

# Baire's Category Theorem and Some Spaces Generated from Real Normed Space<sup>1</sup>

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**Summary.** As application of complete metric space, we proved a Baire's category theorem. Then we defined some spaces generated from real normed space and discussed each of them. In the second section, we showed the equivalence of convergence and the continuity of a function. In other sections, we showed some topological properties of two spaces, which are topological space and linear topological space generated from real normed space.

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The papers [23], [7], [26], [4], [1], [21], [15], [27], [6], [5], [17], [19], [20], [24], [22], [2], [25], [9], [10], [13], [16], [12], [11], [3], [18], [8], and [14] provide the notation and terminology for this paper.

## 1. BAIRE'S CATEGORY THEOREM

The following proposition is true

- (1) Let  $X$  be a non empty metric space and  $Y$  be a sequence of subsets of  $X$ . Suppose  $X$  is complete and  $\bigcup \text{rng } Y = X$  and for every element  $n$  of  $\mathbb{N}$  holds  $Y(n)^c \in \text{the open set family of } X$ . Then there exists an element  $n_0$  of  $\mathbb{N}$  and there exists a real number  $r$  and there exists a point  $x_0$  of  $X$  such that  $0 < r$  and  $\text{Ball}(x_0, r) \subseteq Y(n_0)$ .

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## 2. METRIC SPACE GENERATED FROM REAL NORMED SPACE

Let  $X$  be a real normed space. The distance by norm of  $X$  yields a function from [the carrier of  $X$ , the carrier of  $X$ ] into  $\mathbb{R}$  and is defined as follows:

(Def. 1) For all points  $x, y$  of  $X$  holds (the distance by norm of  $X$ )( $x, y$ ) =  $\|x - y\|$ .

Let  $X$  be a real normed space. The functor  $\text{MetricSpaceNorm } X$  yields a non empty metric space and is defined by:

(Def. 2)  $\text{MetricSpaceNorm } X = \langle \text{the carrier of } X, \text{ the distance by norm of } X \rangle$ .

Next we state several propositions:

- (2) Let  $X$  be a real normed space,  $z$  be an element of  $\text{MetricSpaceNorm } X$ , and  $r$  be a real number. Then there exists a point  $x$  of  $X$  such that  $x = z$  and  $\text{Ball}(z, r) = \{y; y \text{ ranges over points of } X: \|x - y\| < r\}$ .
- (3) Let  $X$  be a real normed space,  $z$  be an element of  $\text{MetricSpaceNorm } X$ , and  $r$  be a real number. Then there exists a point  $x$  of  $X$  such that  $x = z$  and  $\overline{\text{Ball}}(z, r) = \{y; y \text{ ranges over points of } X: \|x - y\| \leq r\}$ .
- (4) Let  $X$  be a real normed space,  $S$  be a sequence of  $X$ ,  $S_1$  be a sequence of  $\text{MetricSpaceNorm } X$ ,  $x$  be a point of  $X$ , and  $x_1$  be a point of  $\text{MetricSpaceNorm } X$ . Suppose  $S = S_1$  and  $x = x_1$ . Then  $S_1$  is convergent to  $x_1$  if and only if for every real number  $r$  such that  $0 < r$  there exists an element  $m$  of  $\mathbb{N}$  such that for every element  $n$  of  $\mathbb{N}$  such that  $m \leq n$  holds  $\|S(n) - x\| < r$ .
- (5) Let  $X$  be a real normed space,  $S$  be a sequence of  $X$ , and  $S_1$  be a sequence of  $\text{MetricSpaceNorm } X$ . If  $S = S_1$ , then  $S_1$  is convergent iff  $S$  is convergent.
- (6) Let  $X$  be a real normed space,  $S$  be a sequence of  $X$ , and  $S_1$  be a sequence of  $\text{MetricSpaceNorm } X$ . If  $S = S_1$  and  $S_1$  is convergent, then  $\lim S_1 = \lim S$ .

## 3. TOPOLOGICAL SPACE GENERATED FROM REAL NORMED SPACE

Let  $X$  be a real normed space. The functor  $\text{TopSpaceNorm } X$  yields a non empty topological space and is defined by:

(Def. 3)  $\text{TopSpaceNorm } X = (\text{MetricSpaceNorm } X)_{\text{top}}$ .

