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Partial Differentiation on Normed Linear Spaces $\mathcal{R}^n$

Noboru Endou
Gifu National College of Technology
Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Keiichi Miyajima
Ibaraki University
Hitachi, Japan

Summary. In this article, we define the partial differentiation of functions of real variable and prove the linearity of this operator [18].

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The notation and terminology used here are introduced in the following papers: [21], [24], [25], [5], [26], [7], [6], [15], [13], [3], [1], [20], [11], [22], [23], [14], [8], [2], [4], [27], [28], [16], [9], [19], [17], [12], and [10].

1. Preliminaries

Let i, n be elements of $\mathbb{N}$. The functor $\mathrm{proj}(i,n)$ yielding a function from $\mathcal{R}^n$ into $\mathbb{R}$ is defined by:

(Def. 1) For every element x of $\mathcal{R}^n$ holds $(\mathrm{proj}(i,n))(x) = x(i)$.

Next we state two propositions:

(1) $\mathrm{dom}\,\mathrm{proj}(1,1) = \mathcal{R}^1$ and $\mathrm{rng}\,\mathrm{proj}(1,1) = \mathbb{R}$ and for every element x of $\mathbb{R}$ holds $(\mathrm{proj}(1,1))(\langle x\rangle) = x$ and $(\mathrm{proj}(1,1))^{-1}(x) = \langle x\rangle$.

(2)(i) $(\mathrm{proj}(1,1))^{-1}$ is a function from $\mathbb{R}$ into $\mathcal{R}^1$,

(ii) $(\mathrm{proj}(1,1))^{-1}$ is one-to-one,

(iii) $\mathrm{dom}((\mathrm{proj}(1,1))^{-1}) = \mathbb{R}$,

(iv) $\mathrm{rng}((\mathrm{proj}(1,1))^{-1}) = \mathcal{R}^1$, and

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(v) there exists a function g from $\mathbb{R}$ into $\mathcal{R}^1$ such that g is bijective and $(\mathrm{proj}(1,1))^{-1} = g$.

One can check that $\mathrm{proj}(1,1)$ is bijective.

Let g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\langle g \rangle$ yields a partial function from $\mathcal{R}^1$ to $\mathcal{R}^1$ and is defined as follows:

(Def. 2) $\langle g \rangle = (\mathrm{proj}(1,1))^{-1} \cdot g \cdot \mathrm{proj}(1,1)$.

Let n be an element of $\mathbb{N}$ and let g be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$. The functor $\langle g \rangle$ yielding a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$ is defined as follows:

(Def. 3) $\langle g \rangle = (\mathrm{proj}(1,1))^{-1} \cdot g$.

Let i, n be elements of $\mathbb{N}$. The functor $\mathrm{Proj}(i,n)$ yielding a function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ is defined as follows:

(Def. 4) For every point x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $(\mathrm{Proj}(i,n))(x) = \langle (\mathrm{proj}(i,n))(x) \rangle$.

Let i be an element of $\mathbb{N}$ and let x be a finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{reproj}(i,x)$ yielding a function is defined as follows:

(Def. 5) $\mathrm{dom}\,\mathrm{reproj}(i,x) = \mathbb{R}$ and for every element r of $\mathbb{R}$ holds $(\mathrm{reproj}(i,x))(r) = \mathrm{Replace}(x,i,r)$.

Let n, i be elements of $\mathbb{N}$ and let x be an element of $\mathcal{R}^n$. Then $\mathrm{reproj}(i,x)$ is a function from $\mathbb{R}$ into $\mathcal{R}^n$.

Let n, i be elements of $\mathbb{N}$ and let x be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. The functor $\mathrm{reproj}(i,x)$ yielding a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is defined by the condition (Def. 6).

(Def. 6) Let r be an element of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Then there exists an element q of $\mathbb{R}$ and there exists an element y of $\mathcal{R}^n$ such that $r = \langle q \rangle$ and $y = x$ and $(\mathrm{reproj}(i,x))(r) = (\mathrm{reproj}(i,y))(q)$.

