

The Relevance of Measure and Probability, and Definition of Completeness of Probability

Bo Zhang
Shinshu University
Nagano, Japan

Hiroshi Yamazaki
Shinshu University
Nagano, Japan

Yatsuka Nakamura
Shinshu University
Nagano, Japan

Summary. In this article, we first discuss the relation between measure defined using extended real numbers and probability defined using real numbers. Further, we define completeness of probability, and its completion method, and also show that they coincide with those of measure.

MML identifier: PROB_4, version: 7.6.01 4.53.937

The articles [18], [20], [2], [3], [5], [1], [12], [15], [21], [8], [19], [17], [4], [9], [14], [23], [6], [11], [16], [22], [10], [7], and [13] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: n denotes a natural number, X denotes a set, A_1 denotes a sequence of subsets of X , S_1 denotes a σ -field of subsets of X , X_1 denotes a sequence of subsets of S_1 , O_1 denotes a non empty set, S_2 denotes a σ -field of subsets of O_1 , A_2 denotes a sequence of subsets of S_2 , and P denotes a probability on S_2 .

Let us consider X , S_1 , X_1 , n . Then $X_1(n)$ is an element of S_1 .

Next we state two propositions:

- (1) $\text{rng } X_1 \subseteq S_1$.
- (2) For every function f holds f is a sequence of subsets of S_1 iff f is a function from \mathbb{N} into S_1 .

The scheme *LambdaSigmaSSeq* deals with a set \mathcal{A} , a σ -field \mathcal{B} of subsets of \mathcal{A} , and a unary functor \mathcal{F} yielding an element of \mathcal{B} , and states that:

There exists a sequence f of subsets of \mathcal{B} such that for every n holds $f(n) = \mathcal{F}(n)$

for all values of the parameters.

Let us consider X . Note that there exists a sequence of subsets of X which is disjoint valued.

Let us consider X, S_1 . Note that there exists a sequence of subsets of S_1 which is disjoint valued.

One can prove the following propositions:

- (3) For all subsets A, B of X there exists A_1 such that $A_1(0) = A$ and $A_1(1) = B$ and for every n such that $n > 1$ holds $A_1(n) = \emptyset$.
- (4) Let A, B be subsets of X . Suppose A misses B and $A_1(0) = A$ and $A_1(1) = B$ and for every n such that $n > 1$ holds $A_1(n) = \emptyset$. Then A_1 is disjoint valued and $\bigcup A_1 = A \cup B$.
- (5) Let S be a non empty set. Then S is a σ -field of subsets of X if and only if the following conditions are satisfied:
 - (i) $S \subseteq 2^X$,
 - (ii) for every sequence A_1 of subsets of X such that for every n holds $A_1(n) \in S$ holds $\bigcup A_1 \in S$, and
 - (iii) for every subset A of X such that $A \in S$ holds $A^c \in S$.
- (6) For all events A, B of S_2 holds $P(A \setminus B) = P(A \cup B) - P(B)$.
- (7) For all events A, B of S_2 such that $A \subseteq B$ and $P(B) = 0$ holds $P(A) = 0$.
- (8) For every n holds $P(A_2(n)) = 0$ iff $P(\bigcup A_2) = 0$.
- (9) For every set A such that $A \in \text{rng } A_2$ holds $P(A) = 0$ iff $P(\bigcup \text{rng } A_2) = 0$.
- (10) For every function s_1 from \mathbb{N} into \mathbb{R} and for every function E_1 from \mathbb{N} into $\overline{\mathbb{R}}$ such that $s_1 = E_1$ holds $(\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}} = \text{Ser } E_1$.
- (11) Let s_1 be a function from \mathbb{N} into \mathbb{R} and E_1 be a function from \mathbb{N} into $\overline{\mathbb{R}}$. If $s_1 = E_1$ and s_1 is upper bounded, then $\sup s_1 = \sup \text{rng } E_1$.
- (12) Let s_1 be a function from \mathbb{N} into \mathbb{R} and E_1 be a function from \mathbb{N} into $\overline{\mathbb{R}}$. If $s_1 = E_1$ and s_1 is lower bounded, then $\inf s_1 = \inf \text{rng } E_1$.
- (13) Let s_1 be a function from \mathbb{N} into \mathbb{R} and E_1 be a function from \mathbb{N} into $\overline{\mathbb{R}}$. If $s_1 = E_1$ and s_1 is non-negative and summable, then $\sum s_1 = \sum E_1$.
- (14) P is a σ -measure on S_2 .