The following propositions are true:

- (7) Let  $X$  be a real normed space and  $V$  be a subset of  $\text{TopSpaceNorm } X$ . Then  $V$  is open if and only if for every point  $x$  of  $X$  such that  $x \in V$  there exists a real number  $r$  such that  $r > 0$  and  $\{y; y \text{ ranges over points of } X: \|x - y\| < r\} \subseteq V$ .

- (8) Let  $X$  be a real normed space,  $x$  be a point of  $X$ , and  $r$  be a real number. Then  $\{y; y \text{ ranges over points of } X: \|x - y\| < r\}$  is an open subset of  $\text{TopSpaceNorm } X$ .
- (9) Let  $X$  be a real normed space,  $x$  be a point of  $X$ , and  $r$  be a real number. Then  $\{y; y \text{ ranges over points of } X: \|x - y\| \leq r\}$  is a closed subset of  $\text{TopSpaceNorm } X$ .
- (10) For every Hausdorff non empty topological space  $X$  such that  $X$  is locally-compact holds  $X$  is Baire.
- (11) For every real normed space  $X$  holds  $\text{TopSpaceNorm } X$  is sequential.  
Let  $X$  be a real normed space. Observe that  $\text{TopSpaceNorm } X$  is sequential. One can prove the following propositions:
- (12) Let  $X$  be a real normed space,  $S$  be a sequence of  $X$ ,  $S_1$  be a sequence of  $\text{TopSpaceNorm } X$ ,  $x$  be a point of  $X$ , and  $x_1$  be a point of  $\text{TopSpaceNorm } X$ . Suppose  $S = S_1$  and  $x = x_1$ . Then  $S_1$  is convergent to  $x_1$  if and only if for every real number  $r$  such that  $0 < r$  there exists an element  $m$  of  $\mathbb{N}$  such that for every element  $n$  of  $\mathbb{N}$  such that  $m \leq n$  holds  $\|S(n) - x\| < r$ .
- (13) Let  $X$  be a real normed space,  $S$  be a sequence of  $X$ , and  $S_1$  be a sequence of  $\text{TopSpaceNorm } X$ . If  $S = S_1$ , then  $S_1$  is convergent iff  $S$  is convergent.
- (14) Let  $X$  be a real normed space,  $S$  be a sequence of  $X$ , and  $S_1$  be a sequence of  $\text{TopSpaceNorm } X$ . If  $S = S_1$  and  $S_1$  is convergent, then  $\text{Lim } S_1 = \{\text{lim } S\}$  and  $\text{lim } S_1 = \text{lim } S$ .
- (15) Let  $X$  be a real normed space,  $V$  be a subset of  $X$ , and  $V_1$  be a subset of  $\text{TopSpaceNorm } X$ . If  $V = V_1$ , then  $V$  is closed iff  $V_1$  is closed.
- (16) Let  $X$  be a real normed space,  $V$  be a subset of  $X$ , and  $V_1$  be a subset of  $\text{TopSpaceNorm } X$ . If  $V = V_1$ , then  $V$  is open iff  $V_1$  is open.
- (17) Let  $X$  be a real normed space,  $U$  be a subset of  $X$ ,  $U_1$  be a subset of  $\text{TopSpaceNorm } X$ ,  $x$  be a point of  $X$ , and  $x_1$  be a point of  $\text{TopSpaceNorm } X$ . Suppose  $U = U_1$  and  $x = x_1$ . Then  $U$  is a neighbourhood of  $x$  if and only if  $U_1$  is a neighbourhood of  $x_1$ .
- (18) Let  $X, Y$  be real normed spaces,  $f$  be a partial function from  $X$  to  $Y$ ,  $f_1$  be a function from  $\text{TopSpaceNorm } X$  into  $\text{TopSpaceNorm } Y$ ,  $x$  be a point of  $X$ , and  $x_1$  be a point of  $\text{TopSpaceNorm } X$ . Suppose  $f = f_1$  and  $x = x_1$ . Then  $f$  is continuous in  $x$  if and only if  $f_1$  is continuous at  $x_1$ .
- (19) Let  $X, Y$  be real normed spaces,  $f$  be a partial function from  $X$  to  $Y$ , and  $f_1$  be a function from  $\text{TopSpaceNorm } X$  into  $\text{TopSpaceNorm } Y$ . Suppose  $f = f_1$ . Then  $f$  is continuous on the carrier of  $X$  if and only if  $f_1$  is continuous.

