

On the Representation of Natural Numbers in Positional Numeral Systems¹

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Summary. In this paper we show that every natural number can be uniquely represented as a base- b numeral. The formalization is based on the proof presented in [11]. We also prove selected divisibility criteria in the base-10 numeral system.

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The notation and terminology used in this paper have been introduced in the following articles: [13], [15], [2], [1], [17], [12], [14], [6], [4], [5], [8], [9], [10], [16], [7], and [3].

1. PRELIMINARIES

One can prove the following propositions:

- (1) For all finite 0-sequences d, e of \mathbb{N} holds $\sum(d \frown e) = \sum d + \sum e$.
- (2) Let S be a sequence of real numbers, d be a finite 0-sequence of \mathbb{N} , and n be a natural number. If $d = S \upharpoonright (n+1)$, then $\sum d = (\sum_{\alpha=0}^n S(\alpha))_{\kappa \in \mathbb{N}}(n)$.
- (3) For all natural numbers k, l, m holds $(k (l^\kappa)_{\kappa \in \mathbb{N}}) \upharpoonright m$ is a finite 0-sequence of \mathbb{N} .
- (4) Let d, e be finite 0-sequences of \mathbb{N} . Suppose $\text{len } d \geq 1$ and $\text{len } d = \text{len } e$ and for every natural number i such that $i \in \text{dom } d$ holds $d(i) \leq e(i)$. Then $\sum d \leq \sum e$.

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- (5) Let d be a finite 0-sequence of \mathbb{N} and n be a natural number. If for every natural number i such that $i \in \text{dom } d$ holds $n \mid d(i)$, then $n \mid \sum d$.
- (6) Let d, e be finite 0-sequences of \mathbb{N} and n be a natural number. Suppose $\text{dom } d = \text{dom } e$ and for every natural number i such that $i \in \text{dom } d$ holds $e(i) = d(i) \bmod n$. Then $\sum d \bmod n = \sum e \bmod n$.

2. REPRESENTATION OF NUMBERS IN THE BASE- b NUMERAL SYSTEM

Let d be a finite 0-sequence of \mathbb{N} and let b be a natural number. The functor $\text{value}(d, b)$ yields a natural number and is defined by the condition (Def. 1).

- (Def. 1) There exists a finite 0-sequence d' of \mathbb{N} such that $\text{dom } d' = \text{dom } d$ and for every natural number i such that $i \in \text{dom } d'$ holds $d'(i) = d(i) \cdot b^i$ and $\text{value}(d, b) = \sum d'$.

Let n, b be natural numbers. Let us assume that $b > 1$. The functor $\text{digits}(n, b)$ yields a finite 0-sequence of \mathbb{N} and is defined as follows:

- (Def. 2)(i) $\text{value}(\text{digits}(n, b), b) = n$ and $(\text{digits}(n, b))(\text{len } \text{digits}(n, b) - 1) \neq 0$ and for every natural number i such that $i \in \text{dom } \text{digits}(n, b)$ holds $0 \leq (\text{digits}(n, b))(i)$ and $(\text{digits}(n, b))(i) < b$ if $n \neq 0$,
- (ii) $\text{digits}(n, b) = \langle 0 \rangle$, otherwise.

One can prove the following two propositions:

- (7) For all natural numbers n, b such that $b > 1$ holds $\text{len } \text{digits}(n, b) \geq 1$.
- (8) For all natural numbers n, b such that $b > 1$ holds $\text{value}(\text{digits}(n, b), b) = n$.

3. SELECTED DIVISIBILITY CRITERIA

One can prove the following propositions:

- (9) For all natural numbers n, k such that $k = 10^n - 1$ holds $9 \mid k$.
- (10) For all natural numbers n, b such that $b > 1$ holds $b \mid n$ iff $(\text{digits}(n, b))(0) = 0$.
- (11) For every natural number n holds $2 \mid n$ iff $2 \mid (\text{digits}(n, 10))(0)$.
- (12) For every natural number n holds $3 \mid n$ iff $3 \mid \sum \text{digits}(n, 10)$.
- (13) For every natural number n holds $5 \mid n$ iff $5 \mid (\text{digits}(n, 10))(0)$.

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