

# Combinatorial Grassmannians

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**Summary.** In the paper I construct the configuration  $G$  which is a partial linear space. It consists of  $k$ -element subsets of some base set as points and  $(k + 1)$ -element subsets as lines. The incidence is given by inclusion. I also introduce automorphisms of partial linear spaces and show that automorphisms of  $G$  are generated by permutations of the base set.

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The articles [15], [17], [3], [14], [7], [11], [13], [8], [18], [19], [4], [12], [16], [9], [5], [6], [10], [2], and [1] provide the notation and terminology for this paper.

## 1. PRELIMINARIES

We follow the rules:  $k, n$  denote elements of  $\mathbb{N}$  and  $X, Y, Z$  denote sets.

One can prove the following propositions:

- (1) For all sets  $a, b$  such that  $a \neq b$  and  $\overline{a} = n$  and  $\overline{b} = n$  holds  $\overline{a \cap b} < n$  and  $n + 1 \leq \overline{a \cup b}$ .
- (2) For all sets  $a, b$  such that  $\overline{a} = n + k$  and  $\overline{b} = n + k$  holds  $\overline{a \cap b} = n$  iff  $\overline{a \cup b} = n + 2 \cdot k$ .
- (3)  $\overline{X} \leq \overline{Y}$  iff there exists a function  $f$  such that  $f$  is one-to-one and  $X \subseteq \text{dom } f$  and  $f^\circ X \subseteq Y$ .
- (4) For every function  $f$  such that  $f$  is one-to-one and  $X \subseteq \text{dom } f$  holds  $\overline{f^\circ X} = \overline{X}$ .
- (5) If  $X \setminus Y = X \setminus Z$  and  $Y \subseteq X$  and  $Z \subseteq X$ , then  $Y = Z$ .
- (6) Let  $Y$  be a non empty set and  $p$  be a function from  $X$  into  $Y$ . Suppose  $p$  is one-to-one. Let  $x_1, x_2$  be subsets of  $X$ . If  $x_1 \neq x_2$ , then  $p^\circ x_1 \neq p^\circ x_2$ .

- (7) Let  $a, b, c$  be sets such that  $\overline{a} = n - 1$  and  $\overline{b} = n - 1$  and  $\overline{c} = n - 1$  and  $\overline{a \cap b} = n - 2$  and  $\overline{a \cap c} = n - 2$  and  $\overline{b \cap c} = n - 2$  and  $2 \leq n$ . Then
- (i) if  $3 \leq n$ , then  $\overline{a \cap b \cap c} = n - 2$  and  $\overline{a \cup b \cup c} = n + 1$  or  $\overline{a \cap b \cap c} = n - 3$  and  $\overline{a \cup b \cup c} = n$ , and
- (ii) if  $n = 2$ , then  $\overline{a \cap b \cap c} = n - 2$  and  $\overline{a \cup b \cup c} = n + 1$ .
- (8) Let  $P_1, P_2$  be projective incidence structures. Suppose the projective incidence structure of  $P_1 =$  the projective incidence structure of  $P_2$ . Let  $A_1$  be a point of  $P_1$  and  $A_2$  be a point of  $P_2$ . Suppose  $A_1 = A_2$ . Let  $L_1$  be a line of  $P_1$  and  $L_2$  be a line of  $P_2$ . If  $L_1 = L_2$ , then if  $A_1$  lies on  $L_1$ , then  $A_2$  lies on  $L_2$ .
- (9) Let  $P_1, P_2$  be projective incidence structures. Suppose the projective incidence structure of  $P_1 =$  the projective incidence structure of  $P_2$ . Let  $A_1$  be a subset of the points of  $P_1$  and  $A_2$  be a subset of the points of  $P_2$ . Suppose  $A_1 = A_2$ . Let  $L_1$  be a line of  $P_1$  and  $L_2$  be a line of  $P_2$ . If  $L_1 = L_2$ , then if  $A_1$  lies on  $L_1$ , then  $A_2$  lies on  $L_2$ .

Let us note that there exists a projective incidence structure which is linear, up-2-rank, and strict and has non-trivial-lines.

