

Basic Properties of Determinants of Square Matrices over a Field¹

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Summary. In this paper I present basic properties of the determinant of square matrices over a field and selected properties of the sign of a permutation. First, I define the sign of a permutation by the requirement

$$\operatorname{sgn}(p) = \prod_{1 \leq i < j \leq n} \operatorname{sgn}(p(j) - p(i)),$$

where p is any fixed permutation of a set with n elements. I prove that the sign of a product of two permutations is the same as the product of their signs and show the relation between signs and parity of permutations. Then I consider the determinant of a linear combination of lines, the determinant of a matrix with permuted lines and the determinant of a matrix with a repeated line. Finally, at the end I prove that the determinant of a product of two square matrices is equal to the product of their determinants.

MML identifier: MATRIX11, version: 7.8.04 4.81.962

The articles [21], [12], [27], [18], [13], [28], [7], [10], [8], [3], [4], [19], [25], [24], [16], [20], [11], [6], [5], [14], [22], [15], [31], [23], [26], [32], [1], [29], [9], [2], [17], and [30] provide the terminology and notation for this paper.

¹This work has been partially supported by the KBN grant 4 T11C 039 24 and the FP6 IST grant TYPES No. 510996.

1. THE SIGN OF A PERMUTATION

For simplicity, we use the following convention: x, X denote sets, i, j, k, l, n, m denote natural numbers, D denotes a non empty set, K denotes a field, a, b denote elements of K , p_1, p, q denote elements of the permutations of n -element set, P_1, P denote permutations of $\text{Seg } n$, F denotes a function from $\text{Seg } n$ into $\text{Seg } n$, p_2, p_3, q_2, p_4 denote elements of the permutations of $(n+2)$ -element set, and P_2 denotes a permutation of $\text{Seg}(n+2)$.

Let X be a set. We introduce $2\text{Set } X$ as a synonym of $\text{TwoElementSets}(X)$.

The following three propositions are true:

- (1) $X \in 2\text{Set } \text{Seg } n$ iff there exist i, j such that $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $i < j$ and $X = \{i, j\}$.
- (2) $2\text{Set } \text{Seg } 0 = \emptyset$ and $2\text{Set } \text{Seg } 1 = \emptyset$.
- (3) For every n such that $n \geq 2$ holds $\{1, 2\} \in 2\text{Set } \text{Seg } n$.

Let us consider n . Observe that $2\text{Set } \text{Seg}(n+2)$ is non empty and finite.

Let us consider n, x and let p_1 be an element of the permutations of n -element set. Note that $p_1(x)$ is natural.

Let us consider K . One can verify that the multiplication of K is unital and the multiplication of K is associative.

Let us consider n, K and let p_2 be an element of the permutations of $(n+2)$ -element set. The functor $\text{Part-sgn}(p_2, K)$ yielding a function from $2\text{Set } \text{Seg}(n+2)$ into the carrier of K is defined by the condition (Def. 1).

(Def. 1) Let i, j be elements of \mathbb{N} such that $i \in \text{Seg}(n+2)$ and $j \in \text{Seg}(n+2)$ and $i < j$. Then

- (i) if $p_2(i) < p_2(j)$, then $(\text{Part-sgn}(p_2, K))(\{i, j\}) = \mathbf{1}_K$, and
- (ii) if $p_2(i) > p_2(j)$, then $(\text{Part-sgn}(p_2, K))(\{i, j\}) = -\mathbf{1}_K$.

One can prove the following proposition

- (4) Let X be an element of $\text{Fin } 2\text{Set } \text{Seg}(n+2)$. Suppose that for every x such that $x \in X$ holds $(\text{Part-sgn}(p_3, K))(x) = \mathbf{1}_K$. Then (the multiplication of K)- $\sum_X \text{Part-sgn}(p_3, K) = \mathbf{1}_K$.

In the sequel s denotes an element of $2\text{Set } \text{Seg}(n+2)$.

