

Several Differentiation Formulas of Special Functions. Part V

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Summary. In this article, we give several differentiation formulas of special and composite functions including trigonometric, polynomial and logarithmic functions.

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The articles [13], [15], [1], [16], [2], [4], [10], [11], [17], [5], [14], [12], [3], [7], [6], [9], and [8] provide the notation and terminology for this paper.

The partial function \sec from \mathbb{R} to \mathbb{R} is defined as follows:

$$\text{(Def. 1)} \quad \sec = \frac{1}{\text{the function } \cos}.$$

The partial function cosec from \mathbb{R} to \mathbb{R} is defined by:

$$\text{(Def. 2)} \quad \operatorname{cosec} = \frac{1}{\text{the function } \sin}.$$

For simplicity, we follow the rules: x, a, b, c are real numbers, n is a natural number, Z is an open subset of \mathbb{R} , and f, f_1, f_2 are partial functions from \mathbb{R} to \mathbb{R} .

One can prove the following propositions:

- (1) If $(\text{the function } \cos)(x) \neq 0$, then \sec is differentiable in x and $(\sec)'(x) = \frac{(\text{the function } \sin)(x)}{(\text{the function } \cos)(x)^2}$.
- (2) If $(\text{the function } \sin)(x) \neq 0$, then cosec is differentiable in x and $(\operatorname{cosec})'(x) = -\frac{(\text{the function } \cos)(x)}{(\text{the function } \sin)(x)^2}$.
- (3) $\left(\frac{1}{x}\right)'_Z = -\frac{1}{x^2}$.
- (4) Suppose $Z \subseteq \operatorname{dom} \sec$. Then \sec is differentiable on Z and for every x such that $x \in Z$ holds $(\sec)'|_Z(x) = \frac{(\text{the function } \sin)(x)}{(\text{the function } \cos)(x)^2}$.

- (5) Suppose $Z \subseteq \text{dom cosec}$. Then cosec is differentiable on Z and for every x such that $x \in Z$ holds $(\text{cosec})'_{|Z}(x) = -\frac{(\text{the function cos})(x)}{(\text{the function sin})(x)^2}$.
- (6) Suppose $Z \subseteq \text{dom}(\text{sec} \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
- $\text{sec} \cdot f$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $(\text{sec} \cdot f)'_{|Z}(x) = \frac{a \cdot (\text{the function sin})(a \cdot x + b)}{(\text{the function cos})(a \cdot x + b)^2}$.
- (7) Suppose $Z \subseteq \text{dom}(\text{cosec} \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
- $\text{cosec} \cdot f$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $(\text{cosec} \cdot f)'_{|Z}(x) = -\frac{a \cdot (\text{the function cos})(a \cdot x + b)}{(\text{the function sin})(a \cdot x + b)^2}$.
- (8) Suppose $Z \subseteq \text{dom}(\text{sec} \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- $\text{sec} \cdot \frac{1}{f}$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $(\text{sec} \cdot \frac{1}{f})'_{|Z}(x) = -\frac{(\text{the function sin})(\frac{1}{x})}{x^2 \cdot (\text{the function cos})(\frac{1}{x})^2}$.
- (9) Suppose $Z \subseteq \text{dom}(\text{cosec} \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- $\text{cosec} \cdot \frac{1}{f}$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $(\text{cosec} \cdot \frac{1}{f})'_{|Z}(x) = \frac{(\text{the function cos})(\frac{1}{x})}{x^2 \cdot (\text{the function sin})(\frac{1}{x})^2}$.
- (10) Suppose $Z \subseteq \text{dom}(\text{sec} \cdot (f_1 + c f_2))$ and $f_2 = \frac{2}{Z}$ and for every x such that $x \in Z$ holds $f_1(x) = a + b \cdot x$. Then
- $\text{sec} \cdot (f_1 + c f_2)$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $(\text{sec} \cdot (f_1 + c f_2))'_{|Z}(x) = \frac{(b+2 \cdot c \cdot x) \cdot (\text{the function sin})(a+b \cdot x+c \cdot x^2)}{(\text{the function cos})(a+b \cdot x+c \cdot x^2)^2}$.
- (11) Suppose $Z \subseteq \text{dom}(\text{cosec} \cdot (f_1 + c f_2))$ and $f_2 = \frac{2}{Z}$ and for every x such that $x \in Z$ holds $f_1(x) = a + b \cdot x$. Then
- $\text{cosec} \cdot (f_1 + c f_2)$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $(\text{cosec} \cdot (f_1 + c f_2))'_{|Z}(x) = -\frac{(b+2 \cdot c \cdot x) \cdot (\text{the function cos})(a+b \cdot x+c \cdot x^2)}{(\text{the function sin})(a+b \cdot x+c \cdot x^2)^2}$.
- (12) Suppose $Z \subseteq \text{dom}(\text{sec} \cdot (\text{the function exp}))$. Then
- $\text{sec} \cdot (\text{the function exp})$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $(\text{sec} \cdot (\text{the function exp}))'_{|Z}(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function sin})((\text{the function exp})(x))}{(\text{the function cos})((\text{the function exp})(x))^2}$.
- (13) Suppose $Z \subseteq \text{dom}(\text{cosec} \cdot (\text{the function exp}))$. Then
- $\text{cosec} \cdot (\text{the function exp})$ is differentiable on Z , and