Let us consider O_1, S_2, P . The functor P2MP yields a σ -measure on S_2 and is defined as follows:

(Def. 1) P2MP = P .

One can prove the following proposition

- (15) Let X be a non empty set, S be a σ -field of subsets of X , and M be a σ -measure on S . If $M(X) = \overline{\mathbb{R}}(1)$, then M is a probability on S .

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . Let us assume that $M(X) = \overline{\mathbb{R}}(1)$. The functor M2P M yielding a probability on S is defined as follows:

(Def. 2) M2P $M = M$.

One can prove the following propositions:

- (16) If A_1 is non-decreasing, then the partial unions of $A_1 = A_1$.
- (17) Suppose A_1 is non-decreasing. Then (the partial diff-unions of A_1)(0) = $A_1(0)$ and for every n holds (the partial diff-unions of A_1)($n+1$) = $A_1(n+1) \setminus A_1(n)$.
- (18) If A_1 is non-decreasing, then for every n holds $A_1(n+1) =$ (the partial diff-unions of A_1)($n+1$) $\cup A_1(n)$.
- (19) If A_1 is non-decreasing, then for every n holds (the partial diff-unions of A_1)($n+1$) misses $A_1(n)$.
- (20) If X_1 is non-decreasing, then the partial unions of $X_1 = X_1$.
- (21) Suppose X_1 is non-decreasing. Then (the partial diff-unions of X_1)(0) = $X_1(0)$ and for every n holds (the partial diff-unions of X_1)($n+1$) = $X_1(n+1) \setminus X_1(n)$.
- (22) If X_1 is non-decreasing, then for every n holds (the partial diff-unions of X_1)($n+1$) misses $X_1(n)$.

Let us consider O_1, S_2, P . We say that P is complete on S_2 if and only if:

(Def. 3) For every subset A of O_1 and for every set B such that $B \in S_2$ holds if $A \subseteq B$ and $P(B) = 0$, then $A \in S_2$.

Next we state the proposition

- (23) P is complete on S_2 iff P2M P is complete on S_2 .

Let us consider O_1, S_2, P . A subset of O_1 is called a set with measure zero w.r.t. P if:

(Def. 4) There exists a set A such that $A \in S_2$ and it $\subseteq A$ and $P(A) = 0$.

We now state three propositions:

- (24) Let Y be a subset of O_1 . Then Y is a set with measure zero w.r.t. P if and only if Y is a set with measure zero w.r.t. P2M P .
- (25) \emptyset is a set with measure zero w.r.t. P .
- (26) Let B_1, B_2 be sets. Suppose $B_1 \in S_2$ and $B_2 \in S_2$. Let C_1, C_2 be sets with measure zero w.r.t. P . If $B_1 \cup C_1 = B_2 \cup C_2$, then $P(B_1) = P(B_2)$.

Let us consider O_1, S_2, P . The functor $\text{COM}(S_2, P)$ yields a non empty family of subsets of O_1 and is defined by the condition (Def. 5).

(Def. 5) Let A be a set. Then $A \in \text{COM}(S_2, P)$ if and only if there exists a set B such that $B \in S_2$ and there exists a set C with measure zero w.r.t. P such that $A = B \cup C$.

Next we state two propositions:

- (27) For every set C with measure zero w.r.t. P holds $C \in \text{COM}(S_2, P)$.
- (28) $\text{COM}(S_2, P) = \text{COM}(S_2, \text{P2M } P)$.

Let us consider O_1, S_2, P and let A be an element of $\text{COM}(S_2, P)$. The functor $\text{P}_{\text{COM}2\text{M}_{\text{COM}}} A$ yields an element of $\text{COM}(S_2, \text{P}2\text{M} P)$ and is defined by:

(Def. 6) $\text{P}_{\text{COM}2\text{M}_{\text{COM}}} A = A$.

Next we state the proposition

(29) $S_2 \subseteq \text{COM}(S_2, P)$.

Let us consider O_1, S_2, P and let A be an element of $\text{COM}(S_2, P)$. The functor $\text{ProbPart} A$ yielding a non empty family of subsets of O_1 is defined by:

(Def. 7) For every set B holds $B \in \text{ProbPart} A$ iff $B \in S_2$ and $B \subseteq A$ and $A \setminus B$ is a set with measure zero w.r.t. P .

We now state several propositions:

(30) For every element A of $\text{COM}(S_2, P)$ holds

$$\text{ProbPart} A = \text{MeasPart} \text{P}_{\text{COM}2\text{M}_{\text{COM}}} A.$$

(31) For every element A of $\text{COM}(S_2, P)$ and for all sets A_1, A_3 such that $A_1 \in \text{ProbPart} A$ and $A_3 \in \text{ProbPart} A$ holds $P(A_1) = P(A_3)$.