The following propositions are true:

- (5) $(\text{Part-sgn}(p_3, K))(s) = \mathbf{1}_K$ or $(\text{Part-sgn}(p_3, K))(s) = -\mathbf{1}_K$.
- (6) For all i, j such that $i \in \text{Seg}(n+2)$ and $j \in \text{Seg}(n+2)$ and $i < j$ and $p_3(i) = q_2(i)$ and $p_3(j) = q_2(j)$ holds $(\text{Part-sgn}(p_3, K))(\{i, j\}) = (\text{Part-sgn}(q_2, K))(\{i, j\})$.
- (7) Let X be an element of $\text{Fin } 2\text{Set } \text{Seg}(n+2)$, given p_3, q_2 , and F be a finite set such that $F = \{s : s \in X \wedge (\text{Part-sgn}(p_3, K))(s) \neq (\text{Part-sgn}(q_2, K))(s)\}$. Then

- (i) if $\text{card } F \bmod 2 = 0$, then (the multiplication of K)- $\sum_X \text{Part-sgn}(p_3, K) =$ (the multiplication of K)- $\sum_X \text{Part-sgn}(q_2, K)$, and
- (ii) if $\text{card } F \bmod 2 = 1$, then (the multiplication of K)- $\sum_X \text{Part-sgn}(p_3, K) = -((\text{the multiplication of } K)\text{-} \sum_X \text{Part-sgn}(q_2, K))$.
- (8) Let P be a permutation of $\text{Seg } n$. Suppose P is a transposition. Let given i, j . Suppose $i < j$. Then $P(i) = j$ if and only if the following conditions are satisfied:
 - (i) $i \in \text{dom } P$,
 - (ii) $j \in \text{dom } P$,
 - (iii) $P(i) = j$,
 - (iv) $P(j) = i$, and
 - (v) for every k such that $k \neq i$ and $k \neq j$ and $k \in \text{dom } P$ holds $P(k) = k$.
- (9) Let given p_3, q_2, p_4, i, j . Suppose $p_4 = p_3 \cdot q_2$ and q_2 is a transposition and $q_2(i) = j$ and $i < j$. Let given s . If $(\text{Part-sgn}(p_3, K))(s) \neq (\text{Part-sgn}(p_4, K))(s)$, then $i \in s$ or $j \in s$.
- (10) Let given p_3, q_2, p_4, i, j, K . Suppose $p_4 = p_3 \cdot q_2$ and q_2 is a transposition and $q_2(i) = j$ and $i < j$ and $\mathbf{1}_K \neq -\mathbf{1}_K$. Then
 - (i) $(\text{Part-sgn}(p_3, K))(\{i, j\}) \neq (\text{Part-sgn}(p_4, K))(\{i, j\})$, and
 - (ii) for every k such that $k \in \text{Seg}(n + 2)$ and $i \neq k$ and $j \neq k$ holds $(\text{Part-sgn}(p_3, K))(\{i, k\}) \neq (\text{Part-sgn}(p_4, K))(\{i, k\})$ iff $(\text{Part-sgn}(p_3, K))(\{j, k\}) \neq (\text{Part-sgn}(p_4, K))(\{j, k\})$.

Let us consider n, K and let p_2 be an element of the permutations of $(n + 2)$ -element set. The functor $\text{sgn}(p_2, K)$ yielding an element of K is defined by:

(Def. 2) $\text{sgn}(p_2, K) = (\text{the multiplication of } K)\text{-} \sum_{\Omega_{2\text{Set Seg}(n+2)}^f} \text{Part-sgn}(p_2, K)$.

The following propositions are true:

- (11) $\text{sgn}(p_3, K) = \mathbf{1}_K$ or $\text{sgn}(p_3, K) = -\mathbf{1}_K$.
- (12) For every element I_1 of the permutations of $(n + 2)$ -element set such that $I_1 = \text{idseq}(n + 2)$ holds $\text{sgn}(I_1, K) = \mathbf{1}_K$.
- (13) For all p_3, q_2, p_4 such that $p_4 = p_3 \cdot q_2$ and q_2 is a transposition holds $\text{sgn}(p_4, K) = -\text{sgn}(p_3, K)$.
- (14) For every element t_1 of the permutations of $(n + 2)$ -element set such that t_1 is a transposition holds $\text{sgn}(t_1, K) = -\mathbf{1}_K$.
- (15) Let P be a finite sequence of elements of A_{n+2} and p_3 be an element of the permutations of $(n + 2)$ -element set such that $p_3 = \prod P$ and for every i such that $i \in \text{dom } P$ there exists an element t_2 of the permutations of $(n + 2)$ -element set such that $P(i) = t_2$ and t_2 is a transposition. Then
 - (i) if $\text{len } P \bmod 2 = 0$, then $\text{sgn}(p_3, K) = \mathbf{1}_K$, and
 - (ii) if $\text{len } P \bmod 2 = 1$, then $\text{sgn}(p_3, K) = -\mathbf{1}_K$.
- (16) Let given i, j, n . Suppose $i < j$ and $i \in \text{Seg } n$ and $j \in \text{Seg } n$. Then there exists an element t_1 of the permutations of n -element set such that t_1 is a

transposition and $t_1(i) = j$.

