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Basic Properties of Determinants of Square Matrices over a Field[1]

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Summary. In this paper I present basic properties of the determinant of square matrices over a field and selected properties of the sign of a permutation. First, I define the sign of a permutation by the requirement

$$\mathrm{sgn}(p) = \prod_{1 \leq i < j \leq n} \mathrm{sgn}(p(j) - p(i)),$$

where p is any fixed permutation of a set with n elements. I prove that the sign of a product of two permutations is the same as the product of their signs and show the relation between signs and parity of permutations. Then I consider the determinant of a linear combination of lines, the determinant of a matrix with permutated lines and the determinant of a matrix with a repeated line. Finally, at the end I prove that the determinant of a product of two square matrices is equal to the product of their determinants.

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The articles [21], [12], [27], [18], [13], [28], [7], [10], [8], [3], [4], [19], [25], [24], [16], [20], [11], [6], [5], [14], [22], [15], [31], [23], [26], [32], [1], [29], [9], [2], [17], and [30] provide the terminology and notation for this paper.

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1. The Sign of a Permutation

For simplicity, we use the following convention: x, X denote sets, i, j, k, l, n, m denote natural numbers, D denotes a non empty set, K denotes a field, a, b denote elements of K, p_1, p, q denote elements of the permutations of n-element set, P_1, P denote permutations of $\mathrm{Seg}\, n$, F denotes a function from $\mathrm{Seg}\, n$ into $\mathrm{Seg}\, n$, p_2, p_3, q_2, p_4 denote elements of the permutations of $(n+2)$-element set, and P_2 denotes a permutation of $\mathrm{Seg}(n+2)$.

Let X be a set. We introduce $\mathrm{2Set}\, X$ as a synonym of $\mathrm{TwoElementSets}(X)$.

The following three propositions are true:

(1) $X \in \mathrm{2Set}\,\mathrm{Seg}\, n$ iff there exist i, j such that $i \in \mathrm{Seg}\, n$ and $j \in \mathrm{Seg}\, n$ and $i < j$ and $X = \{i, j\}$.

(2) $\mathrm{2Set}\,\mathrm{Seg}\, 0 = \emptyset$ and $\mathrm{2Set}\,\mathrm{Seg}\, 1 = \emptyset$.

(3) For every n such that $n \geq 2$ holds $\{1, 2\} \in \mathrm{2Set}\,\mathrm{Seg}\, n$.

Let us consider n. Observe that $\mathrm{2Set}\,\mathrm{Seg}(n+2)$ is non empty and finite.

Let us consider n, x and let p_1 be an element of the permutations of n-element set. Note that $p_1(x)$ is natural.

Let us consider K. One can verify that the multiplication of K is unital and the multiplication of K is associative.

Let us consider n, K and let p_2 be an element of the permutations of $(n+2)$-element set. The functor $\text{Part-sgn}(p_2, K)$ yielding a function from $\mathrm{2Set}\,\mathrm{Seg}(n+2)$ into the carrier of K is defined by the condition (Def. 1).

(Def. 1) Let i, j be elements of $\mathbb{N}$ such that $i \in \mathrm{Seg}(n+2)$ and $j \in \mathrm{Seg}(n+2)$ and $i < j$. Then

(i) if $p_2(i) < p_2(j)$, then $(\text{Part-sgn}(p_2, K))(\{i, j\}) = \mathbf{1}_K$, and

(ii) if $p_2(i) > p_2(j)$, then $(\text{Part-sgn}(p_2, K))(\{i, j\}) = -\mathbf{1}_K$.

One can prove the following proposition

(4) Let X be an element of $\mathrm{Fin}\,\mathrm{2Set}\,\mathrm{Seg}(n+2)$. Suppose that for every x such that $x \in X$ holds $(\text{Part-sgn}(p_3, K))(x) = \mathbf{1}_K$. Then (the multiplication of K)-$\sum_X \text{Part-sgn}(p_3, K) = \mathbf{1}_K$.

In the sequel s denotes an element of $\mathrm{2Set}\,\mathrm{Seg}(n+2)$.

The following propositions are true:

(5) $(\text{Part-sgn}(p_3, K))(s) = \mathbf{1}_K$ or $(\text{Part-sgn}(p_3, K))(s) = -\mathbf{1}_K$.

(6) For all i, j such that $i \in \mathrm{Seg}(n+2)$ and $j \in \mathrm{Seg}(n+2)$ and $i < j$ and $p_3(i) = q_2(i)$ and $p_3(j) = q_2(j)$ holds $(\text{Part-sgn}(p_3, K))(\{i, j\}) = (\text{Part-sgn}(q_2, K))(\{i, j\})$.

