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Regular Expression Quantifiers – m to n Occurrences

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Summary. This article includes proofs of several facts that are supplemental to the theorems proved in [10]. Next, it builds upon that theory to extend the framework for proving facts about formal languages in general and regular expression operators in particular. In this article, two quantifiers are defined and their properties are shown: m to n occurrences (or the union of a range of powers) and optional occurrence. Although optional occurrence is a special case of the previous operator (0 to 1 occurrences), it is often defined in regex applications as a separate operator – hence its explicit definition and properties in the article. Notation and terminology were taken from [13].

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The articles [9], [4], [11], [7], [8], [2], [14], [3], [1], [5], [12], [6], and [10] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: E, x denote sets, A, B, C denote subsets of E^{ω}, a, b denote elements of E^{ω}, and i, k, l, k_1, m, n, m_1 denote natural numbers.

We now state four propositions:

(1) If $m + k \leq i$ and $i \leq n + k$, then there exists m_1 such that $m_1 + k = i$ and $m \leq m_1$ and $m_1 \leq n$.

(2) If $m \leq n$ and $k \leq l$ and $m + k \leq i$ and $i \leq n + l$, then there exist m_1, k_1 such that $m_1 + k_1 = i$ and $m \leq m_1$ and $m_1 \leq n$ and $k \leq k_1$ and $k_1 \leq l$.

(3) If $m < n$, then there exists k such that $m + k = n$ and $k > 0$.

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(4) If $a \frown b = a$ or $b \frown a = a$, then $b = \emptyset$.

2. Addenda to [10]

One can prove the following propositions:

(5) If $x \in A$ or $x \in B$ and if $x \neq \langle\rangle_E$, then $A \frown B \neq \{\langle\rangle_E\}$.

(6) $\langle x\rangle \in A \frown B$ iff $\langle\rangle_E \in A$ and $\langle x\rangle \in B$ or $\langle x\rangle \in A$ and $\langle\rangle_E \in B$.

(7) If $x \in A$ and $x \neq \langle\rangle_E$ and $n > 0$, then $A^n \neq \{\langle\rangle_E\}$.

(8) $\langle\rangle_E \in A^n$ iff $n = 0$ or $\langle\rangle_E \in A$.

(9) $\langle x\rangle \in A^n$ iff $\langle x\rangle \in A$ but $\langle\rangle_E \in A$ and $n > 1$ or $n = 1$.

(10) If $m \neq n$ and $A^m = \{x\}$ and $A^n = \{x\}$, then $x = \langle\rangle_E$.

(11) $(A^m)^n = (A^n)^m$.

(12) $(A^m) \frown A^n = (A^n) \frown A^m$.

(13) If $\langle\rangle_E \in B$, then $A \subseteq A \frown B^l$ and $A \subseteq (B^l) \frown A$.

(14) If $A \subseteq C^k$ and $B \subseteq C^l$, then $A \frown B \subseteq C^{k+l}$.

(15) If $x \in A$ and $x \neq \langle\rangle_E$, then $A^* \neq \{\langle\rangle_E\}$.

(16) If $\langle\rangle_E \in A$ and $n > 0$, then $(A^n)^* = A^*$.

(17) If $\langle\rangle_E \in A$, then $(A^n)^* = (A^*)^n$.

(18) $A \subseteq A \frown B^*$ and $A \subseteq (B^*) \frown A$.

3. Union of a Range of Powers

Let us consider E, A and let us consider m, n. The functor $A^{m,n}$ yields a subset of E^ω and is defined as follows:

(Def. 1) $A^{m,n} = \bigcup\{B : \bigvee_k \ (m \leq k \ \wedge \ k \leq n \ \wedge \ B = A^k)\}$.

One can prove the following propositions:

(19) $x \in A^{m,n}$ iff there exists k such that $m \leq k$ and $k \leq n$ and $x \in A^k$.

(20) If $m \leq k$ and $k \leq n$, then $A^k \subseteq A^{m,n}$.

(21) $A^{m,n} = \emptyset$ iff $m > n$ or $m > 0$ and $A = \emptyset$.

(22) $A^{m,m} = A^m$.

(23) If $m \leq k$ and $l \leq n$, then $A^{k,l} \subseteq A^{m,n}$.

(24) If $m \leq k$ and $k \leq n$, then $A^{m,n} = A^{m,k} \cup A^{k,n}$.

(25) If $m \leq k$ and $k \leq n$, then $A^{m,n} = A^{m,k} \cup A^{k+1,n}$.