- (17) Let p be an element of the permutations of $(k+1)$ -element set. Suppose $p(k+1) \neq k+1$. Then there exists an element t_1 of the permutations of $(k+1)$ -element set such that t_1 is a transposition and $t_1(p(k+1)) = k+1$ and $(t_1 \cdot p)(k+1) = k+1$.
- (18) Let given X, x . Suppose $x \notin X$. Let p_5 be a permutation of $X \cup \{x\}$. If $p_5(x) = x$, then there exists a permutation p of X such that $p_5|_X = p$.
- (19) Let p, q be permutations of X and p_5, q_1 be permutations of $X \cup \{x\}$. If $p_5|_X = p$ and $q_1|_X = q$ and $p_5(x) = x$ and $q_1(x) = x$, then $(p_5 \cdot q_1)|_X = p \cdot q$ and $(p_5 \cdot q_1)(x) = x$.
- (20) For every element t_1 of the permutations of k -element set such that t_1 is a transposition holds $t_1 \cdot t_1 = \text{idseq}(k)$ and $t_1 = t_1^{-1}$.
- (21) Let given p_1 . Then there exists a finite sequence P of elements of A_n such that
- (i) $p_1 = \prod P$, and
 - (ii) for every i such that $i \in \text{dom } P$ there exists an element t_2 of the permutations of n -element set such that $P(i) = t_2$ and t_2 is a transposition.
- (22) K is Fanoian iff $\mathbf{1}_K \neq -\mathbf{1}_K$.
- (23) For every Fanoian field K holds p_2 is even iff $\text{sgn}(p_2, K) = \mathbf{1}_K$ and p_2 is odd iff $\text{sgn}(p_2, K) = -\mathbf{1}_K$.
- (24) For all p_3, q_2, p_4 such that $p_4 = p_3 \cdot q_2$ holds $\text{sgn}(p_4, K) = \text{sgn}(p_3, K) \cdot \text{sgn}(q_2, K)$.
- (25) p is even and q is even or p is odd and q is odd iff $p \cdot q$ is even.
- (26) $(-1)^{\text{sgn}(p_2)} a = \text{sgn}(p_2, K) \cdot a$.
- (27) For every element t_1 of the permutations of $(n+2)$ -element set such that t_1 is a transposition holds t_1 is odd.

Let us consider n . Observe that there exists a permutation of $\text{Seg}(n+2)$ which is odd.

2. THE DETERMINANT OF A LINEAR COMBINATION OF LINES

For simplicity, we follow the rules: p_6 denotes a finite sequence of elements of D , M denotes a matrix over D of dimension $n \times m$, p_7, q_3 denote finite sequences of elements of K , and A, B denote matrices over K of dimension n .

Let us consider l, n, m, D , let M be a matrix over D of dimension $n \times m$, and let p_6 be a finite sequence of elements of D . The functor $\text{ReplaceLine}(M, l, p_6)$ yields a matrix over D of dimension $n \times m$ and is defined as follows:

- (Def. 3)(i) $\text{len } \text{ReplaceLine}(M, l, p_6) = \text{len } M$ and $\text{width } \text{ReplaceLine}(M, l, p_6) = \text{width } M$ and for all i, j such that $\langle i, j \rangle \in$ the indices of M holds

- if $i \neq l$, then $(\text{ReplaceLine}(M, l, p_6))_{i,j} = M_{i,j}$ and if $i = l$, then $(\text{ReplaceLine}(M, l, p_6))_{l,j} = p_6(j)$ if $\text{len } p_6 = \text{width } M$,
- (ii) $\text{ReplaceLine}(M, l, p_6) = M$, otherwise.

Let us consider l, n, m, D , let M be a matrix over D of dimension $n \times m$, and let p_6 be a finite sequence of elements of D . We introduce $\text{RLine}(M, l, p_6)$ as a synonym of $\text{ReplaceLine}(M, l, p_6)$.