- (ii) for every x such that $x \in Z$ holds $(\operatorname{cosec} \cdot (\text{the function exp}))'_{|Z}(x) = \frac{(\text{the function exp})(x) \cdot (\text{the function cos})((\text{the function exp})(x))}{(\text{the function sin})((\text{the function exp})(x))^2}$.
- (14) Suppose $Z \subseteq \operatorname{dom}(\sec \cdot (\text{the function ln}))$. Then
- (i) $\sec \cdot (\text{the function ln})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\sec \cdot (\text{the function ln}))'_{|Z}(x) = \frac{(\text{the function sin})((\text{the function ln})(x))}{x \cdot (\text{the function cos})((\text{the function ln})(x))^2}$.
- (15) Suppose $Z \subseteq \operatorname{dom}(\operatorname{cosec} \cdot (\text{the function ln}))$. Then
- (i) $\operatorname{cosec} \cdot (\text{the function ln})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\operatorname{cosec} \cdot (\text{the function ln}))'_{|Z}(x) = \frac{(\text{the function cos})((\text{the function ln})(x))}{x \cdot (\text{the function sin})((\text{the function ln})(x))^2}$.
- (16) Suppose $Z \subseteq \operatorname{dom}((\text{the function exp}) \cdot \sec)$. Then
- (i) $(\text{the function exp}) \cdot \sec$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function exp}) \cdot \sec)'_{|Z}(x) = \frac{(\text{the function exp})((\sec)(x)) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^2}$.
- (17) Suppose $Z \subseteq \operatorname{dom}((\text{the function exp}) \cdot \operatorname{cosec})$. Then
- (i) $(\text{the function exp}) \cdot \operatorname{cosec}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function exp}) \cdot \operatorname{cosec})'_{|Z}(x) = \frac{(\text{the function exp})((\operatorname{cosec})(x)) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$.
- (18) Suppose $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot \sec)$. Then
- (i) $(\text{the function ln}) \cdot \sec$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function ln}) \cdot \sec)'_{|Z}(x) = \frac{(\text{the function sin})(x)}{(\text{the function cos})(x)}$.
- (19) Suppose $Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot \operatorname{cosec})$. Then
- (i) $(\text{the function ln}) \cdot \operatorname{cosec}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function ln}) \cdot \operatorname{cosec})'_{|Z}(x) = \frac{(\text{the function cos})(x)}{(\text{the function sin})(x)}$.
- (20) Suppose $Z \subseteq \operatorname{dom}(\binom{n}{Z} \cdot \sec)$ and $1 \leq n$. Then
- (i) $\binom{n}{Z} \cdot \sec$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\binom{n}{Z} \cdot \sec)'_{|Z}(x) = \frac{n \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^{n+1}}$.
- (21) Suppose $Z \subseteq \operatorname{dom}(\binom{n}{Z} \cdot \operatorname{cosec})$ and $1 \leq n$. Then
- (i) $\binom{n}{Z} \cdot \operatorname{cosec}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\binom{n}{Z} \cdot \operatorname{cosec})'_{|Z}(x) = \frac{n \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^{n+1}}$.
- (22) Suppose $Z \subseteq \operatorname{dom}(\sec - \operatorname{id}_Z)$. Then
- (i) $\sec - \operatorname{id}_Z$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\sec - \operatorname{id}_Z)'_{|Z}(x) = \frac{(\text{the function sin})(x) - (\text{the function cos})(x)^2}{(\text{the function cos})(x)^2}$.