(7) Let X be an element of $\mathrm{Fin}\,\mathrm{2Set}\,\mathrm{Seg}(n+2)$, given p_3, q_2, and F be a finite set such that $F = \{s : s \in X \wedge (\text{Part-sgn}(p_3, K))(s) \neq (\text{Part-sgn}(q_2, K))(s)\}$. Then

(i) if $\operatorname{card} F \bmod 2 = 0$, then (the multiplication of K)- $\sum_X$ Part-sgn$(p_3, K) =$ (the multiplication of K)- $\sum_X$ Part-sgn(q_2, K), and

(ii) if $\operatorname{card} F \bmod 2 = 1$, then (the multiplication of K)- $\sum_X$ Part-sgn$(p_3, K) =$ $-(($the multiplication of K)- $\sum_X$ Part-sgn$(q_2, K))$.

(8) Let P be a permutation of $\operatorname{Seg} n$. Suppose P is a transposition. Let given i, j. Suppose $i < j$. Then $P(i) = j$ if and only if the following conditions are satisfied:

(i) $i \in \operatorname{dom} P$,

(ii) $j \in \operatorname{dom} P$,

(iii) $P(i) = j$,

(iv) $P(j) = i$, and

(v) for every k such that $k \neq i$ and $k \neq j$ and $k \in \operatorname{dom} P$ holds $P(k) = k$.

(9) Let given p_3, q_2, p_4, i, j. Suppose $p_4 = p_3 \cdot q_2$ and q_2 is a transposition and $q_2(i) = j$ and $i < j$. Let given s. If $(\text{Part-sgn}(p_3, K))(s) \neq (\text{Part-sgn}(p_4, K))(s)$, then $i \in s$ or $j \in s$.

(10) Let given p_3, q_2, p_4, i, j, K. Suppose $p_4 = p_3 \cdot q_2$ and q_2 is a transposition and $q_2(i) = j$ and $i < j$ and $\mathbf{1}_K \neq -\mathbf{1}_K$. Then

(i) $(\text{Part-sgn}(p_3, K))(\{i, j\}) \neq (\text{Part-sgn}(p_4, K))(\{i, j\})$, and

(ii) for every k such that $k \in \operatorname{Seg}(n + 2)$ and $i \neq k$ and $j \neq k$ holds $(\text{Part-sgn}(p_3, K))(\{i, k\}) \neq (\text{Part-sgn}(p_4, K))(\{i, k\})$ iff $(\text{Part-sgn}(p_3, K))(\{j, k\}) \neq (\text{Part-sgn}(p_4, K))(\{j, k\})$.

Let us consider n, K and let p_2 be an element of the permutations of $(n+2)$-element set. The functor $\operatorname{sgn}(p_2, K)$ yielding an element of K is defined by:

(Def. 2) $\operatorname{sgn}(p_2, K) =$ (the multiplication of K)- $\sum_{\Omega^{\mathrm{f}}_{2\mathrm{Set}\,\mathrm{Seg}(n+2)}}$ Part-sgn(p_2, K).

The following propositions are true:

(11) $\operatorname{sgn}(p_3, K) = \mathbf{1}_K$ or $\operatorname{sgn}(p_3, K) = -\mathbf{1}_K$.

(12) For every element I_1 of the permutations of $(n+2)$-element set such that $I_1 = \operatorname{idseq}(n + 2)$ holds $\operatorname{sgn}(I_1, K) = \mathbf{1}_K$.

(13) For all p_3, q_2, p_4 such that $p_4 = p_3 \cdot q_2$ and q_2 is a transposition holds $\operatorname{sgn}(p_4, K) = -\operatorname{sgn}(p_3, K)$.

(14) For every element t_1 of the permutations of $(n+2)$-element set such that t_1 is a transposition holds $\operatorname{sgn}(t_1, K) = -\mathbf{1}_K$.

(15) Let P be a finite sequence of elements of A_{n+2} and p_3 be an element of the permutations of $(n + 2)$-element set such that $p_3 = \prod P$ and for every i such that $i \in \operatorname{dom} P$ there exists an element t_2 of the permutations of $(n + 2)$-element set such that $P(i) = t_2$ and t_2 is a transposition. Then

(i) if $\operatorname{len} P \bmod 2 = 0$, then $\operatorname{sgn}(p_3, K) = \mathbf{1}_K$, and

(ii) if $\operatorname{len} P \bmod 2 = 1$, then $\operatorname{sgn}(p_3, K) = -\mathbf{1}_K$.