The following propositions are true:

- (28) For all l, M, p_6, i such that $i \in \text{Seg } n$ holds if $i = l$ and $\text{len } p_6 = \text{width } M$, then $\text{Line}(\text{RLine}(M, l, p_6), i) = p_6$ and if $i \neq l$, then $\text{Line}(\text{RLine}(M, l, p_6), i) = \text{Line}(M, i)$.
- (29) For all M, p_6 such that $\text{len } p_6 = \text{width } M$ and for every element p' of D^* such that $p_6 = p'$ holds $\text{RLine}(M, l, p_6) = \text{Replace}(M, l, p')$.
- (30) $M = \text{RLine}(M, l, \text{Line}(M, l))$.
- (31) Let given l, p_7, q_3, p_1 . Suppose $l \in \text{Seg } n$ and $\text{len } p_7 = n$ and $\text{len } q_3 = n$. Let M be a matrix over K of dimension n . Then (the multiplication of K) $\otimes (p_1\text{-Path } \text{RLine}(M, l, a \cdot p_7 + b \cdot q_3)) = a \cdot ((\text{the multiplication of } K) \otimes (p_1\text{-Path } \text{RLine}(M, l, p_7))) + b \cdot ((\text{the multiplication of } K) \otimes (p_1\text{-Path } \text{RLine}(M, l, q_3)))$.
- (32) Let given l, p_7, q_3, p_1 . Suppose $l \in \text{Seg } n$ and $\text{len } p_7 = n$ and $\text{len } q_3 = n$. Let M be a matrix over K of dimension n . Then (the product on paths of $\text{RLine}(M, l, a \cdot p_7 + b \cdot q_3))(p_1) = a \cdot (\text{the product on paths of } \text{RLine}(M, l, p_7))(p_1) + b \cdot (\text{the product on paths of } \text{RLine}(M, l, q_3))(p_1)$.
- (33) Let given l, p_7, q_3 . Suppose $l \in \text{Seg } n$ and $\text{len } p_7 = n$ and $\text{len } q_3 = n$. Let M be a matrix over K of dimension n . Then $\text{Det } \text{RLine}(M, l, a \cdot p_7 + b \cdot q_3) = a \cdot \text{Det } \text{RLine}(M, l, p_7) + b \cdot \text{Det } \text{RLine}(M, l, q_3)$.
- (34) If $l \in \text{Seg } n$ and $\text{len } p_7 = n$, then $\text{Det } \text{RLine}(A, l, a \cdot p_7) = a \cdot \text{Det } \text{RLine}(A, l, p_7)$.
- (35) If $l \in \text{Seg } n$, then $\text{Det } \text{RLine}(A, l, a \cdot \text{Line}(A, l)) = a \cdot \text{Det } A$.
- (36) If $l \in \text{Seg } n$ and $\text{len } p_7 = n$ and $\text{len } q_3 = n$, then $\text{Det } \text{RLine}(A, l, p_7 + q_3) = \text{Det } \text{RLine}(A, l, p_7) + \text{Det } \text{RLine}(A, l, q_3)$.

3. THE DETERMINANT OF A MATRIX WITH PERMUTATED LINES AND WITH A REPEATED LINE

Let us consider n, m, D , let F be a function from $\text{Seg } n$ into $\text{Seg } n$, and let M be a matrix over D of dimension $n \times m$. Then $M \cdot F$ is a matrix over D of dimension $n \times m$ and it can be characterized by the condition:

- (Def. 4) $\text{len}(M \cdot F) = \text{len } M$ and $\text{width}(M \cdot F) = \text{width } M$ and for all i, j, k such that $\langle i, j \rangle \in$ the indices of M and $F(i) = k$ holds $(M \cdot F)_{i,j} = M_{k,j}$.