- (23) Suppose $Z \subseteq \text{dom}(-\text{cosec} - \text{id}_Z)$. Then
- $-\text{cosec} - \text{id}_Z$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $(-\text{cosec} - \text{id}_Z)'|_Z(x) = \frac{(\text{the function cos})(x) - (\text{the function sin})(x)^2}{(\text{the function sin})(x)^2}$.
- (24) Suppose $Z \subseteq \text{dom}((\text{the function exp}) \text{ sec})$. Then
- $(\text{the function exp}) \text{ sec}$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $((\text{the function exp}) \text{ sec})'|_Z(x) = \frac{(\text{the function exp})(x)}{(\text{the function cos})(x)} + \frac{(\text{the function exp})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^2}$.
- (25) Suppose $Z \subseteq \text{dom}((\text{the function exp}) \text{ cosec})$. Then
- $(\text{the function exp}) \text{ cosec}$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $((\text{the function exp}) \text{ cosec})'|_Z(x) = \frac{(\text{the function exp})(x)}{(\text{the function sin})(x)} - \frac{(\text{the function exp})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$.
- (26) Suppose $Z \subseteq \text{dom}(\frac{1}{a}(\text{sec} \cdot f) - \text{id}_Z)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x$ and $a \neq 0$. Then
- $\frac{1}{a}(\text{sec} \cdot f) - \text{id}_Z$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $(\frac{1}{a}(\text{sec} \cdot f) - \text{id}_Z)'|_Z(x) = \frac{(\text{the function sin})(a \cdot x) - (\text{the function cos})(a \cdot x)^2}{(\text{the function cos})(a \cdot x)^2}$.
- (27) Suppose $Z \subseteq \text{dom}((-\frac{1}{a})(\text{cosec} \cdot f) - \text{id}_Z)$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x$ and $a \neq 0$. Then
- $(-\frac{1}{a})(\text{cosec} \cdot f) - \text{id}_Z$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $((-\frac{1}{a})(\text{cosec} \cdot f) - \text{id}_Z)'|_Z(x) = \frac{(\text{the function cos})(a \cdot x) - (\text{the function sin})(a \cdot x)^2}{(\text{the function sin})(a \cdot x)^2}$.
- (28) Suppose $Z \subseteq \text{dom}(f \text{ sec})$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
- $f \text{ sec}$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $(f \text{ sec})'|_Z(x) = \frac{a}{(\text{the function cos})(x)} + \frac{(a \cdot x + b) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^2}$.
- (29) Suppose $Z \subseteq \text{dom}(f \text{ cosec})$ and for every x such that $x \in Z$ holds $f(x) = a \cdot x + b$. Then
- $f \text{ cosec}$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $(f \text{ cosec})'|_Z(x) = \frac{a}{(\text{the function sin})(x)} - \frac{(a \cdot x + b) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$.
- (30) Suppose $Z \subseteq \text{dom}((\text{the function ln}) \text{ sec})$. Then
- $(\text{the function ln}) \text{ sec}$ is differentiable on Z , and
 - for every x such that $x \in Z$ holds $((\text{the function ln}) \text{ sec})'|_Z(x) = \frac{1}{(\text{the function cos})(x)} + \frac{(\text{the function ln})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^2}$.
- (31) Suppose $Z \subseteq \text{dom}((\text{the function ln}) \text{ cosec})$. Then

- (i) (the function \ln) cosec is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function } \ln) \text{ cosec})'_{|Z}(x) = \frac{\frac{1}{(\text{the function } \sin)(x)}}{x} - \frac{(\text{the function } \ln)(x) \cdot (\text{the function } \cos)(x)}{(\text{the function } \sin)(x)^2}$.
- (32) Suppose $Z \subseteq \text{dom}(\frac{1}{f} \sec)$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- (i) $\frac{1}{f} \sec$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\frac{1}{f} \sec)'_{|Z}(x) = -\frac{\frac{1}{(\text{the function } \cos)(x)}}{x^2} + \frac{(\text{the function } \sin)(x)}{(\text{the function } \cos)(x)^2}$.
- (33) Suppose $Z \subseteq \text{dom}(\frac{1}{f} \text{cosec})$ and for every x such that $x \in Z$ holds $f(x) = x$. Then
- (i) $\frac{1}{f} \text{cosec}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\frac{1}{f} \text{cosec})'_{|Z}(x) = -\frac{\frac{1}{(\text{the function } \sin)(x)}}{x^2} - \frac{(\text{the function } \cos)(x)}{(\text{the function } \sin)(x)^2}$.
- (34) Suppose $Z \subseteq \text{dom}(\sec \cdot (\text{the function } \sin))$. Then
- (i) $\sec \cdot (\text{the function } \sin)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\sec \cdot (\text{the function } \sin))'_{|Z}(x) = \frac{(\text{the function } \cos)(x) \cdot (\text{the function } \sin)((\text{the function } \sin)(x))}{(\text{the function } \cos)((\text{the function } \sin)(x))^2}$.
- (35) Suppose $Z \subseteq \text{dom}(\sec \cdot (\text{the function } \cos))$. Then
- (i) $\sec \cdot (\text{the function } \cos)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\sec \cdot (\text{the function } \cos))'_{|Z}(x) = -\frac{(\text{the function } \sin)(x) \cdot (\text{the function } \sin)((\text{the function } \cos)(x))}{(\text{the function } \cos)((\text{the function } \cos)(x))^2}$.
- (36) Suppose $Z \subseteq \text{dom}(\text{cosec} \cdot (\text{the function } \sin))$. Then
- (i) $\text{cosec} \cdot (\text{the function } \sin)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\text{cosec} \cdot (\text{the function } \sin))'_{|Z}(x) = -\frac{(\text{the function } \cos)(x) \cdot (\text{the function } \cos)((\text{the function } \sin)(x))}{(\text{the function } \sin)((\text{the function } \sin)(x))^2}$.
- (37) Suppose $Z \subseteq \text{dom}(\text{cosec} \cdot (\text{the function } \cos))$. Then
- (i) $\text{cosec} \cdot (\text{the function } \cos)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\text{cosec} \cdot (\text{the function } \cos))'_{|Z}(x) = \frac{(\text{the function } \sin)(x) \cdot (\text{the function } \cos)((\text{the function } \cos)(x))}{(\text{the function } \sin)((\text{the function } \cos)(x))^2}$.
- (38) Suppose $Z \subseteq \text{dom}(\sec \cdot (\text{the function } \tan))$. Then
- (i) $\sec \cdot (\text{the function } \tan)$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\sec \cdot (\text{the function } \tan))'_{|Z}(x) = \frac{(\text{the function } \sin)((\text{the function } \tan)(x))}{(\text{the function } \cos)(x)^2} \cdot \frac{(\text{the function } \cos)((\text{the function } \tan)(x))}{(\text{the function } \cos)((\text{the function } \tan)(x))^2}$.
- (39) Suppose $Z \subseteq \text{dom}(\sec \cdot (\text{the function } \cot))$. Then
- (i) $\sec \cdot (\text{the function } \cot)$ is differentiable on Z , and