## 2. CONFIGURATION $G$

A partial linear space is an up-2-rank projective incidence structure with non-trivial-lines.

Let  $k$  be an element of  $\mathbb{N}$  and let  $X$  be a non empty set. Let us assume that  $0 < k$  and  $k + 1 \leq \overline{X}$ . The functor  $G_k(X)$  yields a strict partial linear space and is defined by the conditions (Def. 1).

- (Def. 1)(i) The points of  $G_k(X) = \{A; A \text{ ranges over subsets of } X: \overline{A} = k\}$ ,
- (ii) the lines of  $G_k(X) = \{L; L \text{ ranges over subsets of } X: \overline{L} = k + 1\}$ , and
- (iii) the incidence of  $G_k(X) = \subseteq_{2X} \cap \{ \text{the points of } G_k(X), \text{ the lines of } G_k(X) \}$ .

One can prove the following four propositions:

- (10) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $0 < k$  and  $k + 1 \leq \overline{X}$ . Let  $A$  be a point of  $G_k(X)$  and  $L$  be a line of  $G_k(X)$ . Then  $A$  lies on  $L$  if and only if  $A \subseteq L$ .
- (11) For every element  $k$  of  $\mathbb{N}$  and for every non empty set  $X$  such that  $0 < k$  and  $k + 1 \leq \overline{X}$  holds  $G_k(X)$  is Vebleian.
- (12) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $0 < k$  and  $k + 1 \leq \overline{X}$ . Let  $A_1, A_2, A_3, A_4, A_5, A_6$  be points of  $G_k(X)$  and  $L_1, L_2, L_3, L_4$  be lines of  $G_k(X)$ . Suppose that  $A_1$  lies on  $L_1$  and  $A_2$  lies on  $L_1$  and  $A_3$  lies on  $L_2$  and  $A_4$  lies on  $L_2$  and  $A_5$  lies on  $L_1$  and  $A_5$  lies on

$L_2$  and  $A_1$  lies on  $L_3$  and  $A_3$  lies on  $L_3$  and  $A_2$  lies on  $L_4$  and  $A_4$  lies on  $L_4$  and  $A_5$  does not lie on  $L_3$  and  $A_5$  does not lie on  $L_4$  and  $L_1 \neq L_2$  and  $L_3 \neq L_4$ . Then there exists a point  $A_6$  of  $G_k(X)$  such that  $A_6$  lies on  $L_3$  and  $A_6$  lies on  $L_4$  and  $A_6 = A_1 \cap A_2 \cup A_3 \cap A_4$ .

- (13) For every element  $k$  of  $\mathbb{N}$  and for every non empty set  $X$  such that  $0 < k$  and  $k + 1 \leq \overline{X}$  holds  $G_k(X)$  is Desarguesian.

Let  $S$  be a projective incidence structure and let  $K$  be a subset of the points of  $S$ . We say that  $K$  is a clique if and only if:

- (Def. 2) For all points  $A, B$  of  $S$  such that  $A \in K$  and  $B \in K$  there exists a line  $L$  of  $S$  such that  $\{A, B\}$  lies on  $L$ .

Let  $S$  be a projective incidence structure and let  $K$  be a subset of the points of  $S$ . We say that  $K$  is a maximal-clique if and only if:

- (Def. 3)  $K$  is a clique and for every subset  $U$  of the points of  $S$  such that  $U$  is a clique and  $K \subseteq U$  holds  $U = K$ .

Let  $k$  be an element of  $\mathbb{N}$ , let  $X$  be a non empty set, and let  $T$  be a subset of the points of  $G_k(X)$ . We say that  $T$  is a star if and only if:

- (Def. 4) There exists a subset  $S$  of  $X$  such that  $\overline{S} = k - 1$  and  $T = \{A; A \text{ ranges over subsets of } X: \overline{A} = k \wedge S \subseteq A\}$ .

We say that  $T$  is a top if and only if:

- (Def. 5) There exists a subset  $S$  of  $X$  such that  $\overline{S} = k + 1$  and  $T = \{A; A \text{ ranges over subsets of } X: \overline{A} = k \wedge A \subseteq S\}$ .