The following propositions are true:

- (37)(i) The indices of M = the indices of $M \cdot F$, and
(ii) for all i, j such that $\langle i, j \rangle \in$ the indices of M there exists k such that $F(i) = k$ and $\langle k, j \rangle \in$ the indices of M and $(M \cdot F)_{i,j} = M_{k,j}$.
- (38) For every matrix M over D of dimension $n \times m$ and for every F and for every k such that $k \in \text{Seg } n$ holds $\text{Line}(M \cdot F, k) = M(F(k))$.
- (39) $M \cdot \text{idseq}(n) = M$.
- (40) For all p, P_1, q such that $q = p \cdot P_1^{-1}$ holds $p\text{-Path } A \cdot P_1 = (q\text{-Path } A) \cdot P_1$.
- (41) For all p, P_1, q such that $q = p \cdot P_1^{-1}$ holds (the multiplication of K) \otimes ($p\text{-Path } A \cdot P_1$) = (the multiplication of K) \otimes ($q\text{-Path } A$).
- (42) For all p_3, q_2 such that $q_2 = p_3^{-1}$ holds $\text{sgn}(p_3, K) = \text{sgn}(q_2, K)$.
- (43) Let M be a matrix over K of dimension $n + 2$ and given p_2, P_2 . Suppose $p_2 = P_2$. Let given p_3, q_2 . Suppose $q_2 = p_3 \cdot P_2^{-1}$. Then (the product on paths of M)(q_2) = $\text{sgn}(p_2, K) \cdot$ (the product on paths of $M \cdot P_2$)(p_3).
- (44) Let given p_1 . Then there exists a permutation P of the permutations of n -element set such that for every element p of the permutations of n -element set holds $P(p) = p \cdot p_1$.
- (45) For every matrix M over K of dimension $n + 2 \times n + 2$ and for all p_2, P_2 such that $p_2 = P_2$ holds $\text{Det}(M \cdot P_2) = \text{sgn}(p_2, K) \cdot \text{Det } M$.
- (46) For every matrix M over K of dimension n and for all p_1, P_1 such that $p_1 = P_1$ holds $\text{Det}(M \cdot P_1) = (-1)^{\text{sgn}(p_1)} \text{Det } M$.
- (47) Let P_3 be a permutation of the permutations of n -element set and given p_1 . If p_1 is odd and for every p holds $P_3(p) = p \cdot p_1$, then $P_3^\circ \{p : p \text{ is even}\} = \{q : q \text{ is odd}\}$.
- (48) Let given n . Suppose $n \geq 2$. Then there exist finite sets O_1, E_1 such that $E_1 = \{p : p \text{ is even}\}$ and $O_1 = \{q : q \text{ is odd}\}$ and $E_1 \cap O_1 = \emptyset$ and $E_1 \cup O_1 =$ the permutations of n -element set and $\text{card } E_1 = \text{card } O_1$.
- (49) Let given i, j . Suppose $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $i < j$. Let M be a matrix over K of dimension n . Suppose $\text{Line}(M, i) = \text{Line}(M, j)$. Let p, q, t_1 be elements of the permutations of n -element set. Suppose $q = p \cdot t_1$ and t_1 is a transposition and $t_1(i) = j$. Then (the product on paths of M)(q) = $-$ (the product on paths of M)(p).
- (50) Let given i, j . Suppose $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $i < j$. Let M be a matrix over K of dimension n . If $\text{Line}(M, i) = \text{Line}(M, j)$, then $\text{Det } M = 0_K$.
- (51) For all i, j such that $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $i \neq j$ holds $\text{Det RLine}(A, i, \text{Line}(A, j)) = 0_K$.
- (52) For all i, j such that $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $i \neq j$ holds $\text{Det RLine}(A, i, a \cdot \text{Line}(A, j)) = 0_K$.
- (53) For all i, j such that $i \in \text{Seg } n$ and $j \in \text{Seg } n$ and $i \neq j$ holds $\text{Det } A =$

- $\text{Det RLine}(A, i, \text{Line}(A, i) + a \cdot \text{Line}(A, j)).$
- (54) If $F \notin$ the permutations of n -element set, then $\text{Det}(A \cdot F) = 0_K$.

4. THE DETERMINANT OF A PRODUCT OF TWO SQUARE MATRICES

Let K be a non empty loop structure. The functor $\text{addFinS } K$ yielding a binary operation on $(\text{the carrier of } K)^*$ is defined as follows:

- (Def. 5) For all elements p_5, p_3 of $(\text{the carrier of } K)^*$ holds $(\text{addFinS } K)(p_5, p_3) = p_5 + p_3$.

Let K be an Abelian non empty loop structure. One can verify that $\text{addFinS } K$ is commutative.

Let K be an add-associative non empty loop structure. Note that $\text{addFinS } K$ is associative.

