([a, b]) = b - a$ .
- (6)  $\text{vol}([\min(a, b), \max(a, b)]) = |b - a|$ .

## 2. INTEGRABILITY AND THE INTEGRAL OF PARTIAL FUNCTIONS

The following propositions are true:

- (7) If  $A \subseteq \text{dom } f$  and  $f$  is integrable on  $A$  and  $f$  is bounded on  $A$ , then  $|f|$  is integrable on  $A$  and  $|\int_A f(x)dx| \leq \int_A |f|(x)dx$ .
- (8) If  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and  $f$  is integrable on  $[a, b]$  and  $f$  is bounded on  $[a, b]$ , then  $|\int_a^b f(x)dx| \leq \int_a^b |f|(x)dx$ .
- (9) Let  $r$  be a real number. Suppose  $A \subseteq \text{dom } f$  and  $f$  is integrable on  $A$  and  $f$  is bounded on  $A$ . Then  $rf$  is integrable on  $A$  and  $\int_A (rf)(x)dx = r \cdot \int_A f(x)dx$ .
- (10) If  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and  $f$  is integrable on  $[a, b]$  and  $f$  is bounded on  $[a, b]$ , then  $\int_a^b (cf)(x)dx = c \cdot \int_a^b f(x)dx$ .
- (11) Suppose  $A \subseteq \text{dom } f$  and  $A \subseteq \text{dom } g$  and  $f$  is integrable on  $A$  and  $f$  is bounded on  $A$  and  $g$  is integrable on  $A$  and  $g$  is bounded on  $A$ . Then  $f + g$  is integrable on  $A$  and  $f - g$  is integrable on  $A$  and  $\int_A (f + g)(x)dx = \int_A f(x)dx + \int_A g(x)dx$  and  $\int_A (f - g)(x)dx = \int_A f(x)dx - \int_A g(x)dx$ .
- (12) Suppose that  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$  and  $f$  is integrable on  $[a, b]$  and  $g$  is integrable on  $[a, b]$  and  $f$  is bounded on  $[a, b]$  and  $g$  is bounded on  $[a, b]$ . Then  $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

$$\text{and } \int_a^b (f - g)(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

- (13) If  $f$  is bounded on  $A$  and  $g$  is bounded on  $A$ , then  $f g$  is bounded on  $A$ .
- (14) Suppose  $A \subseteq \text{dom } f$  and  $A \subseteq \text{dom } g$  and  $f$  is integrable on  $A$  and  $f$  is bounded on  $A$  and  $g$  is integrable on  $A$  and  $g$  is bounded on  $A$ . Then  $f g$  is integrable on  $A$ .
- (15) Let  $n$  be an element of  $\mathbb{N}$ . Suppose  $n > 0$  and  $\text{vol}(A) > 0$ . Then there exists an element  $D$  of  $\text{divs } A$  such that  $\text{len } D = n$  and for every element  $i$  of  $\mathbb{N}$  such that  $i \in \text{dom } D$  holds  $D(i) = \text{inf } A + \frac{\text{vol}(A)}{n} \cdot i$ .

3. INTEGRABILITY ON A SUBINTERVAL

The following propositions are true:

- (16) Suppose  $\text{vol}(A) > 0$ . Then there exists a DivSequence  $T$  of  $A$  such that
  - (i)  $\delta_T$  is convergent,
  - (ii)  $\lim(\delta_T) = 0$ , and
  - (iii) for every element  $n$  of  $\mathbb{N}$  there exists an element  $T_1$  of  $\text{divs } A$  such that  $T_1$  divides into equal  $n + 1$  and  $T(n) = T_1$ .
- (17) Suppose  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f$  is bounded on  $[a, b]$  and  $[a, b] \subseteq \text{dom } f$  and  $c \in [a, b]$ . Then  $f$  is integrable on  $[a, c]$  and  $f$  is integrable on  $[c, b]$  and  $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ .
- (18) Suppose  $a \leq c$  and  $c \leq d$  and  $d \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f$  is bounded on  $[a, b]$  and  $[a, b] \subseteq \text{dom } f$ . Then  $f$  is integrable on  $[c, d]$  and  $f$  is bounded on  $[c, d]$  and  $[c, d] \subseteq \text{dom } f$ .
- (19) Suppose that  $a \leq c$  and  $c \leq d$  and  $d \leq b$  and  $f$  is integrable on  $[a, b]$  and  $g$  is integrable on  $[a, b]$  and  $f$  is bounded on  $[a, b]$  and  $g$  is bounded on  $[a, b]$  and  $[a, b] \subseteq \text{dom } f$  and  $[a, b] \subseteq \text{dom } g$ . Then  $f + g$  is integrable on  $[c, d]$  and  $f + g$  is bounded on  $[c, d]$ .
- (20) Suppose  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f$  is bounded on  $[a, b]$  and  $[a, b] \subseteq \text{dom } f$  and  $c \in [a, b]$  and  $d \in [a, b]$ . Then  $\int_a^d f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx$ .
- (21) Suppose  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f$  is bounded on  $[a, b]$  and  $[a, b] \subseteq \text{dom } f$  and  $c \in [a, b]$  and  $d \in [a, b]$ . Then  $[\min(c, d), \max(c, d)] \subseteq \text{dom } |f|$  and  $|f|$  is integrable on

$$[\min(c, d), \max(c, d)'] \text{ and } |f| \text{ is bounded on } [\min(c, d), \max(c, d)'] \text{ and}$$

$$\left| \int_c^d f(x) dx \right| \leq \int_{\min(c, d)}^{\max(c, d)} |f|(x) dx.$$