Next we state two propositions:

- (14) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $2 \leq k$  and  $k + 2 \leq \overline{X}$ . Let  $K$  be a subset of the points of  $G_k(X)$ . If  $K$  is a star or a top, then  $K$  is a maximal-clique.
- (15) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $2 \leq k$  and  $k + 2 \leq \overline{X}$ . Let  $K$  be a subset of the points of  $G_k(X)$ . If  $K$  is a maximal-clique, then  $K$  is a star or a top.

### 3. AUTOMORPHISMS

Let  $S_1, S_2$  be projective incidence structures. We consider maps between projective spaces  $S_1$  and  $S_2$  as systems

$\langle$  a point-map, a line-map  $\rangle$ ,

where the point-map is a function from the points of  $S_1$  into the points of  $S_2$  and the line-map is a function from the lines of  $S_1$  into the lines of  $S_2$ .

Let  $S_1, S_2$  be projective incidence structures, let  $F$  be a map between projective spaces  $S_1$  and  $S_2$ , and let  $a$  be a point of  $S_1$ . The functor  $F(a)$  yields a point of  $S_2$  and is defined as follows:

(Def. 6)  $F(a) = (\text{the point-map of } F)(a)$ .

Let  $S_1, S_2$  be projective incidence structures, let  $F$  be a map between projective spaces  $S_1$  and  $S_2$ , and let  $L$  be a line of  $S_1$ . The functor  $F(L)$  yields a line of  $S_2$  and is defined by:

(Def. 7)  $F(L) = (\text{the line-map of } F)(L)$ .

Next we state the proposition

(16) Let  $S_1, S_2$  be projective incidence structures and  $F_1, F_2$  be maps between projective spaces  $S_1$  and  $S_2$ . Suppose for every point  $A$  of  $S_1$  holds  $F_1(A) = F_2(A)$  and for every line  $L$  of  $S_1$  holds  $F_1(L) = F_2(L)$ . Then the map of  $F_1 =$  the map of  $F_2$ .

Let  $S_1, S_2$  be projective incidence structures and let  $F$  be a map between projective spaces  $S_1$  and  $S_2$ . We say that  $F$  preserves incidence strongly if and only if:

(Def. 8) For every point  $A_1$  of  $S_1$  and for every line  $L_1$  of  $S_1$  holds  $A_1$  lies on  $L_1$  iff  $F(A_1)$  lies on  $F(L_1)$ .

The following proposition is true

(17) Let  $S_1, S_2$  be projective incidence structures and  $F_1, F_2$  be maps between projective spaces  $S_1$  and  $S_2$ . Suppose the map of  $F_1 =$  the map of  $F_2$ . If  $F_1$  preserves incidence strongly, then  $F_2$  preserves incidence strongly.

Let  $S$  be a projective incidence structure and let  $F$  be a map between projective spaces  $S$  and  $S$ . We say that  $F$  is automorphism if and only if:

(Def. 9) The line-map of  $F$  is bijective and the point-map of  $F$  is bijective and  $F$  preserves incidence strongly.

Let  $S_1, S_2$  be projective incidence structures, let  $F$  be a map between projective spaces  $S_1$  and  $S_2$ , and let  $K$  be a subset of the points of  $S_1$ . The functor  $F^\circ K$  yielding a subset of the points of  $S_2$  is defined by:

(Def. 10)  $F^\circ K = (\text{the point-map of } F)^\circ K$ .

Let  $S_1, S_2$  be projective incidence structures, let  $F$  be a map between projective spaces  $S_1$  and  $S_2$ , and let  $K$  be a subset of the points of  $S_2$ . The functor  $F^{-1}(K)$  yielding a subset of the points of  $S_1$  is defined as follows:

(Def. 11)  $F^{-1}(K) = (\text{the point-map of } F)^{-1}(K)$ .

Let  $X$  be a set and let  $A$  be a finite set. The functor  $\uparrow(A, X)$  yielding a subset of  $2^X$  is defined as follows:

(Def. 12)  $\uparrow(A, X) = \{B; B \text{ ranges over subsets of } X: \overline{\overline{B}} = \text{card } A + 1 \wedge A \subseteq B\}$ .