The following propositions are true:

- (55) Let A, B be matrices over K . Suppose $\text{width } A = \text{len } B$ and $\text{len } B > 0$. Let given i . Suppose $i \in \text{Seg len } A$. Then there exists a finite sequence P of elements of $(\text{the carrier of } K)^*$ such that $\text{len } P = \text{len } B$ and $\text{Line}(A \cdot B, i) = \text{addFinS } K \odot P$ and for every j such that $j \in \text{Seg len } B$ holds $P(j) = A_{i,j} \cdot \text{Line}(B, j)$.
- (56) Let A, B, C be matrices over K of dimension n and given i . Suppose $i \in \text{Seg } n$. Then there exists a finite sequence P of elements of K such that $\text{len } P = n$ and $\text{Det RLine}(C, i, \text{Line}(A \cdot B, i)) = \text{the addition of } K \odot P$ and for every j such that $j \in \text{Seg } n$ holds $P(j) = A_{i,j} \cdot \text{Det RLine}(C, i, \text{Line}(B, j))$.
- (57) Let X be a set, Y be a non empty set, and given x . Suppose $x \notin X$. Then there exists a function B_1 from $\{Y^X, Y\}$ into $Y^{X \cup \{x\}}$ such that
- (i) B_1 is bijective, and
 - (ii) for every function f from X into Y and for every function F from $X \cup \{x\}$ into Y such that $F \upharpoonright X = f$ holds $B_1(\langle f, F(x) \rangle) = F$.
- (58) Let X be a finite set, Y be a non empty finite set, and given x . Suppose $x \notin X$. Let F be a binary operation on D . Suppose F is commutative and associative and has a unity and an inverse operation. Let f be a function from Y^X into D and g be a function from $Y^{X \cup \{x\}}$ into D . Suppose that for every function H from X into Y and for every element S_1 of $\text{Fin}(Y^{X \cup \{x\}})$ such that $S_1 = \{h; h \text{ ranges over functions from } X \cup \{x\} \text{ into } Y: h \upharpoonright X = H\}$ holds $F\text{-}\sum_{S_1} g = f(H)$. Then $F\text{-}\sum_{\Omega_{Y^X}^f} f = F\text{-}\sum_{\Omega_{Y^{X \cup \{x\}}}^f} g$.
- (59) Let A, B be matrices over D of dimension $n \times m$ and given i . Suppose $i \leq n$ and $0 < n$. Let F be a function from $\text{Seg } i$ into $\text{Seg } n$. Then there exists a matrix M over D of dimension $n \times m$ such that $M = A \cdot (B \cdot$

$(\text{idseq}(n) \cdot F) \upharpoonright \text{Seg } i$ and for every j holds if $j \in \text{Seg } i$, then $M(j) = B(F(j))$ and if $j \notin \text{Seg } i$, then $M(j) = A(j)$.

- (60) Let A, B be matrices over K of dimension n . Suppose $0 < n$. Then there exists a function P from $(\text{Seg } n)^{\text{Seg } n}$ into the carrier of K such that
- (i) for every function F from $\text{Seg } n$ into $\text{Seg } n$ there exists a finite sequence P_4 of elements of K such that $\text{len } P_4 = n$ and for all natural numbers F_1, j such that $j \in \text{Seg } n$ and $F_1 = F(j)$ holds $P_4(j) = A_{j, F_1}$ and $P(F) = ((\text{the multiplication of } K) \otimes (P_4)) \cdot \text{Det}(B \cdot F)$, and
 - (ii) $\text{Det}(A \cdot B) = (\text{the addition of } K) - \sum_{(\text{Seg } n)^{\text{Seg } n}} P$.
- (61) Let A, B be matrices over K of dimension n . Suppose $0 < n$. Then there exists a function P from the permutations of n -element set into the carrier of K such that
- (i) $\text{Det}(A \cdot B) = (\text{the addition of } K) - \sum_{\text{the permutations of } n\text{-element set}} P$, and
 - (ii) for every element p_1 of the permutations of n -element set holds $P(p_1) = ((\text{the multiplication of } K) \otimes (p_1\text{-Path } A)) \cdot (-1)^{\text{sgn}(p_1)} \text{Det } B$.
- (62) For all matrices A, B over K of dimension n such that $0 < n$ holds $\text{Det}(A \cdot B) = \text{Det } A \cdot \text{Det } B$.

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Received March 21, 2007