(32) For every function F from \mathbb{N} into $\text{COM}(S_2, P)$ there exists a sequence B_3 of subsets of S_2 such that for every n holds $B_3(n) \in \text{ProbPart} F(n)$.

(33) Let F be a function from \mathbb{N} into $\text{COM}(S_2, P)$ and B_3 be a sequence of subsets of S_2 . Then there exists a sequence C_3 of subsets of O_1 such that for every n holds $C_3(n) = F(n) \setminus B_3(n)$.

(34) Let B_3 be a sequence of subsets of O_1 . Suppose that for every n holds $B_3(n)$ is a set with measure zero w.r.t. P . Then there exists a sequence C_3 of subsets of S_2 such that for every n holds $B_3(n) \subseteq C_3(n)$ and $P(C_3(n)) = 0$.

(35) Let D be a non empty family of subsets of O_1 . Suppose that for every set A holds $A \in D$ iff there exists a set B such that $B \in S_2$ and there exists a set C with measure zero w.r.t. P such that $A = B \cup C$. Then D is a σ -field of subsets of O_1 .

Let us consider O_1, S_2, P . Then $\text{COM}(S_2, P)$ is a σ -field of subsets of O_1 .

Let us consider O_1, S_2, P . We see that the set with measure zero w.r.t. P is an event of $\text{COM}(S_2, P)$.

Next we state two propositions:

(36) For every set A holds $A \in \text{COM}(S_2, P)$ iff there exist sets A_1, A_3 such that $A_1 \in S_2$ and $A_3 \in S_2$ and $A_1 \subseteq A$ and $A \subseteq A_3$ and $P(A_3 \setminus A_1) = 0$.

(37) Let C be a non empty family of subsets of O_1 . Suppose that for every set A holds $A \in C$ iff there exist sets A_1, A_3 such that $A_1 \in S_2$ and $A_3 \in S_2$ and $A_1 \subseteq A$ and $A \subseteq A_3$ and $P(A_3 \setminus A_1) = 0$. Then $C = \text{COM}(S_2, P)$.

Let us consider O_1, S_2, P . The functor $\text{COM}(P)$ yields a probability on $\text{COM}(S_2, P)$ and is defined as follows:

(Def. 8) For every set B such that $B \in S_2$ and for every set C with measure zero w.r.t. P holds $(\text{COM}(P))(B \cup C) = P(B)$.

One can prove the following propositions:

- (38) $\text{COM}(P) = \text{COM}(P2M P)$.
- (39) $\text{COM}(P)$ is complete on $\text{COM}(S_2, P)$.
- (40) For every event A of S_2 holds $P(A) = (\text{COM}(P))(A)$.
- (41) For every set C with measure zero w.r.t. P holds $(\text{COM}(P))(C) = 0$.
- (42) For every element A of $\text{COM}(S_2, P)$ and for every set B such that $B \in \text{ProbPart } A$ holds $P(B) = (\text{COM}(P))(A)$.

REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(3):589–593, 1990.
- [3] Józef Białas. Completeness of the σ -additive measure. Measure theory. *Formalized Mathematics*, 2(5):689–693, 1991.
- [4] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(1):163–171, 1991.
- [5] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(1):173–183, 1991.
- [6] Józef Białas. The σ -additive measure theory. *Formalized Mathematics*, 2(2):263–270, 1991.
- [7] Józef Białas. Some properties of the intervals. *Formalized Mathematics*, 5(1):21–26, 1996.
- [8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [11] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in \mathcal{E}^2 . *Formalized Mathematics*, 6(3):427–440, 1997.
- [12] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(2):273–275, 1990.
- [13] Andrzej Nędzusiak. Probability. *Formalized Mathematics*, 1(4):745–749, 1990.
- [14] Andrzej Nędzusiak. σ -fields and probability. *Formalized Mathematics*, 1(2):401–407, 1990.
- [15] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [16] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Formalized Mathematics*, 2(4):449–452, 1991.
- [17] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
- [18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [21] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [22] Bo Zhang, Hiroshi Yamazaki, and Yatsuka Nakamura. Inferior limit and superior limit of sequences of real numbers. *Formalized Mathematics*, 13(3):375–381, 2005.
- [23] Bo Zhang, Hiroshi Yamazaki, and Yatsuka Nakamura. Set sequences and monotone class. *Formalized Mathematics*, 13(4):435–441, 2005.

Received November 23, 2005