Let  $k$  be an element of  $\mathbb{N}$  and let  $X$  be a non empty set. Let us assume that  $0 < k$  and  $k + 1 \leq \overline{\overline{X}}$ . Let  $A$  be a finite set. Let us assume that  $\overline{\overline{A}} = k - 1$  and  $A \subseteq X$ . The functor  $\uparrow(A, X, k)$  yields a subset of the points of  $G_k(X)$  and is defined as follows:

(Def. 13)  $\uparrow(A, X, k) = \uparrow(A, X)$ .

The following propositions are true:

- (18) Let  $S_1, S_2$  be projective incidence structures,  $F$  be a map between projective spaces  $S_1$  and  $S_2$ , and  $K$  be a subset of the points of  $S_1$ . Then  $F^\circ K = \{B; B \text{ ranges over points of } S_2: \bigvee_{A: \text{point of } S_1} (A \in K \wedge F(A) = B)\}$ .
- (19) Let  $S_1, S_2$  be projective incidence structures,  $F$  be a map between projective spaces  $S_1$  and  $S_2$ , and  $K$  be a subset of the points of  $S_2$ . Then  $F^{-1}(K) = \{A; A \text{ ranges over points of } S_1: \bigvee_{B: \text{point of } S_2} (B \in K \wedge F(A) = B)\}$ .
- (20) Let  $S$  be a projective incidence structure,  $F$  be a map between projective spaces  $S$  and  $S$ , and  $K$  be a subset of the points of  $S$ . If  $F$  preserves incidence strongly and  $K$  is a clique, then  $F^\circ K$  is a clique.
- (21) Let  $S$  be a projective incidence structure,  $F$  be a map between projective spaces  $S$  and  $S$ , and  $K$  be a subset of the points of  $S$ . Suppose  $F$  preserves incidence strongly and the line-map of  $F$  is onto and  $K$  is a clique. Then  $F^{-1}(K)$  is a clique.
- (22) Let  $S$  be a projective incidence structure,  $F$  be a map between projective spaces  $S$  and  $S$ , and  $K$  be a subset of the points of  $S$ . Suppose  $F$  is automorphism and  $K$  is a maximal-clique. Then  $F^\circ K$  is a maximal-clique and  $F^{-1}(K)$  is a maximal-clique.
- (23) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $2 \leq k$  and  $k+2 \leq \overline{X}$ . Let  $F$  be a map between projective spaces  $G_k(X)$  and  $G_k(X)$ . Suppose  $F$  is automorphism. Let  $K$  be a subset of the points of  $G_k(X)$ . If  $K$  is a star, then  $F^\circ K$  is a star and  $F^{-1}(K)$  is a star.

Let  $k$  be an element of  $\mathbb{N}$  and let  $X$  be a non empty set. Let us assume that  $0 < k$  and  $k+1 \leq \overline{X}$ . Let  $s$  be a permutation of  $X$ . The functor  $\text{incprojmap}(k, s)$  yielding a strict map between projective spaces  $G_k(X)$  and  $G_k(X)$  is defined as follows:

- (Def. 14) For every point  $A$  of  $G_k(X)$  holds  $(\text{incprojmap}(k, s))(A) = s^\circ A$  and for every line  $L$  of  $G_k(X)$  holds  $(\text{incprojmap}(k, s))(L) = s^\circ L$ .

One can prove the following propositions:

- (24) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $k = 1$  and  $k+1 \leq \overline{X}$ . Let  $F$  be a map between projective spaces  $G_k(X)$  and  $G_k(X)$ . Suppose  $F$  is automorphism. Then there exists a permutation  $s$  of  $X$  such that the map of  $F = \text{incprojmap}(k, s)$ .
- (25) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $1 < k$  and  $\overline{X} = k+1$ . Let  $F$  be a map between projective spaces  $G_k(X)$  and  $G_k(X)$ . Suppose  $F$  is automorphism. Then there exists a permutation  $s$  of  $X$  such that the map of  $F = \text{incprojmap}(k, s)$ .
- (26) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $0 < k$

and  $k + 1 \leq \overline{\overline{X}}$ . Let  $T$  be a subset of the points of  $G_k(X)$  and  $S$  be a subset of  $X$ . If  $\overline{\overline{S}} = k - 1$  and  $T = \{A; A \text{ ranges over subsets of } X: \overline{\overline{A}} = k \wedge S \subseteq A\}$ , then  $S = \bigcap T$ .