- (ii) for every x such that $x \in Z$ holds $(\sec \cdot (\text{the function cot}))'_{|Z}(x) = \frac{(\text{the function sin})(\text{the function cot})(x)}{(\text{the function sin})(x)^2} - \frac{(\text{the function cos})(\text{the function cot})(x)}{(\text{the function sin})(\text{the function cot})(x))^2}$.
- (40) Suppose $Z \subseteq \text{dom}(\text{cosec} \cdot (\text{the function tan}))$. Then
- (i) $\text{cosec} \cdot (\text{the function tan})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\text{cosec} \cdot (\text{the function tan}))'_{|Z}(x) = \frac{(\text{the function cos})(\text{the function tan})(x)}{(\text{the function cos})(x)^2} - \frac{(\text{the function sin})(\text{the function tan})(x)}{(\text{the function sin})(\text{the function tan})(x))^2}$.
- (41) Suppose $Z \subseteq \text{dom}(\text{cosec} \cdot (\text{the function cot}))$. Then
- (i) $\text{cosec} \cdot (\text{the function cot})$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $(\text{cosec} \cdot (\text{the function cot}))'_{|Z}(x) = \frac{(\text{the function cos})(\text{the function cot})(x)}{(\text{the function sin})(x)^2} - \frac{(\text{the function sin})(\text{the function cot})(x)}{(\text{the function sin})(\text{the function cot})(x))^2}$.
- (42) Suppose $Z \subseteq \text{dom}((\text{the function tan}) \sec)$. Then
- (i) $(\text{the function tan}) \sec$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function tan}) \sec)'_{|Z}(x) = \frac{1}{(\text{the function cos})(x)} + \frac{(\text{the function tan})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^2}$.
- (43) Suppose $Z \subseteq \text{dom}((\text{the function cot}) \sec)$. Then
- (i) $(\text{the function cot}) \sec$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function cot}) \sec)'_{|Z}(x) = -\frac{1}{(\text{the function sin})(x)} + \frac{(\text{the function cot})(x) \cdot (\text{the function sin})(x)}{(\text{the function cos})(x)^2}$.
- (44) Suppose $Z \subseteq \text{dom}((\text{the function tan}) \text{cosec})$. Then
- (i) $(\text{the function tan}) \text{cosec}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function tan}) \text{cosec})'_{|Z}(x) = \frac{1}{(\text{the function cos})(x)^2} - \frac{(\text{the function tan})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$.
- (45) Suppose $Z \subseteq \text{dom}((\text{the function cot}) \text{cosec})$. Then
- (i) $(\text{the function cot}) \text{cosec}$ is differentiable on Z , and
- (ii) for every x such that $x \in Z$ holds $((\text{the function cot}) \text{cosec})'_{|Z}(x) = -\frac{1}{(\text{the function sin})(x)^2} - \frac{(\text{the function cot})(x) \cdot (\text{the function cos})(x)}{(\text{the function sin})(x)^2}$.

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