(16) Let given i, j, n. Suppose $i < j$ and $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$. Then there exists an element t_1 of the permutations of n-element set such that t_1 is a

transposition and $t_1(i) = j$.

(17) Let p be an element of the permutations of $(k+1)$-element set. Suppose $p(k+1) \neq k+1$. Then there exists an element t_1 of the permutations of $(k+1)$-element set such that t_1 is a transposition and $t_1(p(k+1)) = k+1$ and $(t_1 \cdot p)(k+1) = k+1$.

(18) Let given X, x. Suppose $x \notin X$. Let p_5 be a permutation of $X \cup \{x\}$. If $p_5(x) = x$, then there exists a permutation p of X such that $p_5{\restriction}X = p$.

(19) Let p, q be permutations of X and p_5, q_1 be permutations of $X \cup \{x\}$. If $p_5{\restriction}X = p$ and $q_1{\restriction}X = q$ and $p_5(x) = x$ and $q_1(x) = x$, then $(p_5 \cdot q_1){\restriction}X = p \cdot q$ and $(p_5 \cdot q_1)(x) = x$.

(20) For every element t_1 of the permutations of k-element set such that t_1 is a transposition holds $t_1 \cdot t_1 = \mathrm{idseq}(k)$ and $t_1 = {t_1}^{-1}$.

(21) Let given p_1. Then there exists a finite sequence P of elements of A_n such that

(i) $p_1 = \prod P$, and

(ii) for every i such that $i \in \operatorname{dom} P$ there exists an element t_2 of the permutations of n-element set such that $P(i) = t_2$ and t_2 is a transposition.

(22) K is Fanoian iff $\mathbf{1}_K \neq -\mathbf{1}_K$.

(23) For every Fanoian field K holds p_2 is even iff $\mathrm{sgn}(p_2, K) = \mathbf{1}_K$ and p_2 is odd iff $\mathrm{sgn}(p_2, K) = -\mathbf{1}_K$.

(24) For all p_3, q_2, p_4 such that $p_4 = p_3 \cdot q_2$ holds $\mathrm{sgn}(p_4, K) = \mathrm{sgn}(p_3, K) \cdot \mathrm{sgn}(q_2, K)$.

(25) p is even and q is even or p is odd and q is odd iff $p \cdot q$ is even.

(26) $(-1)^{\mathrm{sgn}(p_2)} a = \mathrm{sgn}(p_2, K) \cdot a$.

(27) For every element t_1 of the permutations of $(n+2)$-element set such that t_1 is a transposition holds t_1 is odd.

Let us consider n. Observe that there exists a permutation of $\mathrm{Seg}(n+2)$ which is odd.

2. The Determinant of a Linear Combination of Lines

For simplicity, we follow the rules: p_6 denotes a finite sequence of elements of D, M denotes a matrix over D of dimension $n \times m$, p_7, q_3 denote finite sequences of elements of K, and A, B denote matrices over K of dimension n.

Let us consider l, n, m, D, let M be a matrix over D of dimension $n \times m$, and let p_6 be a finite sequence of elements of D. The functor $\mathrm{ReplaceLine}(M, l, p_6)$ yields a matrix over D of dimension $n \times m$ and is defined as follows:

(Def. 3)(i) $\operatorname{len} \mathrm{ReplaceLine}(M, l, p_6) = \operatorname{len} M$ and $\operatorname{width} \mathrm{ReplaceLine}(M, l, p_6) = \operatorname{width} M$ and for all i, j such that $\langle i, j \rangle \in$ the indices of M holds if $i \neq l$, then $(\text{ReplaceLine}(M, l, p_6))_{i,j} = M_{i,j}$ and if $i = l$, then $(\text{ReplaceLine}(M, l, p_6))_{l,j} = p_6(j)$ if $\text{len}\, p_6 = \text{width}\, M$,

(ii) $\text{ReplaceLine}(M, l, p_6) = M$, otherwise.

Let us consider l, n, m, D, let M be a matrix over D of dimension $n \times m$, and let p_6 be a finite sequence of elements of D. We introduce $\text{RLine}(M, l, p_6)$ as a synonym of $\text{ReplaceLine}(M, l, p_6)$.