(26) If $m \leq n+1$, then $A^{m,n+1} = A^{m,n} \cup A^{n+1}$.

(27) If $m \leq n$, then $A^{m,n} = A^m \cup A^{m+1,n}$.

(28) $A^{n,n+1} = A^n \cup A^{n+1}$.

(29) If $A \subseteq B$, then $A^{m,n} \subseteq B^{m,n}$.

(30) If $x \in A$ and if $x \neq \langle\rangle_E$ and if $m > 0$ or $n > 0$, then $A^{m,n} \neq \{\langle\rangle_E\}$.

(31) $A^{m,n} = \{\langle\rangle_E\}$ iff $m \leq n$ and $A = \{\langle\rangle_E\}$ or $m = 0$ and $n = 0$ or $m = 0$ and $A = \emptyset$.

(32) $A^{m,n} \subseteq A^*$.

(33) $\langle\rangle_E \in A^{m,n}$ iff $m = 0$ or $m \leq n$ and $\langle\rangle_E \in A$.

(34) If $\langle\rangle_E \in A$ and $m \leq n$, then $A^{m,n} = A^n$.

(35) $(A^{m,n}) \frown A^k = (A^k) \frown A^{m,n}$.

(36) $(A^{m,n}) \frown A = A \frown A^{m,n}$.

(37) If $m \leq n$ and $k \leq l$, then $(A^{m,n}) \frown A^{k,l} = A^{m+k,n+l}$.

(38) $A^{m+1,n+1} = (A^{m,n}) \frown A$.

(39) $(A^{m,n}) \frown A^{k,l} = (A^{k,l}) \frown A^{m,n}$.

(40) $(A^{m,n})^k = A^{m \cdot k, n \cdot k}$.

(41) $(A^{k+1})^{m,n} \subseteq ((A^k)^{m,n}) \frown A^{m,n}$.

(42) $(A^k)^{m,n} \subseteq A^{k \cdot m, k \cdot n}$.

(43) $(A^k)^{m,n} \subseteq (A^{m,n})^k$.

(44) $(A^{k+l})^{m,n} \subseteq ((A^k)^{m,n}) \frown (A^l)^{m,n}$.

(45) $A^{0,0} = \{\langle\rangle_E\}$.

(46) $A^{0,1} = \{\langle\rangle_E\} \cup A$.

(47) $A^{1,1} = A$.

(48) $A^{0,2} = \{\langle\rangle_E\} \cup A \cup A \frown A$.

(49) $A^{1,2} = A \cup A \frown A$.

(50) $A^{2,2} = A \frown A$.

(51) If $m > 0$ and $m \neq n$ and $A^{m,n} = \{x\}$, then for every m_1 such that $m \leq m_1$ and $m_1 \leq n$ holds $A^{m_1} = \{x\}$.

(52) If $m \neq n$ and $A^{m,n} = \{x\}$, then $x = \langle\rangle_E$.

(53) $\langle x\rangle \in A^{m,n}$ iff $\langle x\rangle \in A$ but $m \leq n$ but $\langle\rangle_E \in A$ and $n > 0$ or $m \leq 1$ and $1 \leq n$.

(54) $(A \cap B)^{m,n} \subseteq A^{m,n} \cap B^{m,n}$.

(55) $A^{m,n} \cup B^{m,n} \subseteq (A \cup B)^{m,n}$.

(56) $(A^{m,n})^{k,l} \subseteq A^{m \cdot k, n \cdot l}$.

(57) If $m \leq n$ and $\langle\rangle_E \in B$, then $A \subseteq A \frown B^{m,n}$ and $A \subseteq (B^{m,n}) \frown A$.

(58) If $m \leq n$ and $k \leq l$ and $A \subseteq C^{m,n}$ and $B \subseteq C^{k,l}$, then $A \frown B \subseteq C^{m+k,n+l}$.

(59) $(A^{m,n})^* \subseteq A^*$.

(60) $(A^*)^{m,n} \subseteq A^*$.

(61) If $m \leq n$ and $n > 0$, then $(A^*)^{m,n} = A^*$.

(62) If $m \leq n$ and $n > 0$ and $\langle\rangle_E \in A$, then $(A^{m,n})^* = A^*$.

(63) If $m \leq n$ and $\langle\rangle_E \in A$, then $(A^{m,n})^* = (A^*)^{m,n}$.

(64) If $A \subseteq B^*$, then $A^{m,n} \subseteq B^*$.