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. We say that f is differentiable in x if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and g is differentiable in y.

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. Let us assume that f is differentiable in x. The functor $f'(x)$ yields a function from $\mathcal{R}^m$ into $\mathcal{R}^n$ and is defined as follows:

(Def. 8) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and $f'(x) = g'(y)$.

We now state four propositions:

(3) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$. Suppose $I = (\mathrm{proj}(1,1))^{-1}$. Then

(i) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ such that $x = I(y)$ holds $\|x\| = |y|$,

(ii) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $x = I(a)$ and $y = I(b)$ holds $x + y = I(a + b)$,

(iii) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ and for every real number a such that $x = I(y)$ holds $a \cdot x = I(a \cdot y)$,

(iv) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element a of $\mathbb{R}$ such that $x = I(a)$ holds $-x = I(-a)$, and

(v) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $x = I(a)$ and $y = I(b)$ holds $x - y = I(a - b)$.

(4) Let J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $J = \mathrm{proj}(1, 1)$. Then

(i) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ such that $J(x) = y$ holds $\|x\| = |y|$,

(ii) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $J(x) = a$ and $J(y) = b$ holds $J(x + y) = a + b$,

(iii) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ and for every real number a such that $J(x) = y$ holds $J(a \cdot x) = a \cdot y$,

(iv) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element a of $\mathbb{R}$ such that $J(x) = a$ holds $J(-x) = -a$, and

(v) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $J(x) = a$ and $J(y) = b$ holds $J(x - y) = a - b$.

(5) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$ and J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $I = (\mathrm{proj}(1, 1))^{-1}$ and $J = \mathrm{proj}(1, 1)$. Then

(i) for every rest R of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, $\langle \mathcal{E}^1, \|\cdot\| \rangle$ holds $J \cdot R \cdot I$ is a rest, and

(ii) for every linear operator L from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ holds $J \cdot L \cdot I$ is a linear function.

(6) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$ and J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $I = (\mathrm{proj}(1, 1))^{-1}$ and $J = \mathrm{proj}(1, 1)$. Then

(i) for every rest R holds $I \cdot R \cdot J$ is a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and

(ii) for every linear function L holds $I \cdot L \cdot J$ is a bounded linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$.

In the sequel f is a partial function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, g is a partial function from $\mathbb{R}$ to $\mathbb{R}$, x is a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and y is an element of $\mathbb{R}$.

We now state four propositions:

(7) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and f is differentiable in x, then g is differentiable in y and $g'(y) = (\mathrm{proj}(1, 1) \cdot f'(x) \cdot (\mathrm{proj}(1, 1))^{-1})(1)$.

(8) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and g is differentiable in y, then f is differentiable in x and $f'(x)(\langle 1 \rangle) = \langle g'(y) \rangle$.

(9) If $f = \langle g \rangle$ and $x = \langle y \rangle$, then f is differentiable in x iff g is differentiable in y.

(10) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and f is differentiable in x, then $f'(x)(\langle 1 \rangle) = \langle g'(y) \rangle$.

2. Partial Differentiation

For simplicity, we adopt the following rules: m, n are non empty elements of $\mathbb{N}$, i, j are elements of $\mathbb{N}$, f is a partial function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, g is a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, x is a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and y is an element of $\mathcal{R}^n$.

Let n, m be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. We say that f is partially differentiable in x w.r.t. i if and only if:

(Def. 9) $f \cdot \mathrm{reproj}(i, x)$ is differentiable in $(\mathrm{Proj}(i, m))(x)$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. The functor $\mathrm{partdiff}(f, x, i)$ yielding a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is defined as follows:

(Def. 10) $\mathrm{partdiff}(f, x, i) = (f \cdot \mathrm{reproj}(i, x))'((\mathrm{Proj}(i, m))(x))$.

Let n be a non empty element of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. We say that f is partially differentiable in x w.r.t. i if and only if:

(Def. 11) $f \cdot \mathrm{reproj}(i, x)$ is differentiable in $(\mathrm{proj}(i, n))(x)$.

