

# Riemann Indefinite Integral of Functions of Real Variable<sup>1</sup>

Yasunari Shidama  
Shinshu University  
Nagano, Japan

Noboru Endou  
Gifu National College of Technology  
Japan

Katsumi Wasaki  
Shinshu University  
Nagano, Japan

**Summary.** In this article we define the Riemann indefinite integral of functions of real variable and prove the linearity of that [1]. And we give some examples of the indefinite integral of some elementary functions. Furthermore, also the theorem about integral operation and uniform convergent sequence of functions is proved.

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The papers [24], [25], [3], [23], [5], [13], [2], [26], [7], [21], [8], [10], [4], [17], [16], [15], [14], [19], [20], [6], [9], [11], [18], [12], [27], and [22] provide the terminology and notation for this paper.

## 1. PRELIMINARIES

For simplicity, we adopt the following rules:  $a, b, r$  are real numbers,  $A$  is a non empty set,  $X, x$  are sets,  $f, g, F, G$  are partial functions from  $\mathbb{R}$  to  $\mathbb{R}$ , and  $n$  is an element of  $\mathbb{N}$ .

Next we state a number of propositions:

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- (1) Let  $f, g$  be functions from  $A$  into  $\mathbb{R}$ . Suppose  $\text{rng } f$  is upper bounded and  $\text{rng } g$  is upper bounded and for every set  $x$  such that  $x \in A$  holds  $|f(x) - g(x)| \leq a$ . Then  $\sup \text{rng } f - \sup \text{rng } g \leq a$  and  $\sup \text{rng } g - \sup \text{rng } f \leq a$ .
- (2) Let  $f, g$  be functions from  $A$  into  $\mathbb{R}$ . Suppose  $\text{rng } f$  is lower bounded and  $\text{rng } g$  is lower bounded and for every set  $x$  such that  $x \in A$  holds  $|f(x) - g(x)| \leq a$ . Then  $\inf \text{rng } f - \inf \text{rng } g \leq a$  and  $\inf \text{rng } g - \inf \text{rng } f \leq a$ .
- (3) If  $f|_X$  is bounded on  $X$ , then  $f$  is bounded on  $X$ .
- (4) For every real number  $x$  such that  $x \in X$  and  $f|_X$  is differentiable in  $x$  holds  $f$  is differentiable in  $x$ .
- (5) If  $f|_X$  is differentiable on  $X$ , then  $f$  is differentiable on  $X$ .
- (6) Suppose  $f$  is differentiable on  $X$  and  $g$  is differentiable on  $X$ . Then  $f + g$  is differentiable on  $X$  and  $f - g$  is differentiable on  $X$  and  $f g$  is differentiable on  $X$ .
- (7) If  $f$  is differentiable on  $X$ , then  $r f$  is differentiable on  $X$ .
- (8) Suppose for every set  $x$  such that  $x \in X$  holds  $g(x) \neq 0$  and  $f$  is differentiable on  $X$  and  $g$  is differentiable on  $X$ . Then  $\frac{f}{g}$  is differentiable on  $X$ .
- (9) If for every set  $x$  such that  $x \in X$  holds  $f(x) \neq 0$  and  $f$  is differentiable on  $X$ , then  $\frac{1}{f}$  is differentiable on  $X$ .
- (10) Suppose  $a \leq b$  and  $[a, b] \subseteq X$  and  $F$  is differentiable on  $X$  and  $F'|_X$  is integrable on  $[a, b]$  and  $F'|_X$  is bounded on  $[a, b]$ . Then  $F(b) = \int_a^b (F'|_X)(x) dx + F(a)$ .

## 2. THE DEFINITION OF INDEFINITE INTEGRAL

Let  $X$  be a set and let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathbb{R}$ . The functor  $\text{IntegralFuncs}(f, X)$  yields a set and is defined by the condition (Def. 1).

- (Def. 1)  $x \in \text{IntegralFuncs}(f, X)$  if and only if there exists a partial function  $F$  from  $\mathbb{R}$  to  $\mathbb{R}$  such that  $x = F$  and  $F$  is differentiable on  $X$  and  $F'|_X = f|_X$ .

Let  $X$  be a set and let  $F, f$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$ . We say that  $F$  is an integral of  $f$  on  $X$  if and only if:

- (Def. 2)  $F \in \text{IntegralFuncs}(f, X)$ .

The following propositions are true:

- (11) If  $F$  is an integral of  $f$  on  $X$ , then  $X \subseteq \text{dom } F$ .
- (12) Suppose  $F$  is an integral of  $f$  on  $X$  and  $G$  is an integral of  $g$  on  $X$ . Then  $F + G$  is an integral of  $f + g$  on  $X$  and  $F - G$  is an integral of  $f - g$  on  $X$ .