- (22) Suppose  $a \leq b$  and  $c \leq d$  and  $f$  is integrable on  $[a, b']$  and  $f$  is bounded on  $[a, b']$  and  $[a, b'] \subseteq \text{dom } f$  and  $c \in [a, b']$  and  $d \in [a, b']$ . Then  $[c, d'] \subseteq \text{dom } |f|$  and  $|f|$  is integrable on  $[c, d']$  and  $|f|$  is bounded on  $[c, d']$  and

$$\left| \int_c^d f(x) dx \right| \leq \int_c^d |f|(x) dx \text{ and } \left| \int_d^c f(x) dx \right| \leq \int_c^d |f|(x) dx.$$

- (23) Suppose that  $a \leq b$  and  $c \leq d$  and  $f$  is integrable on  $[a, b']$  and  $f$  is bounded on  $[a, b']$  and  $[a, b'] \subseteq \text{dom } f$  and  $c \in [a, b']$  and  $d \in [a, b']$  and for every real number  $x$  such that  $x \in [c, d']$  holds  $|f(x)| \leq e$ . Then

$$\left| \int_c^d f(x) dx \right| \leq e \cdot (d - c) \text{ and } \left| \int_d^c f(x) dx \right| \leq e \cdot (d - c).$$

- (24) Suppose that  $a \leq b$  and  $f$  is integrable on  $[a, b']$  and  $g$  is integrable on  $[a, b']$  and  $f$  is bounded on  $[a, b']$  and  $g$  is bounded on  $[a, b']$  and  $[a, b'] \subseteq \text{dom } f$  and  $[a, b'] \subseteq \text{dom } g$  and  $c \in [a, b']$  and  $d \in [a, b']$ . Then

$$\int_c^d (f + g)(x) dx = \int_c^d f(x) dx + \int_c^d g(x) dx \text{ and } \int_c^d (f - g)(x) dx = \int_c^d f(x) dx - \int_c^d g(x) dx.$$

- (25) Suppose  $a \leq b$  and  $f$  is integrable on  $[a, b']$  and  $f$  is bounded on  $[a, b']$

and  $[a, b'] \subseteq \text{dom } f$  and  $c \in [a, b']$  and  $d \in [a, b']$ . Then  $\int_c^d (ef)(x) dx =$

$$e \cdot \int_c^d f(x) dx.$$

- (26) Suppose  $a \leq b$  and  $[a, b'] \subseteq \text{dom } f$  and for every real number  $x$  such that  $x \in [a, b']$  holds  $f(x) = e$ . Then  $f$  is integrable on  $[a, b']$  and  $f$  is

bounded on  $[a, b']$  and  $\int_a^b f(x) dx = e \cdot (b - a)$ .

- (27) Suppose  $a \leq b$  and for every real number  $x$  such that  $x \in [a, b']$  holds  $f(x) = e$  and  $[a, b'] \subseteq \text{dom } f$  and  $c \in [a, b']$  and  $d \in [a, b']$ . Then

$$\int_c^d f(x) dx = e \cdot (d - c).$$

## 4. FUNDAMENTAL THEOREM OF CALCULUS

Next we state two propositions:

- (28) Let  $x_0$  be a real number and  $F$  be a partial function from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose that  $a \leq b$  and  $f$  is integrable on  $]a, b[$  and  $f$  is bounded on  $]a, b[$  and  $]a, b[ \subseteq \text{dom } f$  and  $]a, b[ \subseteq \text{dom } F$  and for every real number  $x$  such that  $x \in ]a, b[$  holds  $F(x) = \int_a^x f(x)dx$  and  $x_0 \in ]a, b[$  and  $f$  is continuous in  $x_0$ . Then  $F$  is differentiable in  $x_0$  and  $F'(x_0) = f(x_0)$ .
- (29) Let  $x_0$  be a real number. Suppose  $a \leq b$  and  $f$  is integrable on  $]a, b[$  and  $f$  is bounded on  $]a, b[$  and  $]a, b[ \subseteq \text{dom } f$  and  $x_0 \in ]a, b[$  and  $f$  is continuous in  $x_0$ . Then there exists a partial function  $F$  from  $\mathbb{R}$  to  $\mathbb{R}$  such that  $]a, b[ \subseteq \text{dom } F$  and for every real number  $x$  such that  $x \in ]a, b[$  holds  $F(x) = \int_a^x f(x)dx$  and  $F$  is differentiable in  $x_0$  and  $F'(x_0) = f(x_0)$ .

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## REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [4] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [8] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(4):661–668, 1990.
- [9] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in  $\mathcal{E}^2$ . *Formalized Mathematics*, 6(3):427–440, 1997.
- [10] Noboru Endou and Artur Kornilowicz. The definition of the Riemann definite integral and some related lemmas. *Formalized Mathematics*, 8(1):93–102, 1999.
- [11] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from  $\mathbb{R}$  to  $\mathbb{R}$  and integrability for continuous functions. *Formalized Mathematics*, 9(2):281–284, 2001.
- [12] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Integrability of bounded total functions. *Formalized Mathematics*, 9(2):271–274, 2001.
- [13] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Scalar multiple of Riemann definite integral. *Formalized Mathematics*, 9(1):191–196, 2001.
- [14] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [15] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(2):273–275, 1990.

- [16] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(4):703–709, 1990.
- [17] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [18] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board – part I. *Formalized Mathematics*, 3(1):107–115, 1992.
- [19] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(4):787–791, 1990.
- [20] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(4):797–801, 1990.
- [21] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [22] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
- [23] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [24] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [26] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [27] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

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