(65) If $A \subseteq B^*$, then $B^* = (B \cup A^{m,n})^*$.

(66) $(A^{m,n}) \frown A^* = (A^*) \frown A^{m,n}$.

(67) If $\langle\rangle_E \in A$ and $m \leq n$, then $A^* = (A^*) \frown A^{m,n}$.

(68) $(A^{m,n})^k \subseteq A^*$.

(69) $(A^k)^{m,n} \subseteq A^*$.

(70) If $m \leq n$, then $(A^m)^* \subseteq (A^{m,n})^*$.

(71) $(A^{m,n})^{k,l} \subseteq A^*$.

(72) If $\langle\rangle_E \in A$ and $k \leq n$ and $l \leq n$, then $A^{k,n} = A^{l,n}$.

4. Optional Occurrence

Let us consider E, A. The functor $A?$ yields a subset of E^ω and is defined by:

(Def. 2) $A? = \bigcup\{B : \bigvee_k \ (k \leq 1 \ \wedge \ B = A^k)\}$.

One can prove the following propositions:

(73) $x \in A?$ iff there exists k such that $k \leq 1$ and $x \in A^k$.

(74) If $n \leq 1$, then $A^n \subseteq A?$.

(75) $A? = A^0 \cup A^1$.

(76) $A? = \{\langle\rangle_E\} \cup A$.

(77) $A \subseteq A?$.

(78) $x \in A?$ iff $x = \langle\rangle_E$ or $x \in A$.

(79) $A? = A^{0,1}$.

(80) $A? = A$ iff $\langle\rangle_E \in A$.

Let us consider E, A. One can check that $A?$ is non empty.

We now state a number of propositions:

(81) $A?? = A?$.

(82) If $A \subseteq B$, then $A? \subseteq B?$.

(83) If $x \in A$ and $x \neq \langle\rangle_E$, then $A? \neq \{\langle\rangle_E\}$.

(84) $A? = \{\langle\rangle_E\}$ iff $A = \emptyset$ or $A = \{\langle\rangle_E\}$.

(85) $A^*? = A^*$ and $A?^* = A^*$.

(86) $A? \subseteq A^*$.

(87) $(A \cap B)? = A? \cap B?$.

(88) $A? \cup B? = (A \cup B)?$.

(89) If $A? = \{x\}$, then $x = \langle\rangle_E$.

(90) $\langle x\rangle \in A?$ iff $\langle x\rangle \in A$.

(91) $A? \frown A = A \frown A?$.

(92) $A? ^\frown A = A^{1,2}$.

(93) $A? ^\frown A? = A^{0,2}$.

(94) $A?^k = A?^{0,k}$.

(95) $A?^k = A^{0,k}$.

(96) If $m \leq n$, then $A?^{m,n} = A?^{0,n}$.

(97) $A?^{0,n} = A^{0,n}$.

(98) If $m \leq n$, then $A?^{m,n} = A^{0,n}$.

(99) $A^{1,n}? = A^{0,n}$.

(100) If $\langle\rangle_E \in A$ and $\langle\rangle_E \in B$, then $A? \subseteq A ^\frown B$ and $A? \subseteq B ^\frown A$.

(101) $A \subseteq A ^\frown B?$ and $A \subseteq B? ^\frown A$.

(102) If $A \subseteq C?$ and $B \subseteq C?$, then $A ^\frown B \subseteq C^{0,2}$.

(103) If $\langle\rangle_E \in A$ and $n > 0$, then $A? \subseteq A^n$.

(104) $A? ^\frown A^k = (A^k) ^\frown A?$.

(105) If $A \subseteq B^*$, then $A? \subseteq B^*$.

(106) If $A \subseteq B^*$, then $B^* = (B \cup A?)^*$.

(107) $A? ^\frown A^* = (A^*) ^\frown A?$.

(108) $A? ^\frown A^* = A^*$.

(109) $A?^k \subseteq A^*$.

(110) $A^k? \subseteq A^*$.

(111) $A? ^\frown A^{m,n} = (A^{m,n}) ^\frown A?$.

(112) $A? ^\frown A^k = A^{k,k+1}$.

(113) $A?^{m,n} \subseteq A^*$.

(114) $A^{m,n}? \subseteq A^*$.

(115) $A? = (A \setminus \{\langle\rangle_E\})?$.

(116) If $A \subseteq B?$, then $A? \subseteq B?$.

(117) If $A \subseteq B?$, then $B? = (B \cup A)?$.

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