- (27) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $0 < k$  and  $k + 1 \leq \overline{\overline{X}}$ . Let  $T$  be a subset of the points of  $G_k(X)$ . Suppose  $T$  is a star. Let  $S$  be a subset of  $X$ . If  $S = \bigcap T$ , then  $\overline{\overline{S}} = k - 1$  and  $T = \{A; A \text{ ranges over subsets of } X: \overline{\overline{A}} = k \wedge S \subseteq A\}$ .
- (28) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $0 < k$  and  $k + 1 \leq \overline{\overline{X}}$ . Let  $T_1, T_2$  be subsets of the points of  $G_k(X)$ . If  $T_1$  is a star and  $T_2$  is a star and  $\bigcap T_1 = \bigcap T_2$ , then  $T_1 = T_2$ .
- (29) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $0 < k$  and  $k + 1 \leq \overline{\overline{X}}$ . Let  $A$  be a finite subset of  $X$ . If  $\overline{\overline{A}} = k - 1$ , then  $\uparrow(A, X, k)$  is a star.
- (30) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $0 < k$  and  $k + 1 \leq \overline{\overline{X}}$ . Let  $A$  be a finite subset of  $X$ . If  $\overline{\overline{A}} = k - 1$ , then  $\bigcap \uparrow(A, X, k) = A$ .
- (31) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $0 < k$  and  $k + 3 \leq \overline{\overline{X}}$ . Let  $F$  be a map between projective spaces  $G_{(k+1)}(X)$  and  $G_{(k+1)}(X)$ . Suppose  $F$  is automorphism. Then there exists a map  $H$  between projective spaces  $G_k(X)$  and  $G_k(X)$  such that
- (i)  $H$  is automorphism,
  - (ii) the line-map of  $H =$  the point-map of  $F$ , and
  - (iii) for every point  $A$  of  $G_k(X)$  and for every finite set  $B$  such that  $B = A$  holds  $H(A) = \bigcap (F^\circ \uparrow(B, X, k + 1))$ .
- (32) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $0 < k$  and  $k + 3 \leq \overline{\overline{X}}$ . Let  $F$  be a map between projective spaces  $G_{(k+1)}(X)$  and  $G_{(k+1)}(X)$ . Suppose  $F$  is automorphism. Let  $H$  be a map between projective spaces  $G_k(X)$  and  $G_k(X)$ . Suppose that
- (i)  $H$  is automorphism,
  - (ii) the line-map of  $H =$  the point-map of  $F$ , and
  - (iii) for every point  $A$  of  $G_k(X)$  and for every finite set  $B$  such that  $B = A$  holds  $H(A) = \bigcap (F^\circ \uparrow(B, X, k + 1))$ .
- Let  $f$  be a permutation of  $X$ . If the map of  $H = \text{incprojmap}(k, f)$ , then the map of  $F = \text{incprojmap}(k + 1, f)$ .
- (33) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $2 \leq k$  and  $k + 2 \leq \overline{\overline{X}}$ . Let  $F$  be a map between projective spaces  $G_k(X)$  and  $G_k(X)$ . Suppose  $F$  is automorphism. Then there exists a permutation  $s$  of  $X$  such that the map of  $F = \text{incprojmap}(k, s)$ .
- (34) Let  $k$  be an element of  $\mathbb{N}$  and  $X$  be a non empty set. Suppose  $0 < k$

and  $k + 1 \leq \overline{\overline{X}}$ . Let  $s$  be a permutation of  $X$ . Then  $\text{incprojmap}(k, s)$  is automorphism.

- (35) Let  $X$  be a non empty set. Suppose  $0 < k$  and  $k+1 \leq \overline{\overline{X}}$ . Let  $F$  be a map between projective spaces  $G_k(X)$  and  $G_k(X)$ . Then  $F$  is automorphism if and only if there exists a permutation  $s$  of  $X$  such that the map of  $F = \text{incprojmap}(k, s)$ .

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