The following propositions are true:

(28) For all l, M, p_6, i such that $i \in \text{Seg}\, n$ holds if $i = l$ and $\text{len}\, p_6 = \text{width}\, M$, then $\text{Line}(\text{RLine}(M, l, p_6), i) = p_6$ and if $i \neq l$, then $\text{Line}(\text{RLine}(M, l, p_6), i) = \text{Line}(M, i)$.

(29) For all M, p_6 such that $\text{len}\, p_6 = \text{width}\, M$ and for every element p' of D^* such that $p_6 = p'$ holds $\text{RLine}(M, l, p_6) = \text{Replace}(M, l, p')$.

(30) $M = \text{RLine}(M, l, \text{Line}(M, l))$.

(31) Let given l, p_7, q_3, p_1. Suppose $l \in \text{Seg}\, n$ and $\text{len}\, p_7 = n$ and $\text{len}\, q_3 = n$. Let M be a matrix over K of dimension n. Then (the multiplication of K) $\circledast$ (p_1-Path $\text{RLine}(M, l, a \cdot p_7 + b \cdot q_3)) = a \cdot$ ((the multiplication of K) $\circledast$ (p_1-Path $\text{RLine}(M, l, p_7)$)) $+ b \cdot$ ((the multiplication of K) $\circledast$ (p_1-Path $\text{RLine}(M, l, q_3)$)).

(32) Let given l, p_7, q_3, p_1. Suppose $l \in \text{Seg}\, n$ and $\text{len}\, p_7 = n$ and $\text{len}\, q_3 = n$. Let M be a matrix over K of dimension n. Then (the product on paths of $\text{RLine}(M, l, a \cdot p_7 + b \cdot q_3))(p_1) = a \cdot$ (the product on paths of $\text{RLine}(M, l, p_7))(p_1) + b \cdot$ (the product on paths of $\text{RLine}(M, l, q_3))(p_1)$.

(33) Let given l, p_7, q_3. Suppose $l \in \text{Seg}\, n$ and $\text{len}\, p_7 = n$ and $\text{len}\, q_3 = n$. Let M be a matrix over K of dimension n. Then $\text{Det}\, \text{RLine}(M, l, a \cdot p_7 + b \cdot q_3) = a \cdot \text{Det}\, \text{RLine}(M, l, p_7) + b \cdot \text{Det}\, \text{RLine}(M, l, q_3)$.

(34) If $l \in \text{Seg}\, n$ and $\text{len}\, p_7 = n$, then $\text{Det}\, \text{RLine}(A, l, a \cdot p_7) = a \cdot \text{Det}\, \text{RLine}(A, l, p_7)$.

(35) If $l \in \text{Seg}\, n$, then $\text{Det}\, \text{RLine}(A, l, a \cdot \text{Line}(A, l)) = a \cdot \text{Det}\, A$.

(36) If $l \in \text{Seg}\, n$ and $\text{len}\, p_7 = n$ and $\text{len}\, q_3 = n$, then $\text{Det}\, \text{RLine}(A, l, p_7 + q_3) = \text{Det}\, \text{RLine}(A, l, p_7) + \text{Det}\, \text{RLine}(A, l, q_3)$.

3. The Determinant of a Matrix with Permutated Lines and with a Repeated Line

Let us consider n, m, D, let F be a function from $\text{Seg}\, n$ into $\text{Seg}\, n$, and let M be a matrix over D of dimension $n \times m$. Then $M \cdot F$ is a matrix over D of dimension $n \times m$ and it can be characterized by the condition:

(Def. 4) $\text{len}(M \cdot F) = \text{len}\, M$ and $\text{width}(M \cdot F) = \text{width}\, M$ and for all i, j, k such that $\langle i, j \rangle \in$ the indices of M and $F(i) = k$ holds $(M \cdot F)_{i,j} = M_{k,j}$.

The following propositions are true:

(37)(i) The indices of $M =$ the indices of $M \cdot F$, and

(ii) for all i, j such that $\langle i, j\rangle \in$ the indices of M there exists k such that $F(i) = k$ and $\langle k, j\rangle \in$ the indices of M and $(M \cdot F)_{i,j} = M_{k,j}$.

(38) For every matrix M over D of dimension $n \times m$ and for every F and for every k such that $k \in \operatorname{Seg} n$ holds $\operatorname{Line}(M \cdot F, k) = M(F(k))$.

(39) $M \cdot \operatorname{idseq}(n) = M$.

(40) For all p, P_1, q such that $q = p \cdot {P_1}^{-1}$ holds p-Path $A \cdot P_1 = (q$-Path $A) \cdot P_1$.