Let n be a non empty element of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. The functor $\mathrm{partdiff}(f, x, i)$ yields a real number and is defined by:

(Def. 12) $\mathrm{partdiff}(f, x, i) = (f \cdot \mathrm{reproj}(i, x))'((\mathrm{proj}(i, n))(x))$.

We now state several propositions:

(11) $\mathrm{Proj}(i, n) = (\mathrm{proj}(1, 1))^{-1} \cdot \mathrm{proj}(i, n)$.

(12) If $x = y$, then $\mathrm{reproj}(i, y) \cdot \mathrm{proj}(1, 1) = \mathrm{reproj}(i, x)$.

(13) If $f = \langle g \rangle$ and $x = y$, then $\langle g \cdot \mathrm{reproj}(i, y) \rangle = f \cdot \mathrm{reproj}(i, x)$.

(14) Suppose $f = \langle g \rangle$ and $x = y$. Then f is partially differentiable in x w.r.t. i if and only if g is partially differentiable in y w.r.t. i.

(15) If $f = \langle g \rangle$ and $x = y$ and f is partially differentiable in x w.r.t. i, then $(\mathrm{partdiff}(f, x, i))(\langle 1 \rangle) = \langle \mathrm{partdiff}(g, y, i) \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. We say that f is partially differentiable in x w.r.t. i if and only if the condition (Def. 13) is satisfied.

(Def. 13) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and g is partially differentiable in y w.r.t. i.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. Let us assume that f is partially differentiable in x w.r.t. i. The functor $\mathrm{partdiff}(f, x, i)$ yielding an element of $\mathcal{R}^n$ is defined as follows:

(Def. 14) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and $\mathrm{partdiff}(f, x, i) = (\mathrm{partdiff}(g, y, i))(\langle 1 \rangle)$.

One can prove the following four propositions:

(16) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$. Then F is partially differentiable in x w.r.t. i if and only if G is partially differentiable in y w.r.t. i.

(17) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$ and F is partially differentiable in x w.r.t. i. Then $(\mathrm{partdiff}(F, x, i))(\langle 1 \rangle) = \mathrm{partdiff}(G, y, i)$.

(18) Let g_1 be a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$. Suppose $g_1 = \langle g \rangle$. Then g_1 is partially differentiable in y w.r.t. i if and only if g is partially differentiable in y w.r.t. i.

(19) Let g_1 be a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$. Suppose $g_1 = \langle g \rangle$ and g_1 is partially differentiable in y w.r.t. i. Then $\mathrm{partdiff}(g_1, y, i) = \langle \mathrm{partdiff}(g, y, i) \rangle$.

3. Linearity of Partial Differential Operator

For simplicity, we use the following convention: X is a set, r is a real number, f, f_1, f_2 are partial functions from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, g, g_1, g_2 are partial functions from $\mathcal{R}^n$ to $\mathbb{R}$, h is a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x is a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, y is an element of $\mathcal{R}^n$, and z is an element of $\mathcal{R}^m$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. We say that f is partially differentiable in x w.r.t. i and j if and only if:

(Def. 15) $\mathrm{Proj}(j, n) \cdot f \cdot \mathrm{reproj}(i, x)$ is differentiable in $(\mathrm{Proj}(i, m))(x)$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$.

The functor $\mathrm{partdiff}(f, x, i, j)$ yields a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and is defined by:

(Def. 16) $\mathrm{partdiff}(f, x, i, j) = (\mathrm{Proj}(j, n) \cdot f \cdot \mathrm{reproj}(i, x))'((\mathrm{Proj}(i, m))(x))$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let h be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let z be an element of $\mathcal{R}^m$. We say that h is partially differentiable in z w.r.t. i and j if and only if:

(Def. 17) $\mathrm{proj}(j, n) \cdot h \cdot \mathrm{reproj}(i, z)$ is differentiable in $(\mathrm{proj}(i, m))(z)$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let h be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let z be an element of $\mathcal{R}^m$. The functor $\mathrm{partdiff}(h, z, i, j)$ yielding a real number is defined as follows:

(Def. 18) $\mathrm{partdiff}(h, z, i, j) = (\mathrm{proj}(j, n) \cdot h \cdot \mathrm{reproj}(i, z))'((\mathrm{proj}(i, m))(z))$.