## 4. LINEAR TOPOLOGICAL SPACE GENERATED FROM REAL NORMED SPACE

Let  $X$  be a real normed space. The functor  $\text{LinearTopSpaceNorm } X$  yields a strict non empty real linear topological structure and is defined by the conditions (Def. 4).

- (Def. 4)(i) The carrier of  $\text{LinearTopSpaceNorm } X =$  the carrier of  $X$ ,  
(ii) the zero of  $\text{LinearTopSpaceNorm } X =$  the zero of  $X$ ,  
(iii) the addition of  $\text{LinearTopSpaceNorm } X =$  the addition of  $X$ ,  
(iv) the external multiplication of  $\text{LinearTopSpaceNorm } X =$  the external multiplication of  $X$ , and  
(v) the topology of  $\text{LinearTopSpaceNorm } X =$  the topology of  $\text{TopSpaceNorm } X$ .

Let  $X$  be a real normed space. Note that  $\text{LinearTopSpaceNorm } X$  is add-continuous, mult-continuous, topological space-like, Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

We now state several propositions:

- (20) Let  $X$  be a real normed space,  $V$  be a subset of  $\text{TopSpaceNorm } X$ , and  $V_1$  be a subset of  $\text{LinearTopSpaceNorm } X$ . If  $V = V_1$ , then  $V$  is open iff  $V_1$  is open.  
(21) Let  $X$  be a real normed space,  $V$  be a subset of  $\text{TopSpaceNorm } X$ , and  $V_1$  be a subset of  $\text{LinearTopSpaceNorm } X$ . If  $V = V_1$ , then  $V$  is closed iff  $V_1$  is closed.  
(22) Let  $X$  be a real normed space and  $V$  be a subset of  $\text{LinearTopSpaceNorm } X$ . Then  $V$  is open if and only if for every point  $x$  of  $X$  such that  $x \in V$  there exists a real number  $r$  such that  $r > 0$  and  $\{y; y \text{ ranges over points of } X: \|x - y\| < r\} \subseteq V$ .  
(23) Let  $X$  be a real normed space,  $x$  be a point of  $X$ ,  $r$  be a real number, and  $V$  be a subset of  $\text{LinearTopSpaceNorm } X$ . If  $V = \{y; y \text{ ranges over points of } X: \|x - y\| < r\}$ , then  $V$  is open.  
(24) Let  $X$  be a real normed space,  $x$  be a point of  $X$ ,  $r$  be a real number, and  $V$  be a subset of  $\text{TopSpaceNorm } X$ . If  $V = \{y; y \text{ ranges over points of } X: \|x - y\| \leq r\}$ , then  $V$  is closed.

Let  $X$  be a real normed space. Observe that  $\text{LinearTopSpaceNorm } X$  is  $T_2$  and  $\text{LinearTopSpaceNorm } X$  is sober.

One can prove the following proposition

- (25) Let  $X$  be a real normed space,  $S$  be a family of subsets of  $\text{TopSpaceNorm } X$ ,  $S_1$  be a family of subsets of  $\text{LinearTopSpaceNorm } X$ ,  $x$  be a point of  $\text{TopSpaceNorm } X$ , and  $x_1$  be a point of  $\text{LinearTopSpaceNorm } X$ . Suppose  $S = S_1$  and  $x = x_1$ . Then  $S_1$  is a basis of  $x_1$  if and only if  $S$  is a basis of  $x$ .

Let  $X$  be a real normed space. One can verify the following observations:

- \*  $\text{LinearTopSpaceNorm } X$  is first-countable,
- \*  $\text{LinearTopSpaceNorm } X$  is Frechet, and
- \*  $\text{LinearTopSpaceNorm } X$  is sequential.