(41) For all p, P_1, q such that $q = p \cdot {P_1}^{-1}$ holds (the multiplication of $K) \circledast (p$-Path $A \cdot P_1) =$ (the multiplication of $K) \circledast (q$-Path $A)$.

(42) For all p_3, q_2 such that $q_2 = {p_3}^{-1}$ holds $\operatorname{sgn}(p_3, K) = \operatorname{sgn}(q_2, K)$.

(43) Let M be a matrix over K of dimension $n+2$ and given p_2, P_2. Suppose $p_2 = P_2$. Let given p_3, q_2. Suppose $q_2 = p_3 \cdot {P_2}^{-1}$. Then (the product on paths of $M)(q_2) = \operatorname{sgn}(p_2, K) \cdot$ (the product on paths of $M \cdot P_2)(p_3)$.

(44) Let given p_1. Then there exists a permutation P of the permutations of n-element set such that for every element p of the permutations of n-element set holds $P(p) = p \cdot p_1$.

(45) For every matrix M over K of dimension $n + 2 \times n + 2$ and for all p_2, P_2 such that $p_2 = P_2$ holds $\operatorname{Det}(M \cdot P_2) = \operatorname{sgn}(p_2, K) \cdot \operatorname{Det} M$.

(46) For every matrix M over K of dimension n and for all p_1, P_1 such that $p_1 = P_1$ holds $\operatorname{Det}(M \cdot P_1) = (-1)^{\operatorname{sgn}(p_1)} \operatorname{Det} M$.

(47) Let P_3 be a permutation of the permutations of n-element set and given p_1. If p_1 is odd and for every p holds $P_3(p) = p \cdot p_1$, then ${P_3}^{\circ}\{p : p$ is even$\} = \{q : q$ is odd$\}$.

(48) Let given n. Suppose $n \geq 2$. Then there exist finite sets O_1, E_1 such that $E_1 = \{p : p$ is even$\}$ and $O_1 = \{q : q$ is odd$\}$ and $E_1 \cap O_1 = \emptyset$ and $E_1 \cup O_1 =$ the permutations of n-element set and $\operatorname{card} E_1 = \operatorname{card} O_1$.

(49) Let given i, j. Suppose $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i < j$. Let M be a matrix over K of dimension n. Suppose $\operatorname{Line}(M, i) = \operatorname{Line}(M, j)$. Let p, q, t_1 be elements of the permutations of n-element set. Suppose $q = p \cdot t_1$ and t_1 is a transposition and $t_1(i) = j$. Then (the product on paths of $M)(q) = -$(the product on paths of $M)(p)$.

(50) Let given i, j. Suppose $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i < j$. Let M be a matrix over K of dimension n. If $\operatorname{Line}(M, i) = \operatorname{Line}(M, j)$, then $\operatorname{Det} M = 0_K$.

(51) For all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i \neq j$ holds $\operatorname{Det} \operatorname{RLine}(A, i, \operatorname{Line}(A, j)) = 0_K$.

(52) For all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i \neq j$ holds $\operatorname{Det} \operatorname{RLine}(A, i, a \cdot \operatorname{Line}(A, j)) = 0_K$.

(53) For all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i \neq j$ holds $\operatorname{Det} A =$

Det RLine$(A, i, \mathrm{Line}(A, i) + a \cdot \mathrm{Line}(A, j))$.

(54) If $F \notin$ the permutations of n-element set, then $\mathrm{Det}(A \cdot F) = 0_K$.

4. The Determinant of a Product of Two Square Matrices

Let K be a non empty loop structure. The functor addFinS K yielding a binary operation on (the carrier of $K)^*$ is defined as follows:

(Def. 5) For all elements p_5, p_3 of (the carrier of $K)^*$ holds $(\mathrm{addFinS}\,K)(p_5, p_3) = p_5 + p_3$.

Let K be an Abelian non empty loop structure. One can verify that addFinS K is commutative.

Let K be an add-associative non empty loop structure. Note that addFinS K is associative.

The following propositions are true:

(55) Let A, B be matrices over K. Suppose width $A = \mathrm{len}\,B$ and $\mathrm{len}\,B > 0$. Let given i. Suppose $i \in \mathrm{Seg}\,\mathrm{len}\,A$. Then there exists a finite sequence P of elements of (the carrier of $K)^*$ such that $\mathrm{len}\,P = \mathrm{len}\,B$ and $\mathrm{Line}(A \cdot B, i) = \mathrm{addFinS}\,K \odot P$ and for every j such that $j \in \mathrm{Seg}\,\mathrm{len}\,B$ holds $P(j) = A_{i,j} \cdot \mathrm{Line}(B, j)$.