The following propositions are true:

(20) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$. Then F is differentiable in x if and only if G is differentiable in y.

(21) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. If $F = G$ and $x = y$ and F is differentiable in x, then $F'(x) = G'(y)$.

(22) If $f = h$ and $x = z$, then $\mathrm{Proj}(j, n) \cdot f \cdot \mathrm{reproj}(i, x) = \langle \mathrm{proj}(j, n) \cdot h \cdot \mathrm{reproj}(i, z) \rangle$.

(23) Suppose $f = h$ and $x = z$. Then f is partially differentiable in x w.r.t. i and j if and only if h is partially differentiable in z w.r.t. i and j.

(24) If $f = h$ and $x = z$ and f is partially differentiable in x w.r.t. i and j, then $(\mathrm{partdiff}(f, x, i, j))(\langle 1 \rangle) = \langle \mathrm{partdiff}(h, z, i, j) \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let X be a set. We say that f is partially differentiable on X w.r.t. i if and only if:

(Def. 19) $X \subseteq \mathrm{dom}\, f$ and for every point x of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $x \in X$ holds $f{\restriction}X$ is partially differentiable in x w.r.t. i.

We now state the proposition

(25) If f is partially differentiable on X w.r.t. i, then X is a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let us consider X. Let us assume that f is partially differentiable on X w.r.t. i. The functor $f{\restriction}^i X$ yielding a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is defined by:

(Def. 20) $\mathrm{dom}(f{\upharpoonright}^i X) = X$ and for every point x of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $x \in X$ holds $(f{\upharpoonright}^i X)_x = \mathrm{partdiff}(f, x, i)$.

The following propositions are true:

(26) $(f_1 + f_2) \cdot \mathrm{reproj}(i, x) = f_1 \cdot \mathrm{reproj}(i, x) + f_2 \cdot \mathrm{reproj}(i, x)$ and $(f_1 - f_2) \cdot \mathrm{reproj}(i, x) = f_1 \cdot \mathrm{reproj}(i, x) - f_2 \cdot \mathrm{reproj}(i, x)$.

(27) $r\,(f \cdot \mathrm{reproj}(i, x)) = (r\,f) \cdot \mathrm{reproj}(i, x)$.

(28) Suppose f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i. Then $f_1 + f_2$ is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(f_1 + f_2, x, i) = \mathrm{partdiff}(f_1, x, i) + \mathrm{partdiff}(f_2, x, i)$.

(29) Suppose g_1 is partially differentiable in y w.r.t. i and g_2 is partially differentiable in y w.r.t. i. Then $g_1 + g_2$ is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(g_1 + g_2, y, i) = \mathrm{partdiff}(g_1, y, i) + \mathrm{partdiff}(g_2, y, i)$.

(30) Suppose f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i. Then $f_1 - f_2$ is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(f_1 - f_2, x, i) = \mathrm{partdiff}(f_1, x, i) - \mathrm{partdiff}(f_2, x, i)$.

(31) Suppose g_1 is partially differentiable in y w.r.t. i and g_2 is partially differentiable in y w.r.t. i. Then $g_1 - g_2$ is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(g_1 - g_2, y, i) = \mathrm{partdiff}(g_1, y, i) - \mathrm{partdiff}(g_2, y, i)$.

(32) Suppose f is partially differentiable in x w.r.t. i. Then $r\,f$ is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(r\,f, x, i) = r \cdot \mathrm{partdiff}(f, x, i)$.

(33) Suppose g is partially differentiable in y w.r.t. i. Then $r\,g$ is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(r\,g, y, i) = r \cdot \mathrm{partdiff}(g, y, i)$.

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