Next we state a number of propositions:

- (26) Let  $X$  be a real normed space,  $S$  be a sequence of  $\text{TopSpaceNorm } X$ ,  $S_1$  be a sequence of  $\text{LinearTopSpaceNorm } X$ ,  $x$  be a point of  $\text{TopSpaceNorm } X$ , and  $x_1$  be a point of  $\text{LinearTopSpaceNorm } X$ . Suppose  $S = S_1$  and  $x = x_1$ . Then  $S_1$  is convergent to  $x_1$  if and only if  $S$  is convergent to  $x$ .
- (27) Let  $X$  be a real normed space,  $S$  be a sequence of  $\text{TopSpaceNorm } X$ , and  $S_1$  be a sequence of  $\text{LinearTopSpaceNorm } X$ . If  $S = S_1$ , then  $S_1$  is convergent iff  $S$  is convergent.
- (28) Let  $X$  be a real normed space,  $S$  be a sequence of  $\text{TopSpaceNorm } X$ , and  $S_1$  be a sequence of  $\text{LinearTopSpaceNorm } X$ . If  $S = S_1$  and  $S_1$  is convergent, then  $\text{Lim } S = \text{Lim } S_1$  and  $\text{lim } S = \text{lim } S_1$ .
- (29) Let  $X$  be a real normed space,  $S$  be a sequence of  $X$ ,  $S_1$  be a sequence of  $\text{LinearTopSpaceNorm } X$ ,  $x$  be a point of  $X$ , and  $x_1$  be a point of  $\text{LinearTopSpaceNorm } X$ . Suppose  $S = S_1$  and  $x = x_1$ . Then  $S_1$  is convergent to  $x_1$  if and only if for every real number  $r$  such that  $0 < r$  there exists an element  $m$  of  $\mathbb{N}$  such that for every element  $n$  of  $\mathbb{N}$  such that  $m \leq n$  holds  $\|S(n) - x\| < r$ .
- (30) Let  $X$  be a real normed space,  $S$  be a sequence of  $X$ , and  $S_1$  be a sequence of  $\text{LinearTopSpaceNorm } X$ . If  $S = S_1$ , then  $S_1$  is convergent iff  $S$  is convergent.
- (31) Let  $X$  be a real normed space,  $S$  be a sequence of  $X$ , and  $S_1$  be a sequence of  $\text{LinearTopSpaceNorm } X$ . If  $S = S_1$  and  $S_1$  is convergent, then  $\text{Lim } S_1 = \{\text{lim } S\}$  and  $\text{lim } S_1 = \text{lim } S$ .
- (32) Let  $X$  be a real normed space,  $V$  be a subset of  $X$ , and  $V_1$  be a subset of  $\text{LinearTopSpaceNorm } X$ . If  $V = V_1$ , then  $V$  is closed iff  $V_1$  is closed.
- (33) Let  $X$  be a real normed space,  $V$  be a subset of  $X$ , and  $V_1$  be a subset of  $\text{LinearTopSpaceNorm } X$ . If  $V = V_1$ , then  $V$  is open iff  $V_1$  is open.
- (34) Let  $X$  be a real normed space,  $U$  be a subset of  $\text{TopSpaceNorm } X$ ,  $U_1$  be a subset of  $\text{LinearTopSpaceNorm } X$ ,  $x$  be a point of  $\text{TopSpaceNorm } X$ , and  $x_1$  be a point of  $\text{LinearTopSpaceNorm } X$ . Suppose  $U = U_1$  and  $x = x_1$ . Then  $U$  is a neighbourhood of  $x$  if and only if  $U_1$  is a neighbourhood of  $x_1$ .
- (35) Let  $X, Y$  be real normed spaces,  $f$  be a function from  $\text{TopSpaceNorm } X$  into  $\text{TopSpaceNorm } Y$ ,  $f_1$  be a function from  $\text{LinearTopSpaceNorm } X$  into  $\text{LinearTopSpaceNorm } Y$ ,  $x$  be a point of  $\text{TopSpaceNorm } X$ , and  $x_1$  be a

point of  $\text{LinearTopSpaceNorm } X$ . Suppose  $f = f_1$  and  $x = x_1$ . Then  $f$  is continuous at  $x$  if and only if  $f_1$  is continuous at  $x_1$ .

- (36) Let  $X, Y$  be real normed spaces,  $f$  be a function from  $\text{TopSpaceNorm } X$  into  $\text{TopSpaceNorm } Y$ , and  $f_1$  be a function from  $\text{LinearTopSpaceNorm } X$  into  $\text{LinearTopSpaceNorm } Y$ . If  $f = f_1$ , then  $f$  is continuous iff  $f_1$  is continuous.

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