(56) Let A, B, C be matrices over K of dimension n and given i. Suppose $i \in \mathrm{Seg}\,n$. Then there exists a finite sequence P of elements of K such that $\mathrm{len}\,P = n$ and $\mathrm{Det}\,\mathrm{RLine}(C, i, \mathrm{Line}(A \cdot B, i)) =$ the addition of $K \odot P$ and for every j such that $j \in \mathrm{Seg}\,n$ holds $P(j) = A_{i,j} \cdot \mathrm{Det}\,\mathrm{RLine}(C, i, \mathrm{Line}(B, j))$.

(57) Let X be a set, Y be a non empty set, and given x. Suppose $x \notin X$. Then there exists a function B_1 from $[\!:Y^X, Y:\!]$ into $Y^{X \cup \{x\}}$ such that

(i) B_1 is bijective, and

(ii) for every function f from X into Y and for every function F from $X \cup \{x\}$ into Y such that $F \restriction X = f$ holds $B_1(\langle f, F(x) \rangle) = F$.

(58) Let X be a finite set, Y be a non empty finite set, and given x. Suppose $x \notin X$. Let F be a binary operation on D. Suppose F is commutative and associative and has a unity and an inverse operation. Let f be a function from Y^X into D and g be a function from $Y^{X \cup \{x\}}$ into D. Suppose that for every function H from X into Y and for every element S_1 of $\mathrm{Fin}(Y^{X \cup \{x\}})$ such that $S_1 = \{h; h$ ranges over functions from $X \cup \{x\}$ into $Y\colon h \restriction X = H\}$ holds $F\text{-}\sum_{S_1} g = f(H)$. Then $F\text{-}\sum_{\Omega^{\mathrm{f}}_{Y^X}} f = F\text{-}\sum_{\Omega^{\mathrm{f}}_{Y^{X \cup \{x\}}}} g$.

(59) Let A, B be matrices over D of dimension $n \times m$ and given i. Suppose $i \leq n$ and $0 < n$. Let F be a function from $\mathrm{Seg}\,i$ into $\mathrm{Seg}\,n$. Then there exists a matrix M over D of dimension $n \times m$ such that $M = A {+\cdot} (B \cdot$

$(\mathrm{idseq}(n) + \cdot F)) \restriction \mathrm{Seg}\, i$ and for every j holds if $j \in \mathrm{Seg}\, i$, then $M(j) = B(F(j))$ and if $j \notin \mathrm{Seg}\, i$, then $M(j) = A(j)$.

(60) Let A, B be matrices over K of dimension n. Suppose $0 < n$. Then there exists a function P from $(\mathrm{Seg}\, n)^{\mathrm{Seg}\, n}$ into the carrier of K such that

(i) for every function F from $\mathrm{Seg}\, n$ into $\mathrm{Seg}\, n$ there exists a finite sequence P_4 of elements of K such that $\mathrm{len}\, P_4 = n$ and for all natural numbers F_1, j such that $j \in \mathrm{Seg}\, n$ and $F_1 = F(j)$ holds $P_4(j) = A_{j,F_1}$ and $P(F) =$ ((the multiplication of K) $\circledast (P_4)) \cdot \mathrm{Det}(B \cdot F)$, and

(ii) $\mathrm{Det}(A \cdot B) =$ (the addition of K)-$\sum_{\Omega^{\mathrm{f}}_{(\mathrm{Seg}\, n)^{\mathrm{Seg}\, n}}} P$.

(61) Let A, B be matrices over K of dimension n. Suppose $0 < n$. Then there exists a function P from the permutations of n-element set into the carrier of K such that

(i) $\mathrm{Det}(A \cdot B) =$ (the addition of K)-$\sum_{\Omega^{\mathrm{f}}_{\text{the permutations of } n\text{-element set}}} P$, and

(ii) for every element p_1 of the permutations of n-element set holds $P(p_1) =$ ((the multiplication of K) $\circledast$ (p_1-Path A)) $\cdot (-1)^{\mathrm{sgn}(p_1)} \mathrm{Det}\, B$.

(62) For all matrices A, B over K of dimension n such that $0 < n$ holds $\mathrm{Det}(A \cdot B) = \mathrm{Det}\, A \cdot \mathrm{Det}\, B$.

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