- (13) If  $F$  is an integral of  $f$  on  $X$ , then  $rF$  is an integral of  $rf$  on  $X$ .
- (14) If  $F$  is an integral of  $f$  on  $X$  and  $G$  is an integral of  $g$  on  $X$ , then  $FG$  is an integral of  $fG + Fg$  on  $X$ .
- (15) Suppose for every set  $x$  such that  $x \in X$  holds  $G(x) \neq 0$  and  $F$  is an integral of  $f$  on  $X$  and  $G$  is an integral of  $g$  on  $X$ . Then  $\frac{F}{G}$  is an integral of  $\frac{fG - Fg}{G^2}$  on  $X$ .
- (16) Suppose that
- (i)  $a \leq b$ ,
  - (ii)  $[a, b] \subseteq \text{dom } f$ ,
  - (iii)  $f$  is continuous on  $[a, b]$ ,
  - (iv)  $]a, b[ \subseteq \text{dom } F$ , and
- (v) for every real number  $x$  such that  $x \in ]a, b[$  holds  $F(x) = \int_a^x f(x)dx + F(a)$ .
- Then  $F$  is an integral of  $f$  on  $]a, b[$ .
- (17) Let  $x, x_0$  be real numbers. Suppose  $f$  is continuous on  $[a, b]$  and  $x \in ]a, b[$  and  $x_0 \in ]a, b[$  and  $F$  is an integral of  $f$  on  $]a, b[$ . Then  $F(x) = \int_{x_0}^x f(x)dx + F(x_0)$ .
- (18) Suppose  $a \leq b$  and  $[a, b] \subseteq X$  and  $F$  is an integral of  $f$  on  $X$  and  $f$  is integrable on  $[a, b]$  and  $f$  is bounded on  $[a, b]$ . Then  $F(b) = \int_a^b f(x)dx + F(a)$ .
- (19) Suppose  $a \leq b$  and  $[a, b] \subseteq X$  and  $f$  is continuous on  $X$ . Then  $f$  is continuous on  $[a, b]$  and  $f$  is integrable on  $[a, b]$  and  $f$  is bounded on  $[a, b]$ .
- (20) If  $a \leq b$  and  $[a, b] \subseteq X$  and  $f$  is continuous on  $X$  and  $F$  is an integral of  $f$  on  $X$ , then  $F(b) = \int_a^b f(x)dx + F(a)$ .
- (21) Suppose that  $b \leq a$  and  $[b, a] \subseteq X$  and  $f$  is integrable on  $[b, a]$  and  $g$  is integrable on  $[b, a]$  and  $f$  is bounded on  $[b, a]$  and  $g$  is bounded on  $[b, a]$  and  $X \subseteq \text{dom } f$  and  $X \subseteq \text{dom } g$  and  $F$  is an integral of  $f$  on  $X$  and  $G$  is an integral of  $g$  on  $X$ . Then  $F(a) \cdot G(a) - F(b) \cdot G(b) = \int_b^a (fG)(x)dx + \int_b^a (Fg)(x)dx$ .
- (22) Suppose that  $b \leq a$  and  $[b, a] \subseteq X$  and  $X \subseteq \text{dom } f$  and  $X \subseteq \text{dom } g$  and

$f$  is continuous on  $X$  and  $g$  is continuous on  $X$  and  $F$  is an integral of  $f$  on  $X$  and  $G$  is an integral of  $g$  on  $X$ . Then  $F(a) \cdot G(a) - F(b) \cdot G(b) = \int_b^a (f G)(x)dx + \int_b^a (F g)(x)dx$ .

### 3. EXAMPLES OF INDEFINITE INTEGRAL

We now state several propositions:

(23) The function  $\sin$  is an integral of the function  $\cos$  on  $\mathbb{R}$ .

(24)  $(\text{The function } \sin)(b) - (\text{the function } \sin)(a) = \int_a^b (\text{the function } \cos)(x)dx$ .

(25)  $(-1)(\text{the function } \cos)$  is an integral of the function  $\sin$  on  $\mathbb{R}$ .

(26)  $(\text{The function } \cos)(a) - (\text{the function } \cos)(b) = \int_a^b (\text{the function } \sin)(x)dx$ .

(27) The function  $\exp$  is an integral of the function  $\exp$  on  $\mathbb{R}$ .

(28)  $(\text{The function } \exp)(b) - (\text{the function } \exp)(a) = \int_a^b (\text{the function } \exp)(x)dx$ .

(29)  $\frac{n+1}{\mathbb{Z}}$  is an integral of  $(n+1)\frac{n}{\mathbb{Z}}$  on  $\mathbb{R}$ .

(30)  $(\frac{n+1}{\mathbb{Z}})(b) - (\frac{n+1}{\mathbb{Z}})(a) = \int_a^b ((n+1)\frac{n}{\mathbb{Z}})(x)dx$ .

### 4. UNIFORM CONVERGENT FUNCTIONAL SEQUENCE

We now state the proposition

(31) Let  $H$  be a sequence of partial functions from  $\mathbb{R}$  into  $\mathbb{R}$  and  $r_1$  be a sequence of real numbers. Suppose that

(i)  $a < b$ ,

(ii) for every element  $n$  of  $\mathbb{N}$  holds  $H(n)$  is integrable on  $[a, b]$  and  $H(n)$

is bounded on  $[a, b]$  and  $r_1(n) = \int_a^b H(n)(x)dx$ , and

(iii)  $H$  is uniform-convergent on  $[a, b]$ .

Then  $\lim_{[a,b]} H$  is bounded on  $[a, b]$  and  $\lim_{[a,b]} H$  is integrable on  $[a, b]$

and  $r_1$  is convergent and  $\lim r_1 = \int_a^b \lim_{[a,b]} H(x)dx$ .

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